Unsaturated generic structures

Koichiro IKEDA

Hosei University

RIMS model theory meeting
Nov. 29, 2010

Acknowledgement

This work is in progress with Hirotaka Kikyo.

Our Result

Theorem 2 (I. and Kikyo)

There is a strictly superstable generic structure whose theory is non-trivial.

Baldwin's Question

- generic structure = *ab initio* generic structure
- strictly superstable = superstable but not ω-stable

Question (Baldwin, 1993)

Is there a strictly superstable generic structure?

Theorem 1 (I., 2010)

There is a strictly superstable generic structure.

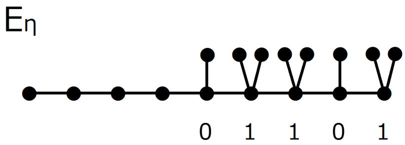
We explain the outline of the proof of this theorem.

Construction

1) The edge relation is binary, and $\alpha = 1$.
   (I.e., $\delta(A) = |A| - |R(*, *)^A|$.)

Construction

1) The edge relation is binary, and $\alpha = 1$.
   (I.e., $\delta(A) = |A| - |R(*,*)^A|$.)
2) For example, for $\eta = (01101)$, E_η is defined as follows.



Construction

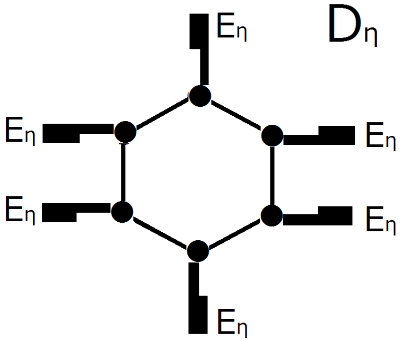
3) Fix a 1-1 onto map $f : 2^{<\omega} \to \{3, 4, 5, \cdots\}$.

Construction

3) Fix a 1-1 onto map $f : 2^{<\omega} \to \{3, 4, 5, \cdots\}$.
4) For example, take $\eta \in 2^{<\omega}$ with $f(\eta) = 6$.

Construction

3) Fix a 1-1 onto map $f : 2^{<\omega} \to \{3, 4, 5, \cdots\}$.
4) For example, take $\eta \in 2^{<\omega}$ with $f(\eta) = 6$.
5) Then D_η is defined as follows.

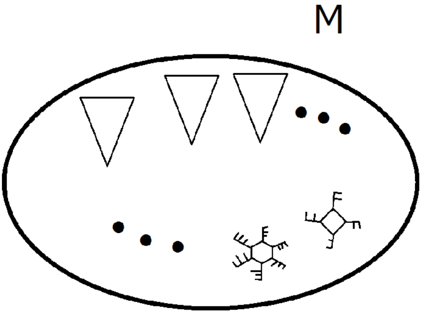


Construction

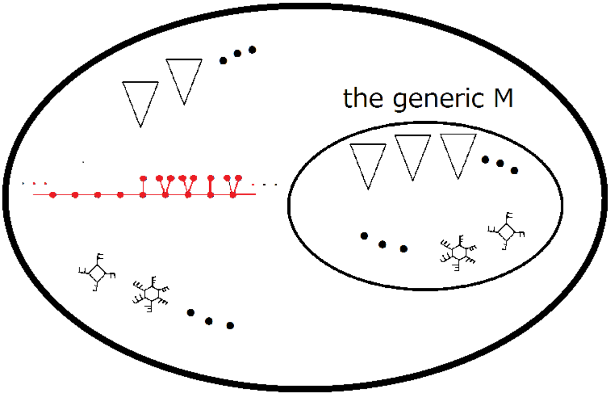
6) Let $\mathbf{K}$ be a class generated by D_η's. ($\mathbf{K}$ is closed under substructures and amalgamation.)

Construction

6) Let $\mathbf{K}$ be a class generated by D_η's. ($\mathbf{K}$ is closed under substructures and amalgamation.)
7) Let M be the $\mathbf{K}$-generic. Then the picture is as follows.



8) Moreover, a big model is as follows.



Lemma

Lemma 1

$\mathbf{Th}(M)$ is not small.

Clear.

Lemma 2

$\mathbf{Th}(M)$ is superstable.

Not clear.

Richness

Definition

Let $D \leq \mathcal{M}$. Then D is said to be rich, if for any finite $A \leq D$ and $B \in K$ with $A \leq B$ there is $B' \cong_A B$ with $B' \leq \mathcal{M}$.

Remark

A generic structure is rich.

Poorness

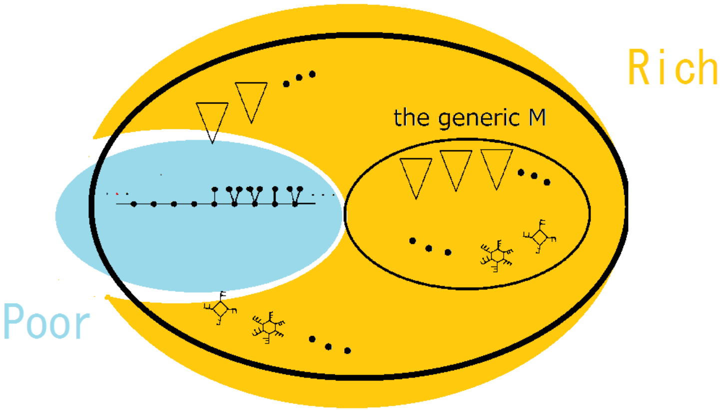
Definition

Let $D \leq \mathcal{M}$. D is said to be poor, if for any finite $A \leq D$ there is $B \in K$ with $A \leq B$ such that there is no $B' \cong_A B$ with $B' \leq \mathcal{M}$.

Remark

A poor set is not rich.

The below structure is divided into the following two areas.



Ultrahomogeneous

Definition

A generic structure $\mathcal{M}$ is said to be ultrahomogeneous, if whenever $A, B \leq \mathcal{M}$ and $A \cong B$, then $\mathbf{tp}(A) = \mathbf{tp}(B)$.

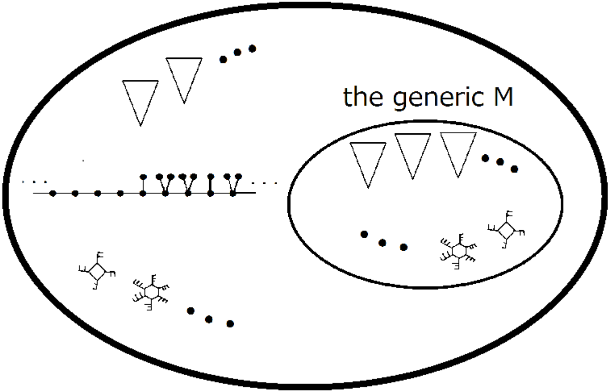
When $\mathcal{M}$ is ultrahomogeneous, it is easy to count the number of types.

Remark

A saturated generic structure is ultrahomogeneous. (Hrushovski's examples are saturated.)

Remark

The below structure is not ultrahomogeneous.



Almost ultrahomogeneous

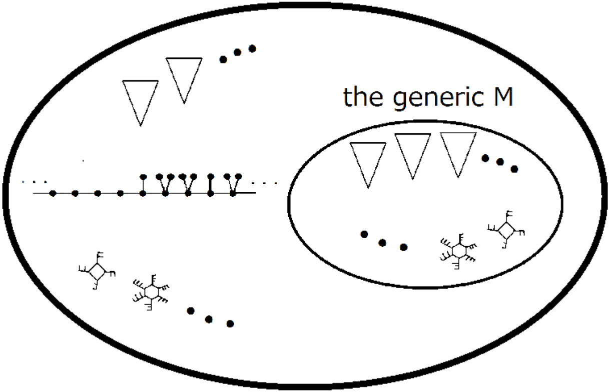
Definition

A generic structure M is said to be almost ultrahomogeneous, if whenever $A, B \leq \mathcal{M}$, $A \cong B$ and $\mathrm{tp}(A_p) = \mathrm{tp}(B_p)$, then $\mathrm{tp}(A) = \mathrm{tp}(B)$.

Almost ultrahomogeneous

Remark

The below structure is almost ultrahomogeneous.



When $\mathcal{M}$ is almost ultrahomogeneous, it is also easy to count the number of types. Then we have

Lemma 2

$\mathbf{Th}(M)$ is superstable.

Hence M is strictly superstable.

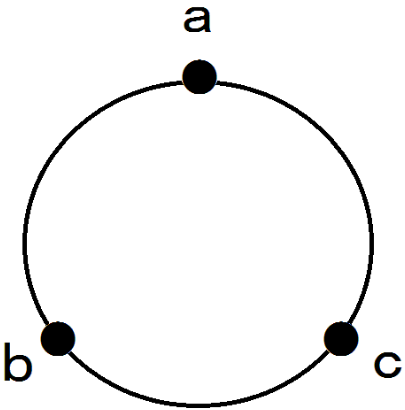
Triviality

Definition

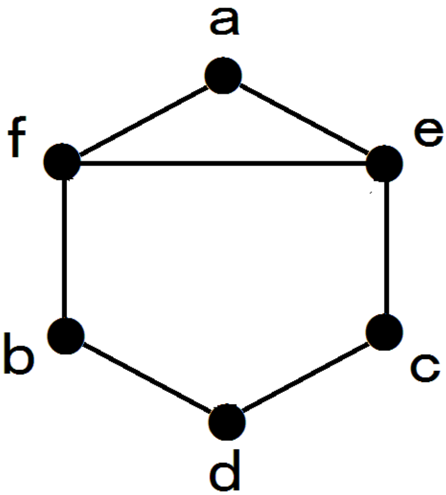
A theory T is said to be trivial, if every pairwise independent set is independent.

1. Hrushovski's strongly minimal structure
   $\Rightarrow$ The edge relation is ternary, and $\alpha = 1$
2. Hrushovski's pseudoplane
   $\Rightarrow$ The edge relation is binary, and $\alpha < 1$
3. The structure of Theorem 1
   $\Rightarrow$ The edge relation is binary, and $\alpha = 1$

Ternary, and $\alpha = 1$



Binary, and $\alpha = 1/2$



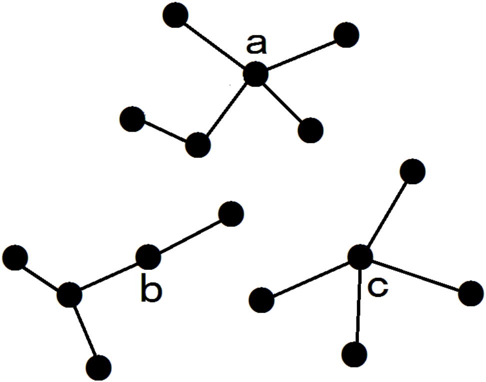
Characterization of Indepencence

Fact

Let B, C be closed and $A = B \cap C$ be algebraically closed. Then the following are equivalent:

1. B and C are independent over A;
2. B and C are free over A, and BC is closed.

Binary, and $\alpha = 1$



Triviality

Definition

A theory T is said to be trivial, if every pairwise independent set is independent.

1. Hrushovski's strongly minimal structure
   $\Rightarrow$ The edge relation is ternary, and $\alpha = \mathbf{1} \Rightarrow$ nontrivial
2. Hrushovski's pseudoplane
   $\Rightarrow$ The edge relation is binary, and $\alpha < \mathbf{1} \Rightarrow$ nontrivial
3. The structure of Theorem 1
   $\Rightarrow$ The edge relation is binary, and $\alpha = \mathbf{1} \Rightarrow$ trivial

Question

Question

Is there a strictly superstable generic structure whose theory is nontrivial?

Answer

Theorem 2(I. and Kikyo)

There is a strictly superstable generic structure whose theory is nontrivial.

Answer

Theorem 2(I. and Kikyo)

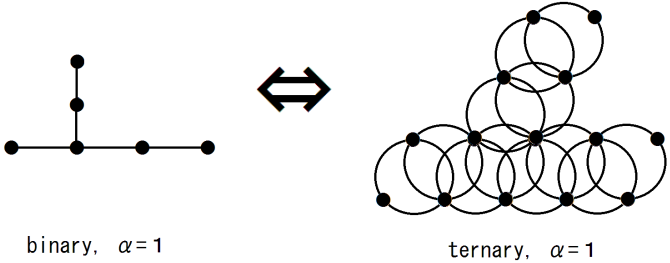
There is a strictly superstable generic structure whose theory is nontrivial.

Remark

The proof is similar to that of Theorem 1. We can construct the following two types of generic structures:

- The edge relation is terary and $\alpha = 1$;
- The edge relation is binary and $\alpha = 2/3$.

Ternary, and $\alpha = 1$



Binary, and $\alpha = 2/3$