A Very Basic Introduction to Model Theory
(モデル理論の超初歩的入門講座)

Akito Tsuboi

This is a very basic introduction to Model Theory. I assume some basic knowledge of naive set theory, which is typically taught to the undergraduate level of mathematics students. Some of them are, for example, cardinality, transfinite induction and Zorn's lemma. I do not assume any specific knowledge of mathematical logic, although some familiarity with it is very helpful.

The latest version of this file is found in: www.math.tsukuba.ac.jp/˜tsuboi/

1 Languages, Structures and Models

1. Languages and formulas.

   A set consisting of constant symbols, function symbols and predicate symbols is called a *language*. (A constant symbol can be considred as a 0-ary function symbol.) Let L be a (formal) language. An L-*formula* is a formal 'proposition' constructed from L, using (individual) variables $x, y, z \dots$ and logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (not) $\to$ (implies), $\forall$ (all) and $\exists$ (some).

   Example: If $L = \{c, F, P\}$, where c is a constant symbol, F is a unary function symbol, and P is a binary predicate symbol, the following are examples of L-formulas:

   $$P(c, x),\ P(F(x), F(y)),\ \forall x[P(x, y) \to \exists z P(x, F(F(x)))],\ \dots$$

   The first two, which do not contain logical symbols, are called *atomic*.

2. Mathematical structures.

Examples of mathematical structures are: $(\mathbb{N}, 0, 1, +, \cdot)$, $(\mathbb{Z}, 0, 1, +\cdot)$, $(\mathbb{R}, 0, 1, +, \cdot)$, $(\mathbb{C}, 0, 1, +, \cdot)$, $(\mathbb{Q}, <)$, $(GL(2, \mathbb{R}), \cdot)$, ... For a language L, which is a set of symbols, we can define the notion of *L-structures* so that each of such examples becomes a structure in our sense.

3. Definable sets.

   Let M be an L-structure. We write $M \models *$, if $*$ is true in M. For example,

   - $(\mathbb{R}, 0, 1, +, \cdot, <) \models \forall x(\, 0 < x \to \exists y(x = y \cdot y \,\wedge\, \neg(y = 0)))$;
   - $(\mathbb{Z}, +, \cdot) \models \forall x \exists y_0, y_1, y_2, y_3(x = y_0 \cdot y_0 + x = y_1 \cdot y_1 + x = y_2 \cdot y_2 + x = y_3 \cdot y_3)$.

   A subset A of M is called a *definable set* if there is an L-formula $\varphi(x, \bar{y})$ and $\bar{b} \in M$ (tuples from M, called parameters) such that

   $$A = \{a \in M : M \models \varphi(a, \bar{b})\}.$$

   Definable sets of M^n is defined similarly. If M is a countable (infinite) structure, there are $2^{\aleph_0}$-many subsets of M. But there are only countably many formulas (with parametes from M). So there are only countably many definable sets of M. In general, if M has the cardinality κ, there are only κ-many definable subsets of M.

4. Definable sets and automorphisms.

   Let A be a definable set of M, defined by a formula with parameters $\bar{b}$. Let $\sigma \in Aut(M/\bar{b})$ be an automorphism of M fixing $\bar{b}$ pointwise. Then

   $$\sigma(A) = A.$$

   Example. Every finite set of M is a definable set: If $A = \{a_1, \ldots, a_n\}$, then A is defined by $x = a_1 \vee \cdots \vee a_n$. However, in the field $\mathbb{C}$, the singleton $\{\sqrt{-1}\}$ is not definable if we dot allow parameters: The mapping $x + iy \mapsto x - iy$ is an automorphism.

5. Let T be a set of L-sentences[1]. If $M \models T$, we say M is a model of T. So, in our terminology, a group is a model of the group axioms.

[1] A sentence is a formula whose variables are all bound by $\forall$ or $\exists$.

2 Compactness Theorem

1. Compactness Theorem.

   Let T be a set of L-sentences. The following two conditions on T are equivalent:

   (a) T has a model;

   (b) Every finite subset of T has a model. (T is finitely satisfiable.)

2. Main Lemma

   Definition 1. Let T^* be a set of L^*-sentences. We say that T^* has the witnessing property if whenever $\varphi(x)$ is an L^*-formula, then there is a constant c in L^* such that '$\exists x\varphi(x) \to \varphi(c)$' $\in T^*$. (In this case, we say c witnesses $\varphi(x)$.)

   Lemma 2. *Let T^* be a set of L^*-sentences with the following conditions:*

   (a) Every finite subset of T^* has a model;

   (b) T^* has the witnessing property;

   (c) T^* is complete, i.e., for all L^*-sentences φ, $\varphi \in T^*$ or $\neg\varphi \in T^*$.

   Then there is a model $M^* \models T$ whose universe is essentially the set of witnessing constants in L^*.

   Proof. Using T^*, we define an L^*-structure M^* by the following:

   - CT =the set of all closed L^*-terms. (A closed term is a term without a variable. Every constant symbol in L^* belongs to CT.)
   - For $s, t \in CT$, $s \sim t \iff s = t$ belongs to T^*. (It will be shown that $\sim$ is an equivalence relation on CT.)
   - $M^* = CT/\sim = \{[t] : t \in CT\}$.
     - $c^{M^*} := [c]$, where c is a constant symbol in L^*;
     - $F^{M^*}([t_1], \ldots, [t_m]) := [F(t_1, \ldots, t_m)]$, where F is an m-ary function symbol in L^*;
     - $P^M = \{([t_1], \ldots, [t_n]) : P(t_1, \ldots, t_n) \in T^*\}$, where P is an n-ary predicate symbol in L^*.

Claim A. *(a) The binary relation $\sim$ is an equivalence relation on CT.*

(b) F^{M^*} is well-defined.

(c) For every $t \in CT$, there is $c \in L$ such that $[t] = [c]$.

(a). Suppose $s \sim t \sim u$. Then, by the definition of $\sim$, we have '$s = t$' $\in T^*$ and '$t = u$' $\in T^*$. Then we have '$s = u$' $\in T^*$, since otherwise, by completeness of T^*, we have '$s \neq u$' $\in T^*$, Then the finite subset $\{s = t, t = u, s \neq u\}$ of T^* must have a model. This is impossible.

(b),(c). Exercise.

Claim B. *The following equivalence holds for all L-formulas $\varphi(x_1, \ldots, x_n)$ and $t_1, \ldots, t_n \in CT$:*

$$M^* \models \varphi([t_1], \ldots, [t_n]) \iff \varphi(t_1, \ldots, t_n) \in T^*.$$

We prove the equivalence by induction on the number k of logical symbols in φ.

$k = 0$: φ is an atomic formula in this case. There are several forms of atomic formulas, but for simplicity we concentrate on the case of $P(x_1, \ldots, x_n)$, where P is a predicate symbol.

$M^* \models P([t_1], \ldots, [t_n]) \iff ([t_1], \ldots, [t_n]) \in P^{M^*} \iff P(t_1, \ldots, t_n) \in T^*$.

$k + 1$: We treat the following two typical cases.

Case 1: φ has the form $\psi \wedge \theta$. Using the interpretation of $\wedge$ and the induction hypothesis, we have

$$\begin{aligned} & M^* \models (\psi \wedge \theta)([t_1], \ldots, [t_n]) \\ \iff & M^* \models \psi([t_1], \ldots, [t_n]) \text{ and } M^* \models \theta([t_1], \ldots, [t_n]) \\ \iff & \psi(t_1, \ldots, t_n) \in T^* \text{ and } \theta(t_1, \ldots, t_n) \in T^*. \end{aligned}$$

By the completeness of T^*, the last line is equivalent to $\psi(t_1, \ldots, t_n) \wedge \theta(t_1, \ldots, t_n) \in T^*$.

Case 2: $\varphi(x_1, \ldots, x_n)$ has the form $\exists y \psi(x, x_1, \ldots, x_n)$.

$$\begin{aligned} & M \models \exists x \psi(x, [t_1], \ldots, [t_n]) \\ \iff & M \models \psi([s], [t_1], \ldots, [t_n]), \text{ for some } s \in CT \\ \iff & \psi(s, t_1, \ldots, t_n) \in T^*, \text{ for some } s \in CT \\ \iff & \exists x \psi(s, t_1, \ldots, t_n) \in T^*. \end{aligned}$$

The equivalence of the first and second lines follows from the interpretation of $\exists$. The equivalence of the second and third lines follows from the induction hypothesis. For the last equivalence, the arrow $\Rightarrow$ follows from the completeness of T^*. $\Leftarrow$ is the most essential. Suppose $\exists x\psi(s, t_1, \ldots, t_n)$ belongs to T^*. Since $\psi(x, t_1, \ldots, t_n)$ is an L^*-formula, it appears in the enumeration $\{\varphi_i(x)\}_i$. So, for some $c \in C$, $T^* \supset T'$ contains $\exists x\psi(x, t_1, \ldots, t_n) \to \psi(c, t_1, \ldots, t_n)$. Then, by the completeness of T^*, $\psi(c, t_1, \ldots, t_n)$ belongs to T^*. Since $c \in CT$, we have the third line as required. (End of Proof of Claim B)

By Claim B, we have the equivalence $M^* \models \psi \iff \psi \in T^*$, for all L-sentences. In particular, (by $\Leftarrow$) we have $M^* \models T$. So T has a model. $\square$

3. Proof of Compactness

Proof. The implication $1 \Rightarrow 2$ is trivial. So we assume 2 and prove 1. For simplicity, we assume L is countable. Let $C = \{c_i : i \in \omega\}$ be a set of new constant symbols and let $L^* = L \cup C$. L^* is also countable. We enumerate all the formulas (with the free variable x) as:

$$\varphi_0(x), \varphi_1(x), \ldots, \varphi_i(x), \ldots \ (i \in \omega).$$

We can assume that, for each i, the new constants in $\varphi_i(x)$ are contained in $\{c_j\}_{j<i}$. Let

$$T' = T \cup \{\exists x\varphi_i(x) \to \varphi_i(c_i) : i \in \omega\}.$$

Claim A. *T' is finitely satisfiable.*

By induction we show that $T_n = T \cup \{\exists x\varphi_i(x) \to \varphi_i(c_i) : i < n\}$ is finitely satisfiable. Suppose we have shown that T_n is finitely satisfiable. Let $F = F_0 \cup \{\exists x\varphi_n(x) \to \varphi_n(c_n)\}$ be a finite subset of T_{n+1}, where $F_0 \subset T_n$. It is sufficient to show that F has a model. By the induction hypothesis, F_0 has a model M. If $M \models \exists x\varphi_n(x)$, then there is $d \in M$ such that $M \models \varphi_n(d)$. In this case, we put $c_n^M = d$. (Notice that c_n does not appear in T_n. So the interpretation of c_n is not yet defined.) If $M \not\models \exists x\varphi_n(x)$, we let c_n^M be an arbitrary element in M. In either case, we have

$$M \models \exists x\varphi_n(x) \to \varphi_n(c_n^M).$$

This shows that F has a model. (End of Claim A)

By Zorn's lemma, we can choose a maximal set T^* of L^*-sentences with the following conditions:

- $T' \subset T^*$;
- T^* is finitely satisfiable.

Claim B. *(a) T^* is complete, i.e., for every L^*-sentence ψ, $\psi \in T^*$ or $\neg\psi \in T^*$.*

(b) $\psi, \theta \in T^* \iff \psi \wedge \theta \in T^*$.

1. Suppose otherwise and choose ψ with $\psi \notin T^*$ and $\neg\psi \notin T^*$. By the maximality of T^*, neither $T^* \cup \{\psi\}$ nor $T^* \cup \{\neg\psi\}$ are finitely satisfiable. So there is a finite set $F \subset T^*$ such that neither $F \cup \{\psi\}$ nor $F \cup \{\psi\}$ have a model. However, if we choose a model M of F, then ψ or $\neg\psi$ must hold in M. This is a contradiction.

2. $\Rightarrow$: Suppose $\psi, \theta \in T^*$ and $\psi \wedge \theta \notin T^*$. Then, by completeness, we have $\neg(\psi \wedge \theta) \in T^*$. Then $\{\psi, \theta, \neg(\psi \wedge \theta)\} \subset T^*$ must have a model, But this is impossible. $\Leftarrow$: A similar argument. (End of Proof of Claim B)

Now T^* satisfies the conditions in Main Lemma (Lemma 2). So, there is a model $M^* \models T^*$. The reduct $M^*|L$ is a model of T. $\square$

4. Some Applications

Example 3. First we introduce a terminology. A class $\mathcal{C}$ of L-structures is called an elementary class, if there is a set T of L-sentences such that $\mathcal{C} = \{M : M \models T\}$. Let $L = L_{gp} = \{e, * \cdot *, *^{-1}\}$.

(a) The class of all groups is an elementary class, because, for an L-structure G, we have

$$G \text{ is a group} \iff G \models \text{the group axioms}.$$

The group axioms are Associativity $(\forall xyz(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$, Identity $(\forall x(x \cdot e = e \cdot x = x)$ and Inverse $(\forall x(x \cdot x^{-1} = x^{-1} \cdot x = e)$. (Notice that each of the three axioms is an L-sentence.)

(b) The class of all *finite* groups is not an elementary class. By way of a contradiction, suppose that T axiomatizes the class:

$$G \text{ is a finite group} \iff G \models T.$$

Let χ_n denote a sentence stating that there are at least n elements (in the universe). For example, χ_3 is the sentence $\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1)$. We put

$$T^* = T \cup \{\chi_n : n \in \omega\}.$$

Since there are arbitrarily large finite groups, T^* is finitely satisfiable. (Notice that $\mathbb{Z}/n\mathbb{Z}$ is a finite group of cardinality n.) So, by Compactness Theorem, there is a model $G^* \models T^*$. Since $G^* \models T$, G^* is a finite group. However, we have $G^* \models \chi_n$ for all n. So the universe of G^* must be an infinite set. This is a contradiction.

Example 4. Let $\mathbb{N} = (\mathbb{N}, 0, 1, \dots, <)$. We show that the set $2\mathbb{N}$ of all even numbers is not definable. Recall that every definable set is setwise fixed by all automorphisms. We want to use this fact to show the non-definability of $2\mathbb{N}$. However, since $\mathbb{N}$ is rigid (having no non-trivial automorphisms), we cannot apply it directly. Let

$$T = \{\text{ all sentences hold in } \mathbb{N}\} \cup \{0 < c, 1 < c, 2 < c, \dots\},$$

where c is a new constant symbol. Notice that T is finitely satisfiable. So, by compactness, there is a model $M \models T$. M contains (a copy of) $\mathbb{N}$.

Now we assume for contradiction that $2\mathbb{N}$ is definable using a formula $\varphi(x)$. In $\mathbb{N}$, if x satisfies $\varphi(x)$, then $x+1$ (the next element of x) does not satisfy φ. Since this property is written by a sentence, it is also true in M. Let $\sigma : M \to M$ be the mapping

$$\sigma(a) = \begin{cases} a & \text{if } a \in \mathbb{N}, \\ a+1 & \text{otherwise.} \end{cases}$$

Clearly, σ preserves $<$, so it is an automorphism (fixing $\mathbb{N}$ pointwise). Hence σ must preserve φ. This is a contradiction.

Example 5. We write $M \prec N$ if M is a substructure of N such that, for all $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$, $M \models \varphi(a_1, \dots, a_n) \iff N \models \varphi(a_1, \dots, a_n)$. N is called an elementary extension of M. Let M_i $(i < \alpha)$ satisfy

$$M_0 \prec M_1 \prec \cdots \prec M_i \prec \dots.$$

Then each M_i is an elementary substructure of $\bigcup_{i<\alpha} M_i$.

Proof. For each element $a \in \bigcup_{i<\alpha} M_i$, add a constant c_a to L. Let L^* be the augmented lauguage. Let $T^* = \bigcup_{i<\alpha}\{\varphi : M_i \models \varphi\}$. Clearly, T^* is finitely satisfiable and complete. Moreover, since each element has a name in L^*, T^* has the witnessing property. So, by Main Lemma, there is a model $M^* \models T^*$ whose universe is $\{[c_a] : a \in \bigcup_{i<\alpha} M_i\}$. Notice that $a \neq b$ implies $[c_a] \neq [c_b]$. So the universe of M^* is identified with $\bigcup_{i<\alpha} M_i$. In other words, $\bigcup_{i<\alpha} M_i \models T^*$. □

3 Construction of Large Models and Small Models

1. Types.

   Let M be an L-structure and $A \subset M$. A formula with parameters from A is called an $L(A)$-formula. A definable set defined by an $L(A)$-formula is called an A-definable set.

   Definition 6. Let M be an L-structure and let $A \subset M$. Let $\Sigma(\bar{x})$ be a set of $L(A)$-formulas whose free variables[2] are contained in $\bar{x}$.

   (a) $\Sigma(\bar{x})$ is finitely satisfiable in M, if whenever $\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x}) \in \Sigma(\bar{x})$ then $M \models \exists \bar{x}(\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}))$.

   (b) $\Sigma(\bar{x})$ is complete (with respect to $L(A)$), if $\varphi(\bar{x}) \in \Sigma(\bar{x})$ or $\neg\varphi(\bar{x}) \in \Sigma(\bar{x})$, for all $L(A)$-formulas $\varphi(\bar{x})$.

   (c) $\Sigma(\bar{x})$ is a type over A (in M), if it is complete and finitely satisfiable.

   Remark 7. Every finitely satisfiable set $\Sigma(\bar{x})$ of $L(A)$-formulas can be extended to a type $p(\bar{x})$ over A. (Use Zorn's lemma.)

   The set of all types $p(\bar{x})$ over A is denoted by $S(A)$. $p(\bar{x}) \in S(A)$ is called an n-type if $|\bar{x}| = n$ (the length is n).

   Proposition 8. *Let M be an L-structure. Let $p(x) \in S(A)$, where $A \subset M$. Then there is an extension $M^* \supset M$ such that*

   (a) M^* is an elementary extension of M, i.e. for all $L(M)$-sentences φ, $M \models \varphi \iff M^* \models \varphi$.

[2] A variable in a formula is called a free variable if it is not bound by quantifiers.

(b) *M^* realizes $p(x)$, i.e. there is an element $d \in M$ that satisfies all $\psi(x) \in p(x)$.*

2. Models realizing many types

3. Models realizing few types

4. Some Applications

4 Unstable Theories

1. Indiscernible sequence

2. Instability

   x, y, ... will be used to denote finite tuples of variables. $\varphi^{\text{if} X}$ is the formula φ if X holds and otherwise $\neg\varphi$. 2^ω is the set of all $\{0,1\}$-sequences of length ω.

   Definition 9. (a) We say $\varphi(x, y)$ has the independence property if there is a sequence $\{a_i\}_{i\in\omega}$ such that

   (*) $\{\varphi(x, a_i)^{\text{if } \eta(i)=1} : i \in \omega\}$ is consistent for all $\eta \in 2^\omega$.

   We say T has the independence property if some formula φ has the independence property.

   (b) We say $\varphi(x, y)$ has the strict order property if there is a sequence $\{a_i\}_{i\in\omega}$ such that

   (**) $\mathcal{M} \models \forall x[\varphi(x, a_i) \rightarrow \varphi(x, a_j)] \iff i \leq j$, for all $i, j \in \omega$.

   We say T has the strict order property if some formula φ has the independence property.

3. Unstable formulas

5 Exercises with Detailed Hints

1. Exercise for Compactness

   (a) Let $L = \{0, S\}$, where 0 is a constant symbol and S is a unary function symbol. Let T be the set of following L-sentences.

- $\forall x \forall y (S(x) = S(y) \to x = y)$. ($S$ is one-to-one.)
- $\forall y (y \neq 0 \to \exists x (S(x) = y))$. (Every non-zero element belongs to the image of S.)
- $\forall x (S(x) \neq 0)$. (0 does not belong to the image.)
- $\forall x (S^n(x) \neq x)$ $(n = 1, 2, ...)$. (S has no loops.)

i. Show that T has infinitely many (pairwise non-isomorphic) countable models. (Hint: $\mathbb{N}$ becomes a model of T, if we define S on $\mathbb{N}$ by $S(n) = n + 1$. This is the standard model of T. For $n \in \omega$, let $M_n = \mathbb{N} \sqcup \underbrace{\mathbb{Z} \sqcup \cdots \sqcup \mathbb{Z}}_{n}$ and define S on each $\mathbb{Z}$-chain by $S(n) = n + 1$. M_n's are non-isomorphic.)

ii. Show that T is complete (any two models are elementarily equivalent).
Hint: Let M and N be two (countable) models of T. By Löwenheim-Skolem theorem, there are elementary extensions $M^* \succ M$ and $N^* \succ N$ of size $\aleph_1$. Either of M^* and N^* consists of the standard part $(\cong (\mathbb{N}, S))$ plus $\aleph_1$-many $\mathbb{Z}$-parts. So we have $M^* \cong N^*$, and hence $M^* \equiv N^*$.

(b) An element $a \in G$, where G is a group, is called a torsion element if there is an integer $n > 0$ such that $a^n = e$, the least such n is called the order of a. Show the following:

i. The class $\mathcal{C}$ of all torsion free groups is an elementary class. (G is torsion free if no element other than e is of finite order.)
Hint: The statement 'the order of x is n' can be expressed by an L-formula.

ii. The class $\mathcal{D}$ of all torsion groups is not an elementary class. (G is called a torsion group if every element in G is a torsion element.)
Hint: Suppose that T axiomatizes $\mathcal{D}$ and derive a contradiction. Consider $L^* = L_{gp} \cup \{c\}$, where c is a new constant.

1. Show that if T has the independence property then T is unstable.

Hint: Let $\kappa \geq \omega$ be given. By compactness the sequence $\{a_i\}$ can be assumed to have the length κ. For each $\eta \in 2^\kappa$, $p_\eta(x) = \{\varphi(x, a_i)^{\text{if } \eta(i) = 1} : i \in \omega\}$ is a (possibly incomplete) type over the a_i's. It is clear that if $\eta \neq \nu$ then p_η and p_ν are contradictory. So T is not κ-stable.

2. Show that if T has the strict order property then T is unstable.

   Hint: Let $\kappa \geq \omega$ be given and choose the minimum λ such that $2^\lambda > \kappa$. Then we have $2^{<\lambda} \leq \kappa$. By giving the lexicographic order on $2^{\leq\lambda}$, we can assume $2^{\leq\lambda}$ is an ordered set. By compactness the sequence $\{a_i\}$ in (**) can be assumed to be indexed by the set $2^{<\lambda}$. For each $\nu \in 2^\lambda$, $p_\nu(x) = \{\varphi(x, a_\eta)^{\text{if } \eta \geq \nu} : \eta \in 2^{<\lambda}\}$ is a (possibly incomplete) type over the a_η's.

3. Show that if φ has the independence property $\{a_i\}_{i\in\omega}$ in (*) can be chosen as an indiscernible sequence.

4. Show that if φ has the strict order property $\{a_i\}_{i\in\omega}$ in (**) can be chosen as an indiscernible sequence.

5. Suppose that T is unstable. Show that T has either the strict order property or the independence property.

   Hint: Let $\varphi(x, y)$ and $\{a_i : i \in \omega\}$ witness the unstability. (We can assume $\{a_i\}_i$ is indiscernible, and it can be extended to an indiscernible subsequence $\{a_i : i \in \mathbb{Q}\}$.) Suppose that T does not have the independence property. Then there exist $n \in \omega$ and $F \subset n$ such $\bigwedge_{i\in F} \neg\varphi(x, a_i) \wedge \bigwedge_{i\in n\smallsetminus F} \varphi(x, a_i)$ is inconsistent. However, $\bigwedge_{i<|F|} \neg\varphi(x, a_i) \wedge \bigwedge_{|F|\leq i<n} \varphi(x, a_i)$ is consistent. (It is realized by $a_{|F|-1/2}$.) Since a permutation is a product of swaps $(k, k+1)$, this shows that there is a decomposition $F_0 \sqcup F_1 \sqcup \{k, k+1\}$ of n such that (inside the set defined by $\bigwedge_{i\in F_0} \neg\varphi(x, a_i) \wedge \bigwedge_{i\in F_1} \varphi(x, a_i)$) (i) $\varphi(x, a_k) \wedge \neg\varphi(x, a_{k+1})$ is inconsistent, but (ii) $\varphi(x, a_{k+1}) \wedge \neg\varphi(x, a_k)$ is consistent. Hence T has the strict order property.

6. Suppose that T is a NIP theory (i.e., T does not have the independence property). Let $I = \{a_i\}_{i\in\omega}$ a non-trivial indiscernible sequence. Let $\varphi = \varphi(x, b)$ be a formula. Show that the set

   $$I_\varphi = \{a_i \in I : \mathcal{M} \models \varphi(a_i, b)\}$$

   is either finite or co-finite.

   Hint: Suppose otherwise. Then both I_φ and $I_{\neg\varphi}$ are infinite. Let $\eta \in 2^\omega$ be arbitrary. We can choose an increasing function $f : \omega \to \omega$ such that $a_{f(n)} \in I_\varphi$ if $\eta(n) = 1$ and otherwise $a_{f(n)} \in I_{\neg\varphi}$. For this f, we can show that

$$\{\varphi(a_{f(i)}, y)^{\text{if } \eta(i)=1} : i \in \omega\}$$

is satisfied by b. Since I is indiscernible, this shows that $\{\varphi(a_i, y)^{\text{if } \eta(i)=1} : i \in \omega\}$ is consistent. Hence T has the independence property. A contradiction.

7. Suppose that T is NIP. Let $I = \{a_i\}_{i\in\omega}$ be a (non-trivial) indiscernible sequence and let A be an arbitrary set. We define the average type of I over A by:

   $$Av(I/A) = \{\varphi(x) \in L(A) : I_\varphi \text{ is co-finite}\}.$$

   Show that $Av(I/A)$ is a complete type over A.

   Hint: Let F be a finite subset of $Av(I/A)$. For each $\varphi(x) \in F$, I_φ is a co-finite set. So, if we choose $a \in \bigcap_{\varphi\in F} I_\varphi$, then a satisfies all the formulas in F. This argument shows that $Av(I/A)$ is finitely satisfiable. The completeness follows from the fact that I_φ is co-finite or finite.

8. Suppose that T is a NIP theory. Show that the statement in Exercise 6 can be generalized to the following form: For every infinite indiscernible sequence $I = \{a_i : i < \kappa\}$ and $\varphi(x, a)$, either I_φ or $I_{\neg\varphi}$ is a bounded set.

9. Suppose that $\varphi(x, y)$ has the independence property. Show that $\psi(y, x) := \varphi(x, y)$ has the independence property.

   Hint: First choose $I = \{a_i : i \in \omega\}$ witnessing the fact that $\varphi(x, y)$ has the independence property. Let $p_0 < p_1 < \dots$ be a sequence of prime numbers. For each $i \in \omega$, let $D_i \subset \omega$ be the multiples of p_i. Then choose $d_i \models \{\varphi(x, a_j)^{\text{if } j \in D_i} : j \in \omega\}$, for each i. For each finite set $F \subset \omega$, let

   $$\Phi_F := \{\varphi(d_i, y) : i \in F\} \cup \{\neg\varphi(d_i, y) : i \in \omega \smallsetminus F\}.$$

   It is sufficient to show that Φ_F is realized. You can show that it is realized by a_{n_F}, where $n_F = \prod_{i\in F} p_i$. (Notice the equvalence $\varphi(d_i, a_{n_F}) \iff n_F \in D_i \iff p_i | n_F \iff i \in F$.)

10. Suppose that $\psi(x, y) := \varphi_0(x, y) \vee \varphi_1(x, y)$ has the independence property, witnessed by an indiscernible sequence $I = \{a_i\}_{i \in \omega}$. Show that either $\varphi_0(x, y)$ or $\varphi_1(x, y)$ has the independence property (witnessed by the same I).

    Hint: Let $d \models \{\psi(x, a_i)^{\text{if } i \text{ is even}} : i \in \omega\}$. For even i, $\varphi_0(d, a_i)$ or $\varphi_1(d, a_i)$ holds. We can assume, by condesation, for all even i, $\varphi_0(d, a_i)$ holds. Then d realizes $\{\varphi_0(x, a_i)^{\text{if } i \text{ is even}} : i \in \omega\}$.

Definition 10. [(i)]

1. Let us say that T has the k-independence property (k-IP) if there is a formula $\varphi(x, y)$ ($lh(x) = k$) having the independence property.

2. Let $I = \{a_i : i \in \omega\}$ be an indiscernible sequence, A a set, and $n \in \omega$.

    (a) $Av^n(I/A) = \{\varphi(x_0, \dots, x_{n-1}) \in L(A) : (\exists m \in \omega)(\forall i_0, \dots, i_{n-1} \in \omega \setminus m)[i_0 < \cdots < i_{n-1} \Rightarrow \varphi(a_{i_0}, \dots, a_{i_{n-1}})]\}$.

    (b) $Av^*(I/A) = \bigcup_{n \in \omega} p_n(x_0, \dots, x_{n-1})$, where $p_n = Av^n(I/A)$.

11. Show the following:

    (a) $Av^n(I/A)$ is finitely satisfiable.

    (b) Suppose that T does not have the k-IP and $|A| = k$. Then $Av^n(I/A)$ is a complete type over A.

    Hint: (a) is clear, since any formula in the average type is satisfied by 'almost all' n-tuples from I. (b): Similar argument as in Exercise 7 can apply. For simplicity, we assume $n = 2$ and consider $\varphi(xy, z)$, where $|z| = k$. By way of contradiction, neither $\varphi(xy, A)$ nor $\neg\varphi(xy, A)$ belongs to the average type. Then, we can find a sequence $i_0 < j_0 < i_1 < j_1 < \cdots < i_l < j_l < \cdots < \omega$ such that $\varphi(a_{i_l} a_{j_l}, A)^{\text{if } l \text{ is even}}$ holds, for all l. Then T has $|A|$-independence property. A contradiction.

12. Suppose that T does not have the k-IP. Let I be an indiscernible sequence and let A a k-element set. Show that $I^* \models Av^*(I/A)$ is an indiscernible sequence over A.

    Hint: Notice that if $n < m$, $p_n = Av^n(I/A)$ and $p_m = Av^m(I/A)$. Then $p_n(x_{i_0}, \dots, x_{i_{n-1}}) \subset p_m(x_0, \dots, x_{m-1})$ holds for all $i_0 < \cdots < i_{n-1} < m$.

13. Show that k-IP implies 1-IP.

    Hint: For simplicity, $k = 2$. Suppose $\varphi(xy, z)$ with $lh(xy) = 2$ has the independence property witnessed by $I = \{a_i\}_{i\in\omega}$. We derive a contradiction by assuming 1-NIP. Let $de \models \{\varphi(xy, a_i)^{\text{ if } i \text{ is even}} : i \in \omega\}$. By 1-NIP, $Av^*(I/e)$ is defined. Let $\{a_i^*\}_{i\in\omega} \models Av^*(I/e)$. Then $\{a_i^*e\}_{i\in\omega}$ is an indiscernible sequence. Since $\exists x\big[\bigwedge_{i<n} \varphi(x, e, a_{N+i})^{\text{ if } i \text{ is even}}\big]$ holds for all even $N \in \omega$, $\exists x\big[\bigwedge_{i<n} \varphi(x, e, a_i^*)^{\text{ if } i \text{ is even}}\big]$ must hold (for all n). This shows that $\varphi(x, e, a_i^*)$'s witness the 1-IP. A contradiction.

14. Suppose that T has the strict order property. Show that the strict order property is witnessed by a formula $\varphi(x, y)$ satisfying $|x| = 1$.

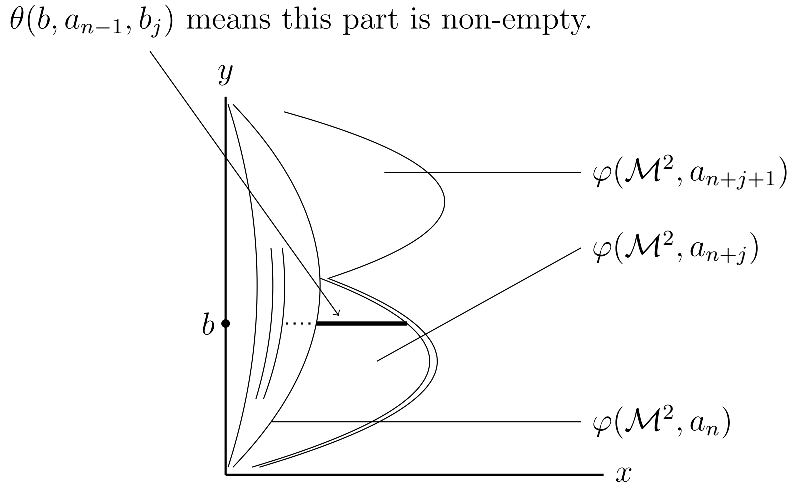
    Hint: Let $\psi(\bar{x}, u)$ and $I = \{a_i\}_{i\in\omega}$ (indiscernibles) witness the condition of the s.o.p. We assume $|\bar{x}| = 2$ and $\bar{x} = xy$. Notice that

    (*) $\{\psi(\mathcal{M}^2, a_i)\}_{i\in\omega}$ is a strictly increasing sequence of definable sets.

    By the symmetry (of x and y) , we can assume that there is $b \in \mathcal{M}$ such that $\psi(\mathcal{M}, ba_i) \subsetneq \psi(\mathcal{M}, ba_{i+1})$ for at least one i.

    (Case 1) $\exists b\, \exists^\infty i$ s.t. $\psi(\mathcal{M}, ba_i) \subsetneq \psi(\mathcal{M}, ba_{i+1})$. In this case, a subsequence of $\{\psi(x, ba_i)\}_{i\in\omega}$ witnesses the strict order property.

    (Case 2) For every b, there are only finitely many i's such that $\psi(\mathcal{M}, ba_i) \subsetneq \psi(\mathcal{M}, ba_{i+1})$. In this case, by compactness, we see that the numbers $n_b = |\{i : \psi(\mathcal{M}, ba_i) \subsetneq \psi(\mathcal{M}, ba_{i+1})\}|$ have a finite least upper bound, say n. Consider the formula $\theta(y, u, v) := \exists x[\neg\psi(x, y, u) \wedge \psi(x, y, v)]$. Then, since I being indiscernible, $\bigwedge_{i<n} \theta(y, a_ia_{i+1})$ is consistent but $\bigwedge_{i<n+1} \theta(y, a_ia_{i+1})$ is not. Now consider the formulas $\varphi(y, b_j) := \bigwedge_{i<n-1} \theta(y, a_ia_{i+1}) \wedge \theta(a_{n-1}, b_j)$, where $b_j = a_{n+j}$ $(j \in \omega)$. By (*) and the definition of θ, $\{\varphi(\mathcal{M}, b_j)\}_j$ forms an increasing sequence of definable sets. It is left to show that it is strictly increasing. However, this is clear, since if $b \models \bigwedge_{i<n-1} \theta(y, a_ia_{i+1}) \wedge \theta(y, b_j, b_{j+1})$ then (i) $b \models \theta(y, a_{n-1}, b_{j+1})$ (x witnessing $\theta(b, b_j, b_{j+1})$ works) and (ii) $b \models \neg\theta(y, b_j)$ (by the choice of n).



Definition 11. Let $X = \{x_{i,j}\}_{i,j\in\omega}$ be a set of variables. Let $\Gamma(X)$ be a set of L-formulas whose free variables belong to X. Let us say that Γ has the sub-array property if there is a realization $A = \{a_{ij}\}_{i,j\in\omega}$ of Γ such that whenever $f, g : \omega \to \omega$ are strictly increasing, then $A_{f,g} = \{a_{f(i),g(j)}\}_{i,j}$ realizes Γ.

14. Suppose that $\Gamma(\{x_{ij}\}_{i,j\in\omega})$ has the sub-array property. Then a realization $A = \{a_{ij}\}_{i,j}$ of Γ can be chosen as an indiscernible array in the following sense:

    (*) For all finite subsets $I = \{i_0 < \cdots < i_{m-1}\}$, $I' = \{i_0 < \cdots < i'_{m-1}\}$, $J = \{j_0 < \cdots < j_{n-1}\}$ and $J = \{j'_0 < \cdots < j'_{n-1}\}$ of ω, $\{a_{ij}\}_{i\in I, j\in J}$ and $\{a_{ij}\}_{i\in I', j\in J'}$ have the same L-type.

    Hint: For each i, let $X_i = (x_{i,j})_{j\in\omega}$ be the i-th row vector of X. Then

    $$\Gamma = \Gamma\begin{pmatrix} X_0 \\ X_1 \\ \vdots \end{pmatrix}$$

    has the subsequence property. So, for A realizing Γ, we can assume the row vectors A_i's form an indiscernible sequence. Similarly, we can assume the column vectors form an indiscernible sequence.

15. Generalize the above to the case when $X = \{x_\eta : \eta \in \omega^n\}$.