On the Model Theory of quantum 2-tori

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Table of Contents

1. Quantum Tori
2. Aim and conjectures
3. Details

Quantum Torus

In the 1980s and 1990s, Fields medallist Alain Connes introduced a range of new geometric objects designed to put quantum physics on a firm mathematical foundation.

One of the most important examples is the ”quantum torus”, an abstract, quantum version of the traditional doughnut-shaped torus.

Quantum Torus, contnd

While a classical torus is easy to visualise, it is impossible to picture a quantum torus in the same way.

This is because quantum geometry replaces the traditional notions of shape, area, curvature and so on with a more abstract concept of a mathematical ”space”.

Quantum Torus, contnd

All the same, the quantum torus is fundamental to ongoing efforts to model the quantum universe.

excerpts from
18 July 2007, NewScientist.com news service,
Richard Elwes

Torus

Algebraically,

- A (classical) torus is just $\mathbb{F}^* \times \mathbb{F}^*$ for $\mathbb{F}$ an acf.

$\mathbb{F}$-algebra $\mathcal{A}_q^2$

Consider a non-cummutative $\mathbb{F}$-algebra

$$\mathcal{A}_q^2(\mathbb{F}) = \mathbb{F}[x^{\pm}, y^{\pm}]/\langle xy - qyx \rangle$$

where $\mathbb{F}$ is an acf of char 0, and either

- q is a root of unity, or

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- q is a root of unity, or
- $q \in \Gamma$ which is an inf. cyclic group.

Quantum torus (Finite Version)

Assumptions

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Conclusion

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Conclusion

- Then $\mathbf{Max}\, Z \simeq \mathbb{F}^* \times \mathbb{F}^*$

Aim

- Find quantum algebras that are also analytic Zariski structures.

Analytic Zariski structures

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- (M, C) is called an *analytic Zariski structure* if it satisfies a number of axioms characterizing *analytic sets*.
- Unlike (algebraic) Zariski structures, the topology on M given by C may not be Noetherian.
- Non-elementary nature of model theory.

From the point of view of infinitary logic

- Describe the properties of quantum tori.
- Examine categoricity issue
- Examine quasi-minimality issue

Conjecture 1

- The 1st-order theory of quantum 2-torus is super-stable.

Conjecture 2

- Quntum tori are analytic Zariski structures.

Concrete case

Consider first the case when $\mathbb{F}$ is the complex numbers $\mathbb{C}$.

$\mathbb{C}$-algebra $\mathcal{A}_q$ and a group $\Gamma_q(\mathbb{C})$

- $\mathcal{A}_q$ is a $\mathbb{C}$-algebra generated by operators U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV$$

where $q = e^{2\pi i h}$ with $h \in \mathbb{R}$.

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  where $q = e^{2\pi ih}$ with $h \in \mathbb{R}$.
- $\Gamma_q(\mathbb{C}) = q^{\mathbb{Z}}$ is a multiplicative subgroup of $\mathbb{C}^*$.

Intuitively, $\cdots$

The quantum torus $T_q^2(\mathbb{C})$ is a structure consisting of two 2-dimensional objects $\mathbf{U}$ and $\mathbf{V}$ with two operations U and V satisfying certain (non-cummutative) properties.

Two objects $\mathbf{U}$ and $\mathbf{V}$.

Describe the properties of following $\mathbf{U}$ and $\mathbf{V}$.

- Both $\mathbf{U}$ and $\mathbf{V}$ are two dimensional objects.

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- Both $\mathbf{U}$ and $\mathbf{V}$ are two dimensional objects.
- Both $\mathbf{U}$ and $\mathbf{V}$ are bases for an ambient module which we do not give any formal description in the theory.

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Two operators U and V.

- The operator U moves each element (vector) of $\mathbf{U}$ on its fibre, say *vertically*.
- The operator V moves each element of $\mathbf{U}$ to another element of $\mathbf{U}$, say *holizontally*.
- The operator V does the same actions on $\mathbf{U}$ and $\mathbf{V}$.

Recall, intuitively, $\cdots$

The quantum torus $T^2_q(\mathbb{C})$ is a structure consisting of two 2-dimensional objects $\mathbf{U}$ and $\mathbf{V}$ with two operations U and V satisfying certain (non-cummutative) properties.

Categoricity

Proposition (Zilber)

Suppose $\mathbb{F}$ and $\mathbb{F}'$ are isomorphic algebraically closed fields of characteristic zero. Let $\Gamma = q^{\mathbb{Z}}$ and $\Gamma' = q'^{\mathbb{Z}}$ be its infinite multiplicative subgroups. Then $T^2_q(\mathbb{F})$ and $T^2_{q'}(\mathbb{F}')$ are isomorphic as quantum 2-tori.

So we describe this ispmorphism type using the language $\mathcal{L}_q$ given below.

First-order language $\mathcal{L}_q$

- Unary predicates $\mathbf{U}$ and $\mathbf{V}$ for the sorts $\mathbf{U}_\phi$ and $\mathbf{V}_\phi$.
- Two more unary predicates for the field $\mathbb{F}$ and its multiplicative subgroup $\boldsymbol{\Gamma}$ generated by $\boldsymbol{q}$.

First-order theory of $T_q^2(\mathbb{C})$.

Definable subsets in a model of the theory of this 3-sorted structure $(\mathbf{U}, \mathbf{V}, \mathbb{F})$ are determined by the actions U and V on each sort $\mathbf{U}$ and $\mathbf{V}$.

Informal descriptions of the theory

What we can say about the operations U and V are basically the number of times we apply these operations, thus this part can be expressed by positive quantifier free formulas.

However we need an existential quantifer in order to express an equivalence relation $\sim_E$ defined later.

- The complexity of the definable sets is the boolean combination of positive quantifier free formula modulo existential quantifier.
  (Near model completeness).

Formal description of the quantum torus

Let $\mathcal{L}_q = \mathcal{L}_{T_q^2} = \{\mathbf{U}, \mathbf{V}, \mathbb{F}, \mathbf{\Gamma}, U, V, q, 0, 1, T_q\}$ where

- $\mathbf{U}, \mathbf{V}, \mathbb{F}, \mathbf{\Gamma}$ are unary predicates,
- U, V are 4-ary relations,
- q is a constant symbol,
- T_q is a ternary relation symbol corresponding to the pairing function.

Consider the language $\mathcal{L}_\Gamma = \{+, \cdot, \Gamma, 0, 1, q\}$.
Let T_Γ be the $\mathcal{L}_\Gamma$-theory consisting

1. $(\mathbb{F}, +, -, \cdot, 0, 1)$ is an algebraically closed field of characteristic zero.
2. $(\Gamma, \cdot, 1) \equiv (\mathbb{Z}, +, 0)$
3. $q \neq 0$

Conjecture

The first-order theory $\mathbf{Th}(\mathbb{F}, +, -, \cdot, \Gamma.0, 1, q)$ *is superstable.*

Conjecture

The quantum 2-torus $T_q^2(\mathbb{C})$ is an analytic Zariski structure.

Note that $\mathbf{Th}(\mathbb{F}, +, -, \cdot, \Gamma, 0, 1, q) = T_\Gamma$ is axiomatized by

- $\mathbf{ACF_0}$
- $(\Gamma, \cdot, 1)$ is a multiplicative group
- $\forall x\, (\Gamma(x) \rightarrow \Gamma(q \cdot x))$
- $\forall x\, \exists y\, (\Gamma(x) \rightarrow (\Gamma(y) \wedge x = q \cdot y))$

Superstability of T_Γ comes from the fact that $\mathbf{Th}(\mathbb{Z}, +, 0)$ is super-stable.

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- Construct a geometric structure *quantum torus* consisting two 2-dimensional objects associated with the algebra and the group.
- Describe the properties of *quantum tori* in first-order way.
- Study the *quantum tori* from the analytic Zariski geometry point of view.