Existentially closed models of the theory of differential fields with a cyclic automorphism

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Motivation

Let C be any field and choose an arbitrary element $q \in C \setminus \{0, 1\}$. Let $\mathbb{k}_0$ denote the prime field included in C, and set $\mathbb{k} = \mathbb{k}_0(q)$, the subfield of C generated by q over $\mathbb{k}_0$.

Definition. The q-integer, the q-factorial and q-binomial, respectively, denotes

$$[k]_q = \frac{q^k - 1}{q - 1},\ [0]_q = 0,$$

$$[k]_q! = [k]_q[k-1]_q \cdots [1]_q,\ [0]_q! = 1,$$

$$\binom{m}{n}_q = \frac{[m]_q!}{[n]_q![m-n]_q!},$$

where $k, m, n \in \mathbb{N}$ with $m \geq n$.

Suppose that R is a field containing $\mathbb{k}(t)$ and $\sigma_q : R \to R$ is a ring automorphism such that it is an extension of the q-difference operator $f(t) \mapsto f(qt)$ on $\mathbb{k}(t)$.

Definition (C.Hardouin). We say that a sequence $\delta_R^* = (\delta_R^{(k)})_{k\in\mathbb{N}}$ of maps on R is an iterative q-difference operator on R if it satisfies the following condition:

1. $\delta_R^{(0)} = \mathrm{id}_R$,
2. $\delta_R^{(1)} = \frac{1}{(q-1)t}(\sigma_q - \mathrm{id}_R)$,
3. $\delta_R^{(k)}(x+y) = \delta_R^{(k)}(x) + \delta_R^{(k)}(y),\ x, y \in R$,
4. $\delta_R^{(k)}(xy) = \sum_{i+j=k} \sigma_q^i \circ \delta_R^j(x)\delta_R^{(i)}(y),\ x, y \in R$,
5. $\delta_R^{(i)} \circ \delta_R^{(j)} = \binom{i+j}{i}_q \delta_R^{(i+j)}$

Remark. Assume that q is not a root of unity. Then,

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1} \neq 0$$

for all $k > 0$. If $\delta_R^* = (\delta_R^{(k)})_{k\in\mathbb{N}}$ is an iterative q-difference operator on R, conditions 1, 2 and 5 above require

$$\delta_R^{(1)} = \frac{1}{(q-1)t}(\sigma_q - \mathrm{id}_R), \quad \delta_R^{(k)} = \frac{1}{[k]_q!}(\delta_R^{(1)})^k, \ k \in \mathbb{N}.$$

Conversely, if we define $\delta_R^{(k)}$ by above, then $\delta_R^* = (\delta_R^{(k)})_{k\in\mathbb{N}}$ forms an iterative q-difference operator on R. Therefore under the assumption, an iterative q-difference ring is nothing but a difference field (R, σ_q).

From now on, we assume q is a root of unity of order $N > 1$.

Fact (Masuoka and Y., 2013).

1. For any iterative q-difference field $(R, (\delta_R^{(k)})_{k \in \mathbb{N}})$, the q-difference operator σ_q on R is of order N, that is $\sigma_R^N = \mathrm{id}_R$.
2. There is the smallest iterative q-difference field $\Bbbk(t)$.

Theorem (Masuoka and Y., 2013). There is a functor

$$\mathcal{F} : \{\mathsf{I}q\text{D-fields}\} \to \{\text{models of } DF_\sigma\}$$

and satisfies the following properties:

1. $\mathcal{F}$ is a strictly embedding,
2. for any model (R, σ) of DF_σ there is $\mathcal{F}^{-1}(R)$ whenever $R \supset \mathcal{F}(\mathbb{k}(t))$ and $\sigma^N = \mathrm{id}_R$, and
3. $\mathcal{F}$ has a good property for model theory.

DF_{C_N} denotes the theory $DF_\sigma \cup \left\{\forall x(\sigma^N(x) = x)\right\}$.

Corollary. Suppose that q is a root of unity of order $N > 1$. Then the theory $IqDF$ and the theory $DF_{C_N} \cup \mathrm{Diag}(\mathcal{F}(\Bbbk(t)))$ have same model theoretical property.

To study $IqDF$, first, to study about DF_{C_N}.

Question. Does the theory DF_{C_N} admit a model companion?

Introduction

Definition. Let K be a field and δ be a additive map on K. We say that (K, δ) is a differential field if δ satisfies the Leibnitz rule:

$$\delta(ab) = a\delta(b) + \delta(a)b, \quad \text{for all } a, b \in K.$$

The language of differential fields, denoted by L_δ, is the language of rings with a new unary function symbol δ. DF denotes the theory of differential fields (of characteristic 0) in the language L_δ.

T_σ

Suppose that T is a theory in a language L. L_σ denotes the language L with a new unary function symbol σ. We consider the theory

$$T_\sigma = T \cup \text{“}\sigma \text{ is an automorphism”}.$$

Example. Let $K = \mathbb{Q}(X), \delta = \frac{d}{dX}$, and $\sigma(X) = X + 1$. Then

- $K \models Fld$,
- $(K, \delta) \models DF$,
- $(K, \sigma) \models Fld_\sigma$, and
- $(K, \delta, \sigma) \models DF_\sigma$.

Model companion

Let T be a theory in a language L.

A model M of T is existentially closed if for any extension $N \models T$ of M and quantifier-free formula $\varphi(x)$ over M,

$$\text{if} \quad N \models \exists x\varphi(x) \quad \text{then} \quad M \models \exists\varphi(x).$$

Definition. Suppose that T is a $\forall\exists$-theory. We say that T admits a model companion if the class

$$\mathcal{K} = \{M \models T \mid M \text{ is existentially closed.}\}$$

is elementary.

Fact.

1. (Tarski) *Fld* admits a model companion. $\rightarrow$ *ACF*.
2. (Robinson) *DF* admits a model companion. $\rightarrow$ *DCF*
3. (Macintyre) Fld_σ admits a model companion. $\rightarrow$ *ACFA*.
4. (Hrushovski) DF_σ admits a model companion. $\rightarrow$ *DCFA*

Example. The theory of groups does not admit model companion.

Remark. The automorphism σ of any model of $ACFA$ (or $DCFA$) does not have finite order.

Suppose that $(K, \sigma) \models ACFA$. Let $n(> 0)$ be a natural number. We define $\widetilde{\sigma}$ on $K(X_1, \ldots, X_{n+1})$ by

$$\widetilde{\sigma}(X_i) = X_{i+1} \quad (i < n),$$
$$\widetilde{\sigma}(X_{n+1}) = X_1,$$
$$\widetilde{\sigma}|_K = \sigma.$$

Then $(K(X_1, \ldots, X_{n+1}), \widetilde{\sigma})$ is a model of Fld_σ extending (K, σ) and $K(X_1, \ldots, X_{n+1}) \models \exists x \bigwedge_{0<i<n+1} \sigma^i(x) \neq x$.

Some results

First approach

We want to construct an existentially closed model of DF_{C_N}, where DF_{C_N} is $DF_\sigma \cup \left\{\forall x(\sigma^N(x) = x)\right\}$.

Let (K, δ, σ) be a model of $DCFA$.

We consider the fixed field

$$K^{\langle \sigma^N \rangle} := \{a \in K \mid \sigma^N(a) = a\}$$

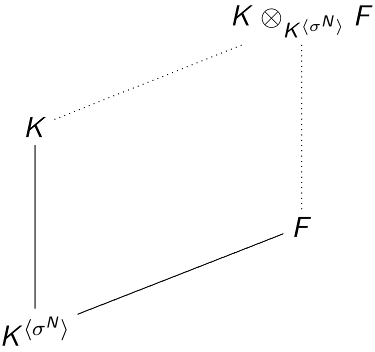
of σ^N in K. Then $(K^{\langle \sigma^N \rangle}, \delta|_{K^{\langle \sigma^N \rangle}}, \sigma|_{K^{\langle \sigma^N \rangle}})$ is naturally a model of DF_{C_N}, since $\sigma \circ \delta = \delta \circ \sigma$ in K.

Question. Is $(K^{\langle \sigma^N \rangle}, \delta|_{K^{\langle \sigma^N \rangle}}, \sigma|_{K^{\langle \sigma^N \rangle}})$ an existentially closed model of DF_{C_N}?

Answer. I don't know, but it has the property close to existentially closed.

Suppose that (F, δ, σ) is an extension of $(K^{\langle \sigma^N \rangle}, \delta|_{K^{\langle \sigma^N \rangle}}, \sigma|_{K^{\langle \sigma^N \rangle}})$ and $\varphi(x)$ is a quantifier-free formula over $K^{\langle \sigma^N \rangle}$ such that

$$(F, \delta', \sigma') \models \exists x \varphi(x).$$



We can define derivation and difference on $K \otimes_{K^{\langle \sigma^N \rangle}} F$. We assume $K \otimes_{K^{\langle \sigma^N \rangle}} F$ is an integral domain. Then there is a model K' of DF_σ such that

$$K \subset K' \quad \text{and} \quad K' \models \exists x(\varphi(x) \wedge \sigma^N(x) = x).$$

Since K is existentially closed, $K \models \exists x(\varphi(x) \wedge \sigma^N(x) = x)$. This means $K^{\langle \sigma^N \rangle} \models \exists x \varphi(x)$.

Probrem. Is $K \otimes_{K^{\langle \sigma^N \rangle}} F$ always an integral domain?

In my opinion, there is little possibility that the answer is yes.

Second approach

We will modify N.Sjögren's argument in "The Model Theory of Field with a Group action".
In this paper, he show the following theorem:

Theorem (N.Sjögren, 2005). Fld_{C_N} admits a model companion.

More precisely, a model (K, σ) of Fld_{C_N} is existentially closed iff it holds the following properties

1. K and $K^{\langle\sigma\rangle}$ are pseudo-algebraically closed,
2. $\mathrm{Gal}(K \cap (K^{\langle\sigma\rangle})^{alg} / K^{\langle\sigma\rangle}) \simeq C_N (\simeq \mathbb{Z}/N\mathbb{Z})$,
3. $\mathrm{Gal}((K^{\langle\sigma\rangle})^{alg} / K^{\langle\sigma\rangle}) \simeq \mathrm{Gal}(K^{alg}/K) \simeq \mathbb{Z}_N$

Remark. To prove $\Rightarrow$, We need some knowledge of pro-finite group. To prove $\Leftarrow$, on the other hand, the following lemma is essentially:

Lemma. The above conditions 2 and 3 imply that (K, σ) has no algebraic C_N-field extension.

(Sketch of proof of $\Leftarrow$)

- Suppose K' be a extension of K and $\varphi(x) :=$ "$f(x) = 0$" is an $L_\sigma(K)$-formula such that $K' \models f(a) = 0$ for some $a \in K'$.
- Since σ is of order N, there is finite tuple $b \in K$ such that $K = K^{\langle\sigma\rangle}(b)$ and $K' = K'^{\langle\sigma\rangle}(b)$
- We write $a_i = \sum_j c_{ij} b_j$, and consider $V = V(c/K)$.
- Since K has no algebraic G-extension, V is absolutely irreducible, so V has K-rational point c'.
- We put $a'_i = \sum_j c'_{ij} b_j \in K$, then $K \models f(a') = 0$.

By modifying the Sjögre's proof, we can prove the following result.

Lemma. Suppose that (K, δ, σ) is an existentially closed model of DF_{C_N}. Then the following properties hold:

1. K and $K^{\langle\sigma\rangle}$ are pseudo-differentially closed,
2. $C_K = \{a \in K \mid \delta(a) = 0\}$ and $C_{K^{\langle\sigma\rangle}} = (C_K)^{\langle\sigma\rangle}$ are pseudo-algebraically closed.

Comments.

1. If the property " one has no differentially algebraic C_N-field extension" is possible to imply from the common first-order property among existentially closed models of DF_{C_N}, then (I think that) it is possible to prove that DF_{C_N} admits a model companion.
2. To do this, we need more knowledge of differential Galois group. (However, I luck it now...)

References

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