On locally o-minimal structures

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joint works with

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Outline

1. local o-minimality
2. uniform local o-minimalily and strong local o-minimality
3. local monotonicity
4. local cell decomposition property

Let L be a language containing $<$.

Let $\mathcal{M} = (M, <, \ldots)$ be an L-structure expanding a dense linear ordering $<$.

Definition 1.1

- $A \subseteq M$ is said to be convex in $\mathcal{M}$ if for any $a, b \in A$, we have $(a, b) \subseteq A$.
- If additionally $\sup A, \inf A \in M \cup \{-\infty, \infty\}$, then A is called an interval in $\mathcal{M}$.

Example 1.2

Let $Q_1 = (\mathbb{Q}, <)$.

- $(-1, 1)$, $[-1, 1]$, $[-1, 1)$, $(-1, 1]$, $\{1\}$ are intervals.
- $\left(-\sqrt{2}, \sqrt{2}\right) \cap \mathbb{Q}$ is not an interval but a convex set.

Example 1.3

Let $Q_2 = (\mathbb{Q} \times \mathbb{Q}, <)$, where $<$ is the lexicographic ordering.

- $\{0\} \times \mathbb{Q}$ is not an interval but a convex set.

Let $\mathcal{M} = (M, <, \ldots)$ be an L-structure expanding a dense linear ordering $<$.

Definition 1.4

- $\mathcal{M}$ is said to be o-minimal if
  any definable subset of M is a finite union of intervals.
- $\mathcal{M}$ is said to be weakly o-minimal if
  any definable subset of M is a finite union of convex sets.

The notion of local o-minmality and strongly local o-minimality was introduced by C. Toffalori and K. Vozoris.

Definition 1.5

- $\mathcal{M}$ is said to be locally o-minimal if for any $a \in M$ and any definable $X \subseteq M$, there is an open interval $I \ni a$ such that $X \cap I$ is a finite union of intervals.

The notion of local o-minmality and strongly local o-minimality was introduced by C. Toffalori and K. Vozoris.

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- $\mathcal{M}$ is said to be locally o-minimal if for any $a \in M$ and any definable $X \subseteq M$, there is an open interval $I \ni a$ such that $X \cap I$ is a finite union of intervals.
- $\mathcal{M}$ is said to be strongly locally o-minimal if for any $a \in M$, there is an open interval $I \ni a$ such that for any definable $X \subseteq M$, $X \cap I$ is a finite union of intervals.

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Definition 1.5

- $\mathcal{M}$ is said to be locally o-minimal if for any $a \in M$ and any definable $X \subseteq M$, there is an open interval $I \ni a$ such that $X \cap I$ is a finite union of intervals.
- $\mathcal{M}$ is said to be strongly locally o-minimal if for any $a \in M$, there is an open interval $I \ni a$ such that for any definable $X \subseteq M$, $X \cap I$ is a finite union of intervals.
- $\mathcal{M}$ is said to be uniformly locally o-minimal if for any $a \in M$ and any formula $\varphi(x, y) \in L$, there is an open interval $I \ni a$ such that $\varphi(M, b) \cap I$ is a finite union of intervals for any $b \in M$.

A typical example of locally o-minimal structures is the following structure.

Example 1.6 (Marker and Steinhorn)

$\mathcal{R} = (\mathbb{R}, <, +, \sin(x))$ is strongly locally o-minimal.

Facts

In locally o-minimal structures, the following are known.

Proposition 1.7 (Toffalori and Vozoris)

Any weakly o-minimal structure is locally o-minimal.

Proposition 1.8 (Toffalori and Vozoris)

Local o-minimality is preserved under elementary equivalence.

Remark 1.9 (Toffalori and Vozoris)

Strong local o-minimality is not preserved under elementary equivalence.

Uniformly locally o-minimal structures

Proposition 2.1 (Kawakami, Takeuchi, Tsuboi, and T.)

Let $\mathcal{M}$ be a uniformly locally o-minimal structure. Suppose that $\mathcal{M}$ is ω-saturated. Then, $\mathcal{M}$ is strongly locally o-minimal.

Uniformly locally o-minimal structures

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Let $\mathcal{M}$ be a uniformly locally o-minimal structure. Suppose that $\mathcal{M}$ is ω-saturated. Then, $\mathcal{M}$ is strongly locally o-minimal.

Proof.

- **Let $a \in M$ and $\varphi(x, y) \in L$.**
- **By the uniformity of $\mathcal{M}$, there is an open interval $I \ni a$ such that for any $b \in M$, we can take $n_b \in \mathbb{N}$ so that $\varphi(\mathcal{M}, b) \cap I$ is a union of n_b many intervals.**
- **By the saturation of $\mathcal{M}$, the set $\{n_b : b \in M\}$ is uniformly bounded, denoted by $n_\varphi \in \mathbb{N}$.**

Proof.

- **Let $\theta_\varphi(u, v)$ ≡ for any $z \in M$, the set $\{x \in (u, v) : \mathcal{M} \models \varphi(x, z)\}$ is a union of at most n_φ many intervals.**
- **Let $\Gamma(u, v) \equiv \{u < a < v\} \cup \{\theta_\varphi(u, v) : \varphi \in L\}$.**

Proof.

- Let $\theta_\varphi(u, v) \equiv$ for any $z \in M$, the set $\{x \in (u, v) : \mathcal{M} \models \varphi(x, z)\}$ is a union of at most n_φ many intervals.
- Let $\Gamma(u, v) \equiv \{u < a < v\} \cup \{\theta_\varphi(u, v) : \varphi \in L\}$.
- By compactness, $\Gamma(u, v)$ is consistent.
- By the saturation of $\mathcal{M}$, there are c, $d \in M$ such that $\mathcal{M} \models \Gamma(c, d)$.
- The open interval (c, d) witnesses to the strong local o-minimality of $\mathcal{M}$.

□

We show that there is an ω-saturated locally o-minimal structure that is not uniformly locally o-minimal.

Example 2.2

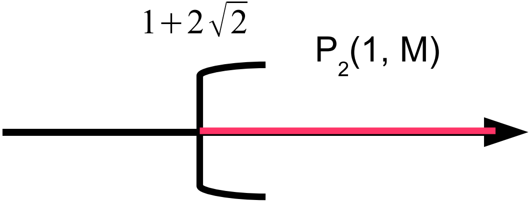
Let $L = \{<, P_q\}_{q \in \mathbb{Q}^+}$ and $M := (\mathbb{Q}, <, P_q)_{q \in \mathbb{Q}^+}$.

Here, $P_q(a, b) \iff a + \sqrt{2} \cdot q \leq b$ in $\mathbb{R}$.

Let $M^* \succ M$ be ω-saturated.

Then, M^* is locally o-minimal but not uniformly locally o-minimal.

For example, when $a = 1$, $q = 2$,



Uniformly locally o-minimal structures

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Let $L = \{<, P_q\}_{q \in \mathbb{Q}^+}$ and $M := (\mathbb{Q}, <, P_q)_{q \in \mathbb{Q}^+}$.
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Uniformly locally o-minimal structures

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Here, $P_q(a, b) \iff a + \sqrt{2} \cdot q \leq b$ in $\mathbb{R}$.

Proof.

$\mathrm{Th}(M)$ admits elimination of quantifiers.

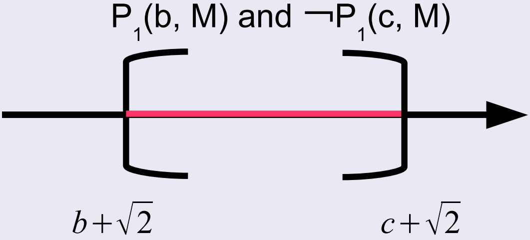
So, M is weakly o-minimal and hence locally o-minimal. □

Proof.

However, M is not uniformly locally o-minimal.

∵) For any open interval $I = (b, c) \ni 0$,

we have $P_1(b, M) \wedge P_1(c, M) \neq \emptyset$.



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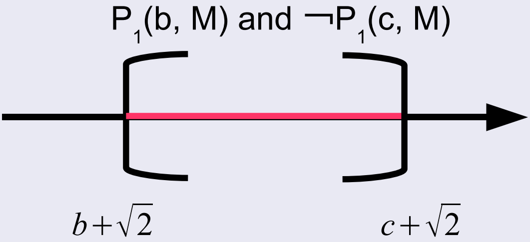


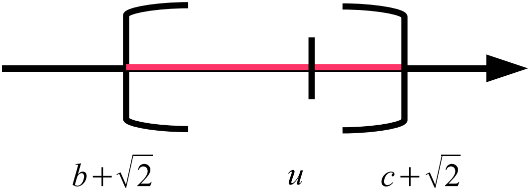
We take $u \in M$ such that $P_1(b, u) \wedge P_1(c, u)$.

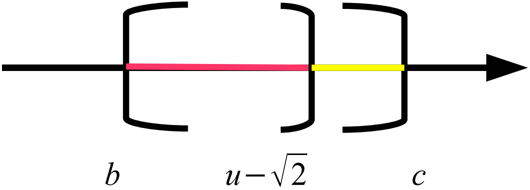
Then, the set $P_1(M, u)$ divedes into two convexes C_1 and C_2.

Neither C_1 nor C_2 are intervals. □

Uniformly locally o-minimal structures

$P_1(b, M)$ and $\neg P_1(c, M)$





Local monotonicity

Definition 3.1

- A local o-minimal strucuture $\mathcal{M} = (M, <, \ldots)$ have local monotonicity if for any definable $X \subseteq M$, any definable $f : X \to M$ and any $a \in M$ there are an open interval $I \ni a$ and intervals $X_0, X_1, \ldots, X_n$ such that any $f|X_i$ is constant, strictly increasing, or strictly decreasing.
- If additionally any $f|X_i$ is continuous, we have local monotonicity with continuity.

Local monotonicity

In general, locally o-minimal structures do not have local monotonicity. However, the following holds.

Fact 1 (Toffalori and Vozoris)

Any strongly locally o-minimal structure satisfies local monotonicity.

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Fact 1 (Toffalori and Vozoris)

Any strongly locally o-minimal structure satisfies local monotonicity.

Fact 2 (Toffalori and Vozoris)

Any locally o-minimal expansion $(\mathbb{R}, <, \ldots)$ *of* $(\mathbb{R}, <)$ *is strongly locally o-minimal.*

Fact 3 (Kawakami and T.)

Any locally o-minimal expansion $(\mathbb{R}, <, \ldots)$ *of* $(\mathbb{R}, <)$ *satisfies local monotonicity with continuity.*

In general, a strongly locally o-minimal structure dose not satisfy local monotonicity with continuity.

Example 3.2

$M = (\mathbb{Q} \times \mathbb{Q}, <, 0, f(x), E(x, y))$.

Here, $<$ is the lexicograhic ordering, $\mathbf{0} := (0, 0)$,

and for any (p, q), (p_1, q_1), $(p_2, q_2) \in \mathbb{Q} \times \mathbb{Q}$,

$f((p, q)) := (q, 0)$ and $E((p_1, q_1), (p_2, q_2)) \iff p_1 = p_2$.

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$f((p, q)) := (q, 0)$ and $E((p_1, q_1), (p_2, q_2)) \iff p_1 = p_2$.

Then, for $x = (x_1, x_2), y = (y_1, y_2) \in M$,

$$x < y \iff x_1 < y_1 \quad \text{or } (x_1 = y_1 \wedge x_2 < y_2)$$

$$\iff (\neg E(x, y) \wedge x < y) \text{ or } (E(x, y) \wedge f(x) < f(y)).$$

Hence, $\mathrm{Th}(M)$ admits elimination of quantifiers.

Example 3.3

$M = (\mathbb{Q} \times \mathbb{Q}, <, 0, f(x), E(x, y))$.

$f((p, q)) := (q, 0)$ and $E((p_1, q_1), (p_2, q_2)) \iff p_1 = p_2$.

Let $a = (a_1, a_2), b = (b_1, b_2) \in M$.

- The set $\{x \in M : f(x) = (a_1, 0)\} = \mathbb{Q} \times \{a_1\}$ is discrete.
- The set $\{x \in M : E(x, a)\} = \{a_1\} \times \mathbb{Q}$ is convex.

Hence, M is strongly locally o-minimal.

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$M = (\mathbb{Q} \times \mathbb{Q}, <, 0, f(x), E(x, y))$.

$f((p, q)) := (q, 0)$ and $E((p_1, q_1), (p_2, q_2)) \iff p_1 = p_2$.

Let $a = (a_1, a_2), b = (b_1, b_2) \in M$.

- The set $\{x \in M : f(x) = (a_1, 0)\} = \mathbb{Q} \times \{a_1\}$ is discrete.
- The set $\{x \in M : E(x, a)\} = \{a_1\} \times \mathbb{Q}$ is convex.

Hence, M is strongly locally o-minimal.

However, for open interval (a, b),

the set $f((a, b)) = (a_1, b_1) \times \{0\}$ is discrete.

Therefore, for any $c \in (a, b)$, $f(c)$ is not continuous.

M dose not satisfy local monotonicity with continuity.

Local cell decomposition property

For every $n \in \mathbb{N}$, we inductively introduce cells in M^n.

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Definition 4.1

Let $C \subseteq M$.

- $C = \{a\}$, where $a \in M$, is called a **0-cell** and $\mathbf{dim}\, C := 0$
- If C is an open interval, the C is called a **1-cell** and $\mathbf{dim}\, C := 1$.

Definition 4.2

Suppose that $C \subseteq M^n$ is a cell with $\dim C = k$.

- Let $f : C \to M$ be definable and continuous. Then $C_1 := \text{graph}(f) := \{(x, f(x)) : x \in C\}$ is a k-cell in M^{n+1} and we put $\dim C_1 = k$.
- Let g, h be definable continuous functions from C to $M \cup \{\pm\infty\}$ with $g < h$ on C. Then $C_2 := (g, h)_C := \{(x, y) : x \in C, g(x) < y < h(x)\}$ is a $(k + 1)$-cell in M^{n+1} and we put $\dim C_2 = k + 1$.

Local cell decomposition property

Definition 4.3

1. $\mathcal{M}$ is said to have the cell decomposition property if for any $n \in \mathbb{N}$ and any definable $X \subseteq M^n$, there is a finite partition of X into cells.
2. $\mathcal{M}$ is said to have the local cell decomposition property if for any $n \in \mathbb{N}$, any $a \in M^n$, and any definable $X \subseteq M^n$, there is an open box $B \subseteq M^n$ containing a and a finite partition of $X \cap B$ into cells.

Local cell decomposition property

Theorem 4.4 (Knight, Pillay and Steinhorn)

Any o-mininal structure has the cell decomposition property.

Theorem 4.5 (Kawakami and T.)

Let $\mathcal{R} = (\mathbb{R}, <, \cdots)$ *be a locally o-minimal expansion of* $(\mathbb{R}, <)$.

1. $\mathcal{R}$ *has the local cell decomposition property.*
2. *Let* $n \in \mathbb{N}$. *Let* $X \subseteq \mathbb{R}^n$ *be a cell,* $f : X \to \mathbb{R}$ *a definable function and* $a \in \mathbb{R}^n$. *Then, there exists an open box* $B \subseteq \mathbb{R}^n$ *containing* a *and a finite partition* P *of* $X \cap B$ *into cells such that for any* $Y \in P$, $f|Y$ *is continuous.*

Actually, Theorem 4.5(1) holds in strongly locally o-minimal structures.

Proposition 4.6 (Kawakami, Takeuchi, Tsuboi, and T.)

Any strongly locally o-minimal structure M has the local cell decomposition property.

To show Proposition 4.6, we first give a characterization of strong local o-minimality.

Definition 4.7

Let M be an L-structure and $A \subseteq M$.

1. $\mathbf{Def}^n(A, M) := \{D \cap A^n : D \subseteq M^n \text{ is } M\text{-definable}\}$.
   $\mathbf{Def}(A, M) := \bigcup_{n \in \omega} \mathbf{Def}^n(A, M)$.
2. The structure $A_{\mathbf{def}} := (A, (X)_{X \in \mathbf{Def}(A,M)})$.

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2. The structure $A_{\mathbf{def}} := (A, (X)_{X \in \mathbf{Def}(A,M)})$.

Remark 4.8

If $A \subseteq M$ is definable, then $\mathbf{Def}(A_{\mathbf{def}}) = \mathbf{Def}(A, M)$,

that is, for any $X \subseteq A$,

X is definable in $A_{\mathbf{def}} \iff X$ is definable in M.

Tsuboi gave the following characterization of strong local o-minimality at Model theory summer school 2010 (August).

Proposition 4.9

The following are equivalent.

1. M is strongly locally o-minimal.
2. For any $a_1, \ldots, a_n \in M$, there are intervals $I_1 = (b_1, c_1], \ldots,$ $I_n = (b_n, c_n]$ with $a_i \in (b_i, c_i)$ such that, by putting $I := \bigcup_{i=1,\ldots,n} I_i$, I_{def} is o-minimal.

Local cell decomposition property

Proposition 4.10 (Kawakami, Takeuchi, Tsuboi, and T.)

Any strongly locally o-minimal structure M has the local cell decomposition property.

Proof.

Let $a = (a_1, a_2, \ldots, a_n) \in M^n$ and $X \subseteq M^n$ definable.

By the strong local o-minimality of M,

there are intervals $I_1 = (b_1, c_1], \ldots, I_n = (b_n, c_n]$ with $a_i \in (b_i, c_i)$ such that $I_{\mathrm{def}} := (I_1 \cup I_2 \cup \cdots \cup I_n)_{\mathrm{def}}$ is o-minimal.

Proof.

Let $a = (a_1, a_2, \ldots, a_n) \in M^n$ and $X \subseteq M^n$ definable.

By the strong local o-minimality of M,

there are intervals $I_1 = (b_1, c_1], \ldots, I_n = (b_n, c_n]$ with $a_i \in (b_i, c_i)$ such that $I_{\mathrm{def}} := (I_1 \cup I_2 \cup \cdots \cup I_n)_{\mathrm{def}}$ is o-minimal.

We put $B := (b_1, c_1) \times \cdots \times (b_n, c_n)$ $(\subseteq I^n)$. Then, $(a_1, \ldots, a_n) \in B$, and B is definable in I_{def} because B is definable in M.

Local cell decomposition property

Proof.

Let $a = (a_1, a_2, \ldots, a_n) \in M^n$ and $X \subseteq M^n$ definable.

By the strong local o-minimality of M,

there are intervals $I_1 = (b_1, c_1], \ldots, I_n = (b_n, c_n]$ with $a_i \in (b_i, c_i)$ such that $I_{\mathrm{def}} := (I_1 \cup I_2 \cup \cdots \cup I_n)_{\mathrm{def}}$ is o-minimal.

We put $B := (b_1, c_1) \times \cdots \times (b_n, c_n)$ ($\subseteq I^n$). Then, $(a_1, \ldots, a_n) \in B$,

and B is definable in I_{def} because B is definable in M.

Hence, by the o-minimality of I_{def},

there is a finite partition $\mathcal{P}$ of $X \cap B$ into cells in I_{def}.

Since B is an open box, any $Y \in \mathcal{P}$ is also a cell in M.

This proprosition is proved.

□

A diagram