Indiscernibility on Trees

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This talk is a summary of some of the key ideas in the paper that Byunghan Kim and I are writing at the moment.

In model theory, we often encounter the following question:

Let $\mathcal{M}$ be a model. Suppose we have shown that there exists a sequence $\langle \bar{a}_i \in \mathcal{M} \mid i < \omega \rangle$ having some property A.

Question

Can we choose such a sequence to be also indiscernible?

It is well known that the answer is often yes.

Brief review of indiscernible sequence.

Definition

Let $\mathcal{M}$ be any model.

1. A sequence $\langle \bar{a}_i \in \mathcal{M} \mid i \in I \rangle$ is called an indiscernible sequence if
   1. I is a linearly ordered set,
   2. $tp(\bar{a}_{i_0}\bar{a}_{i_1}\cdots\bar{a}_{i_d})$ depends only on the order-type of the sequence $\langle i_0, \cdots, i_d \rangle$.
2. We say a sequence $\langle \bar{a}_i \in \mathcal{M} \mid i < \omega \rangle$ is **modelled by** a sequence $\langle \bar{b}_i \in \mathcal{M} \mid i < \omega \rangle$ if, for any finite set $\Delta(\bar{x}_0, \cdots, \bar{x}_d)$ of $\mathcal{L}$-formulas and any finite sequence $\langle i_k \in \omega \mid k < d \rangle$, we can find a finite sequence $\langle j_k \in \omega \mid k < d \rangle$ such that
   1. $\langle i_0, \cdots, i_d \rangle$ and $\langle j_0, \cdots, j_d \rangle$ have the same order type,
   2. $tp_\Delta(\bar{b}_{i_0}, \cdots, \bar{b}_{i_d}) = tp_\Delta(\bar{a}_{j_0}, \cdots, \bar{a}_{j_d})$.

One of the key properties of indiscernible sequence is the following:

Theorem

Any sequence $\langle \bar{a}_i \mid i < \omega \rangle$ can be modelled by some indiscernible sequence $\langle \bar{b}_i \mid i < \omega \rangle$.

Proof.

Easy (Ramsey's Theorem and compactness.) □

Indeed, it is this theorem that often allows us to choose an indiscernible sequence having some desired properties.

The main idea of this talk is that we can generalize the notion of indiscernible sequence to sequences of the form $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$, and prove a generalized version of the theorem above.

This idea was originally developed by Shelah and Džamonja in their paper *On $\triangleleft^*$ maximality* (APAL, 2004). (They worked with sequences indexed by the binary tree ${}^{\omega>}2$.) We are also influenced by Lynn Scow who gave a detailed exposition on it in her recent PhD thesis (2010).

We have revised their proofs (and corrected errors). In doing so, we could

1. significantly clarify the argument by introducing some new notions and terminologies,
2. apply the main lemma to a couple of concrete problems.
3. Our result generalizes the original result from ${}^{\omega>}2$ to ${}^{\kappa>}\lambda$, for any cardinals $\kappa \geq \omega$ and $\lambda \geq 2$.

Review of Tree

Definition

1. A partially ordered set T is called a **tree** if, $\forall\, x \in T$

$$Pred\,(x) := \{\, y < x \mid y \in T \,\} \quad \text{is linearly ordered.}$$

2. A tree T is called **finitistic** if
    1. T has the least element,
    2. $\forall\, x \in T,\ Pred(x)$ is a finite set,
    3. $\forall\, n < \omega,\ \{\, x \in T \mid |Pred(x)| = n \,\}$ is a finite set.

Definition

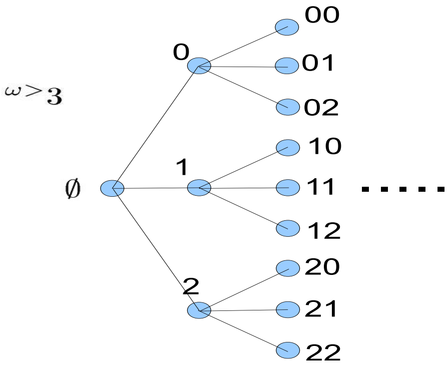
For $n < \omega$,

$$^{\omega>}n := \{ f : m \to n \mid f \text{ is a function}, \ m < \omega \}$$

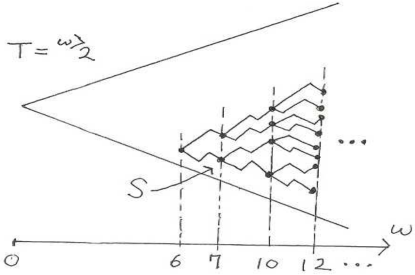
Note

1. $^{\omega>}n$, ordered by inclusion, is a finitistic tree.
2. For $\eta, \nu \in {}^{\omega>}n$, $(\eta \cap \nu)$ denotes the greatest common lower bound of η and ν.
3. Any function $h : m \to n$ can be represented as a finite sequence $\langle h(0), h(1), \cdots, h(m-1) \rangle$.

Example



Strong subtree (Definition by picture) Let T be a finitistic tree without maximal elements. For example, let $T = {}^{\omega>}2$.



S is a **strong subtree of** T **witnessed by** $\{6, 7, 10, 12, \cdots\} \subseteq \omega$.

Halpern-Läuchli Theorem is a kind of Ramsey's theorem for trees.

Theorem (Halpern-Läuchli, strong subtree version)

In any k-coloring of the Cartesian product $\prod_{i=1}^{n} T_i$ of finitistic trees without maximal element, there exist strong subtrees $\{ S_i \subseteq T_i \}_{i=1}^{n}$, all witnessed by the same infinite set $B \subseteq \omega$, such that all elements of the set

$$\{ \bar{e} \in \prod_{i<d} S_i \mid \text{ All } e_i\text{'s are on the same level. } \}$$

are in the same color.

There are a number of equivalent versions of Halpern-Läuchli Theorem. For more details on these equivalent versions, please refer to:

Introduction to Ramsey Spaces by S. Todorcevic (Princeton University Press, 2010).

Some special cases implied by Halpern-Läuchli Theorem:

Special case 1

In any k-coloring of the Cartesian product $\prod_{i=1}^{n} \omega$, there exists an infinite subset $B \subseteq \omega$ such that all the elements in the set

$$\{ (b, \cdots, b) \in \prod_{i=1}^{n} \omega \mid b \in B \}$$

are in the same color.

Special case 2

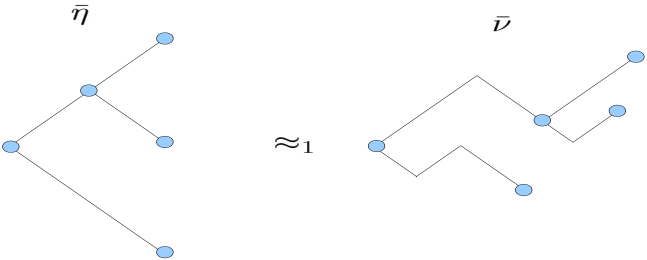
In any k-coloring of the tree ${}^{\omega>}n$, there exists a monochromatic strong subtree.

Definition

Let $\bar{\eta} = \langle \eta_0, \cdots, \eta_{d-1} \rangle$ and $\bar{\nu} = \langle \nu_0, \cdots, \nu_{d-1} \rangle$ be tuples in ${}^{\omega>}n$. We say $\bar{\eta} \approx_1 \bar{\nu}$ if

1. both $\bar{\eta}$ and $\bar{\nu}$ are $\cap$-closed (i.e. closed under the $\cap$-operation),
2. $\forall\, i, j < d$ and $\forall\, t < n$
    1. $\eta_i \trianglelefteq \eta_j$ iff $\nu_i \trianglelefteq \nu_j$, (Partial order)
    2. $\eta_i{}^\frown\langle t \rangle \trianglelefteq \eta_j$ iff $\nu_i{}^\frown\langle t \rangle \trianglelefteq \nu_j$ (Directionality)

Example: $\approx_1$-equivalent tuples in ${}^{\omega>}2$.



Definition

1. We say a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is **1-fully tree indiscernible** (or 1-fti, for short) if, $\forall\, \bar{\eta}, \bar{\nu} \in {}^{\omega>}n$,

$$\bar{\eta} \approx_1 \bar{\nu} \quad \Rightarrow \quad tp(\bar{a}_{\bar{\eta}}) = tp(\bar{a}_{\bar{\nu}})$$

   where $\bar{a}_{\bar{\eta}}$ denotes the sequence $\langle \bar{a}_{\eta_0}, \cdots, \bar{a}_{\eta_{d-1}} \rangle$.

2. We say a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is **1-modelled** by a sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ if,
   for any $\cap$-closed tuple $\bar{\eta} \in {}^{\omega>}n$ and any finite set Δ of $\mathcal{L}$-formulas, we can find $\bar{\nu} \in {}^{\omega>}n$ such that $\bar{\eta} \approx_1 \bar{\nu}$ and $tp_\Delta(\bar{b}_{\bar{\eta}}) = tp_\Delta(\bar{a}_{\bar{\nu}})$.

Our main goal is to prove:

Main Lemma

Any sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 1-modelled by some 1-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

Strategy of proving Main Lemma

- We define an auxiliary, technical notion $\approx_0$**-equivalence** that is stronger than the $\approx_1$-equivalence.
- We also define notions **0-fti** and 0**-modelling property** that are analogous to the 1-fti and 1-modelling property defined earlier.

The point is that we have more control over $\approx_0$-equivalent tuples than over $\approx_1$-equivalent tuples.

Before formally defining $\approx_0$-equivalence, we need a terminology:

Terminology

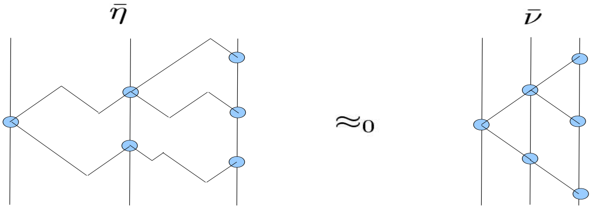
We say a tuple $\bar{\eta} = \langle \eta_0, \cdots, \eta_{d-1} \rangle$ in ${}^{\omega>}n$ is **closed** if it is $\cap$-closed, contains the root $\langle\rangle$, and is closed under level-restriction. i.e. $\forall\, i, j < d$, $\exists\, k < d$ such that $\eta_i\lceil_{|\eta_j|} = \eta_k$.

Definition

Let $\bar{\eta} = \langle \eta_0, \cdots, \eta_{d-1} \rangle$ and $\bar{\nu} = \langle \nu_0, \cdots, \nu_{d-1} \rangle$ be tuples in ${}^{\omega>}n$. We say $\bar{\eta} \approx_0 \bar{\nu}$ if

1. both $\bar{\eta}$ and $\bar{\nu}$ are closed tuples,
2. $\forall\, i, j < d$ and $\forall\, t < n$,
   1. $\eta_i \trianglelefteq \eta_j$ iff $\nu_i \trianglelefteq \nu_j$, (Partial order)
   2. $\eta_i{}^\frown\langle t \rangle \trianglelefteq \eta_j$ iff $\nu_i{}^\frown\langle t \rangle \trianglelefteq \nu_j$ (Directionality)
   3. $|\eta_i| < |\eta_j|$ iff $|\nu_i| < |\nu_j|$. (Length relation)

Example: $\approx_0$-equivalent tuples in ${}^{\omega>}2$.



Definition

1. We say a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is **0-fti** if, $\forall \bar{\eta}, \bar{\nu} \in {}^{\omega>}n$,

$$\bar{\eta} \approx_0 \bar{\nu} \quad \Rightarrow \quad tp(\bar{a}_{\bar{\eta}}) = tp(\bar{a}_{\bar{\nu}})$$

2. We say a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is **0-modelled** by a sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ if,
   for any closed tuple $\bar{\eta} \in {}^{\omega>}n$ and any finite set Δ of $\mathcal{L}$-formulas,
   we can find $\bar{\nu} \in {}^{\omega>}n$ such that $\bar{\eta} \approx_0 \bar{\nu}$ and $tp_\Delta(\bar{b}_{\bar{\eta}}) = tp_\Delta(\bar{a}_{\bar{\nu}})$.

Our intermediate goal is to prove:

Intermediate Main Lemma

Any sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 0-modelled by some 0-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

We need to define one more technical notion '$\approx_{(m,s)}$-equivalence.'

Notation

For $m < \omega$ and a tuple $\bar{\eta} := \langle \eta_0, \cdots, \eta_{d-1} \rangle$ in ${}^{\omega>}n$,

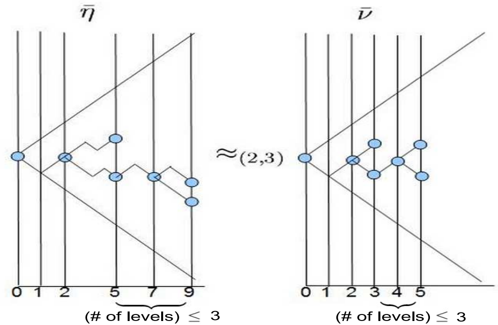
1. $L(\bar{\eta}) := \{ |\eta_i| \mid i < d \}$,
2. $u_m(\bar{\eta}) := \{ i \in L(\bar{\eta}) \mid i > m \}$

Definition

Let $\bar{\eta} = \langle \eta_0, \cdots, \eta_{d-1} \rangle$ and $\bar{\nu} = \langle \nu_0, \cdots, \nu_{d-1} \rangle$ be tuples in ${}^{\omega>}n$. For $m, s < \omega$, we say $\bar{\eta} \approx_{(m,s)} \bar{\nu}$ if

1. $\bar{\eta} \approx_0 \bar{\nu}$,
2. $m \in L(\bar{\eta}) \cap L(\bar{\nu})$,
3. $|u_m(\bar{\eta})| = |u_m(\bar{\nu})| \leq s$,
4. $|\eta_i| \leq m$ iff $|\nu_i| \leq m$, for each $i < |\bar{\eta}|$. And if both sides of the biconditional are true, then $\eta_i = \nu_i$

Example: $\approx_{(2,3)}$-equivalent tuples in ${}^{\omega>}2$.



Note $\bar{\eta}$ and $\bar{\nu}$ are identical up to level 2.

Clearly, it is also true that $\bar{\eta} \approx_{(2,s)} \bar{\nu}$ for every $s \geq 3$.

Definition

Let $m, s < \omega$ and Δ a finite set of $\mathcal{L}$-formulas.

1. We say a sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is (m, s, Δ)**-indiscernible** if, $\forall \bar{\eta}, \bar{\nu} \in {}^{\omega>}n, \quad \bar{\eta} \approx_{(m,s)} \bar{\nu} \quad \Rightarrow \quad tp_\Delta(\bar{a}_{\bar{\eta}}) = tp_\Delta(\bar{a}_{\bar{\nu}})$.
2. $(< \omega, s, \Delta)$-indiscernible $\Leftrightarrow$ (m, s, Δ)-indiscernible, $\forall\, m < \omega$.

Note

1. Any $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is trivially $(< \omega, 0, \Delta)$-indiscernible.
2. 0-fti $\Leftrightarrow$ $(< \omega, s, \Delta)$-indiscernible for every $s < \omega$ and Δ.

Key Technical Lemma

Suppose $T := \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is a $(<\omega, s, \Delta)$-indiscernible sequence. Then, $\forall\, m < \omega$, there exists a $(m, s{+}1, \Delta)$-indiscernible sequence $S := \langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ such that $S \leq^m T$.

$S \leq^m T$ basically means that S can be naturally embedded onto a strong subtree of T, in such a way that S and T are identical up to (and including) the m-th levels.

The proof of this lemma is quite technical and relies on Halpern-Läuchli Theorem.

Corollary

Suppose a sequence $T = \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is $(<\omega, s, \Delta)$-indiscernible. Then there exists a $(<\omega, s{+}1, \Delta)$-indiscernible sequence $S = \langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ such that $S \leq^0 T$.

Proof of Corollary

Suppose $T = \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is a $(<\omega, s, \Delta)$-indiscernible sequence.

For convenience, let us call sequences of the form $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ **parameterized trees**.

Using Key Technical Lemma, we can build a sequence $T_0, T_1, \cdots$ of parameterized trees such that

1. $\cdots \leq^3 T_2 \leq^2 T_1 \leq^1 T_0 \leq^0 T$,
2. each T_i is $(\leq i, s{+}1, \Delta)$-indiscernible.

Condition (1) allows us define $S := \lim_{i\to\infty} T_i$.

Then $S \leq^0 T$ and S is $(<\omega, s+1, \Delta)$-indiscernible. $\square$

Recall that any $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ is trivially $(<\omega, 0, \Delta)$-indiscernible.

Corollary

Let $T := \langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ be any sequence. Then, given any finite set Δ of $\mathcal{L}$-formulas, there exists a sequence $S_1^\Delta, S_2^\Delta, \cdots$ of parameterized trees such that,

1. $\cdots \leq^0 S_2^\Delta \leq^0 S_1^\Delta \leq^0 S_0^\Delta := T$,
2. each S_i^Δ is $(<\omega, i, \Delta)$-indiscernible.

Recall that $0\text{-}fti \Leftrightarrow (<\omega, s, \Delta)$-indiscernible for every s and Δ.

Applying compactness, we obtain

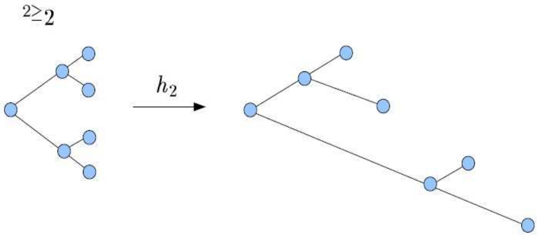
Intermediate Main Lemma

Any sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 0-modelled by some $0\text{-}fti$ sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

The final stage of proving Main Lemma

For each $m < \omega$, define a map $h_m : {}^{m\geq}n \to {}^{\omega>}n$ as follows:

Example $h_2 : {}^{2\geq}2 \to {}^{\omega>}2$



Note h_m preserves partial order and directionality. Moreover,
$\forall\, \eta, \nu \in {}^{m\geq}n, \quad \eta <_{lex} \nu \iff |h_m(\eta)| < |h_m(\nu)|,$
where $<_{lex}$ is the lexicographic order in ${}^{\omega>}n$. (e.g. $010 <_{lex} 02$.)

We further observe that,

if $\bar{\eta} \approx_1 \bar{\nu} \in {}^{m\geq}n$, then $h_m(\bar{\eta}) \approx_1 h_m(\bar{\nu})$ and, $\forall\, i, j < |\bar{\eta}|$,

$|h_m(\eta_i)| < |h_m(\eta_j)| \;\Leftrightarrow\; \eta_i <_{lex} \eta_j \;\Leftrightarrow\; \nu_i <_{lex} \nu_j \;\Leftrightarrow\; |h_m(\nu_i)| < |h_m(\nu_j)|$

Notation $\bar{\eta} \approx_1^* \bar{\nu} \;\Leftrightarrow\;$ (1) $\bar{\eta} \approx_1 \bar{\nu}$ and (2) both $\bar{\eta}$ and $\bar{\nu}$ contain the root $\langle\rangle$.

Then, the only thing that prevents us from saying

$$\bar{\eta} \approx_1^* \bar{\nu} \quad \Rightarrow \quad h_m(\bar{\eta}) \approx_0 h_m(\bar{\nu})$$

is that $h_m(\bar{\eta})$ and $h_m(\bar{\nu})$ may not be level-closed.

But we can easily remedy this by taking the 'level-closures' of $h_m(\bar{\eta})$ and $h_m(\bar{\nu})$, respectively. i.e. we let

$cl(h_m(\bar{\eta})) =$ the smallest level-closed tuple containing $h_m(\bar{\eta})$.

Proposition

$$\bar{\eta} \approx_1^* \bar{\nu} \in {}^{m\geq}n \quad \Rightarrow \quad cl(h_m(\bar{\eta})) \approx_0 cl(h_m(\bar{\nu}))$$

Corollary

Let $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ be any 0-fti sequence. Then, for each $m < \omega$,

$$\langle \bar{a}_{h_m(\eta)} \mid \eta \in {}^{m\geq}n \rangle$$

is 1^*-fti. i.e. $\bar{\eta} \approx_1^* \bar{\nu} \in {}^{m\geq}n \quad \Rightarrow \quad tp(\bar{a}_{h_m(\bar{\eta})}) = tp(\bar{a}_{h_m(\bar{\nu})})$.

Applying compactness, we obtain

Corollary

Any 0-fti sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 1^*-modelled by some 1^*-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

We can finally prove Main Lemma.

Main Lemma

Any sequence $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 1-modelled by some 1-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

Proof.

By Intermediate Main Lemma, $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 0-modelled by some 0-fti sequence $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

By the preceding corollary, $\langle \bar{b}_\eta \mid \eta \in {}^{\omega>}n \rangle$ can be 1^*-modelled by some 1^*-fti sequence $\langle \bar{c}_\eta \mid \eta \in {}^{\omega>}n \rangle$.

Then, $\langle \bar{c}_{\langle 0 \rangle \frown \eta} \mid \eta \in {}^{\omega>}n \rangle$ is a 1-fti sequence 1-modelling $\langle \bar{a}_\eta \mid \eta \in {}^{\omega>}n \rangle$. □

Remark

The notions $\approx_0, \approx_1$ 0-modelling and 1-modelling clearly make sense even for sequences $\langle \bar{a}_\eta \mid \eta \in {}^{\kappa >}\lambda \rangle$ where $\kappa \geq \omega$ and $\lambda \geq 2$. Then, by compactness argument, we can extend Main Lemma to this context.

Applications of Main Lemma

Definition

We say a theory T has k-TP1 ($k \geq 2$) if it allows an $\mathcal{L}$-formula $\varphi(\bar{x}\,\bar{y})$ to witness a sequence $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}\omega \,\rangle$ satisfying

1. whenever $\eta_0 \lhd \cdots \lhd \eta_{d-1} \in {}^{\omega>}\omega$, $\bigcap_{i<d} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is consistent,
2. whenever $\eta_0, \cdots, \eta_{k-1} \in {}^{\omega>}\omega$ are pairwise incomparable elements, $\bigcap_{i<k} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is not consistent.

We have been able to apply Main Lemma to show

Theorem

A theory T has 2-TP1 iff it has k-TP1 for some $k \geq 2$.

Remark

1. In proving this theorem, we used an idea of Shelah and Usvyatsov who proved a similar theorem in their paper *More on SOP_1 and SOP_2*, APAL, 2008.
2. Our definition of k-TP1 is a generalization of the notion 'TP1' originally defined by Shelah.

Application 2

Definition

Consider tree ${}^{\omega>}\omega$. We say $\eta_0, \cdots, \eta_{k-1} \in {}^{\omega>}\omega$ are

1. **siblings** if they are distinct elements sharing the same immediate predecessor. (i.e. there exist $\nu \in {}^{\omega>}\omega$ and distinct $t_0, \cdots, t_{k-1} < \omega$ such that $\nu^\frown\langle t_i \rangle = \eta_i$ for each $i < k$.)
2. **distant siblings** if there exist $\nu \in {}^{\omega>}\omega$ and distinct $t_0, \cdots, t_{k-1} < \omega$ such that $\nu^\frown\langle t_i \rangle \trianglelefteq \eta_i$ for each $i < k$.

Definition

We say a theory T has **weak k-TP1** ($k \geq 2$) if it allows an $\mathcal{L}$-formula $\varphi(\bar{x}\,\bar{y})$ to witness a sequence $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}\omega \,\rangle$ satisfying

1. whenever $\eta_0 \lhd \cdots \lhd \eta_{d-1} \in {}^{\omega>}\omega$,
   $\bigcap_{i<d} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is consistent,
2. whenever $\eta_0, \cdots, \eta_{k-1} \in {}^{\omega>}\omega$ are distant siblings,
   $\bigcap_{i<k} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is <u>not</u> consistent.

The following notion was defined by Shelah and Džamonja:

Definition

A theory T is said to have SOP_1 if it allows an $\mathcal{L}$-formula to witness a sequence $\langle\, \bar{a}_\eta \mid \eta \in {}^{\omega>}2 \,\rangle$ satisfying

1. whenever $\eta_0 \lhd \cdots \lhd \eta_{d-1} \in {}^{\omega>}2$,
   $\bigcap_{i<d} \varphi(\bar{x}\,\bar{a}_{\eta_i})$ is consistent,
2. whenever $\eta^\frown\langle 0\rangle \trianglelefteq \nu \in {}^{\omega>}2$,
   $\varphi(\bar{x}\,\bar{a}_{\eta^\frown\langle 1\rangle}) \wedge \varphi(\bar{x}\,\bar{a}_\nu)$ is <u>not</u> consistent.

We have been able to apply Main Lemma to prove:

Theorem

If a theory T has weak k-TP1 for some $k \geq 2$, then T has SOP_1.

Shelah and Džamonja also defined the notion SOP_2, which turns out to be equivalent to k-TP1 ($\Leftrightarrow$ 2-TP1). We have the following picture:

$$SOP_2(\Leftrightarrow k\text{-TP1}) \Rightarrow \text{Weak } k\text{-TP1} \Rightarrow SOP_1 \Rightarrow TP$$

where TP denotes the tree property characterizing non-simple theories.

Shelah and Usvyatsov showed (2008) that

The implication $SOP_1 \Rightarrow TP$ can not be reversed.

Open problem

$SOP_1 \Leftrightarrow SOP_2$?

Reference

1. M. Džamonja and S. Shelah, 'On $\triangleleft^*$-maximality', *Annals of Pure and Applied Logic* 125 (2004) 119-158.
2. J. D. Halpern and H. Läuchli, 'A partition theorem', *Transactions of AMS* 124 (1966) 360-367.
3. L. Scow, *Characterization Theorems by Generalized Indiscernibles*, Ph. D. Thesis, Univ. of California, Berkeley (2010).
4. S. Shelah, 'Toward classifying unstable theories', *Annals of Pure and Applied Logic* 80 (1996) 229-255.
5. S. Shelah and A. Usvyatsov, 'More on SOP_1 and SOP_2', *Annals of Pure and Applied Logic* 155 (2008) 16-31.
6. S. Todorcevic, *Introduction to Ramsey Spaces*, *Annals of Mathematical Studies* 174, Princeton University Press, Princeton (2010).