A transverse condition of definable C^rG maps

Tomohiro Kawakami

Wakayama University

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Ordered fields

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  (1) For any $x, y, z \in R$, if $x < y$, then $x + z < y + z$.
  (2) For any $x, y, z \in R$, if $x < y$ and $z > 0$, then $xz < yz$.

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  (2) For any $y_1, \dots, y_m \in R$,
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  A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.
  (1) [Intermediate value property] For every $f(x) \in R[x]$, if $a < b$ and $f(a) \neq f(b)$, then $f([a, b]_R)$ contains $[f(a), f(b)]_R$ if $f(a) < f(b)$ or $[f(b), f(a)]_R$ if $f(b) < f(a)$, where $[a, b]_R = \{x \in R | a \leq x \leq b\}$.
  (2) The ring $R[i] = R[x]/(x^2 + 1)$ is an algebraically closed field.

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  (3) $\mathbf{R}^S_{an} := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where S is a subset of $\mathbb{R}$, f ranges over all restricted analytic functions as in (2)., and the function $x^r : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

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  (5) $\mathbf{R}_{an,exp} := (\mathbb{R}, +, \cdot, <, (f), exp)$, where (f) and exp denote as above.

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  The topology of R is the interval topology and the topology of R^n is the product topology.
  In this presentation, everything is considered in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, \dots,)$ of a real closed field $(R, +, \cdot, <)$ unless otherwise stated.

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- The development of o-minimality has been influenced by real analytic geometry and it is based on the following four things.
  (1) Adaptation of methods of real analytic geometry and Nash setting to the o-minimal setting.
  (2) Construction of new interesting examples of o-minimal structures.
  (3) New insights originated from model-theoretic methods into the real analytic setting and Nash setting.
  (4) O-minimal structures give a generalization, a uniform treatment and new tools.

An example of non-Archimedean o-minimal structures

- The field $\mathbb{R}[X]^\wedge$ of Puiseux series with real coefficients, namely the set of expressions $\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$.
  $\mathbb{R}[X]^\wedge$ is non-Archimedean.

Facts on o-minimal structures

Theorem

(1) The characteristic of every real closed field is $\mathbf{0}$.
(2) For any cardinality $\kappa \geq \aleph_0$, *there exist* 2^{κ} *many non-isomorphic real closed fields with cardinality* κ.
(3) There exists uncountably many o-minimal expansions of the field $\mathbb{R}$ *of real numbers.*

Definably compact and definably connected

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  A definable subset X of R^n is definably compact if for any definable function $f : [0, 1)_R \to X$, there exists the limit $\lim_{x \to 1-0} f(x)$ exists in X, where $[0, 1)_R = \{x \in R | 0 \leq x < 1\}$.

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  A definable subset X of R^n is definably connected if there do not exist two non-empty definable open subsets Y, Z of X such that $X = Y \cup Z$ and $Y \cap Z = \emptyset$.
  A compact definable set is definably compact, but a definably compact set is not necessarily compact. A connected definable set is definable connected, but a definably connected set is not necessarily connected. For example if
  $R = \mathbb{R}_{alg} := \{x \in \mathbb{R} | x$ is algebraic over $\mathbb{Q}\}$, then
  $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact and definably connected but neither compact nor connected.

Closed bounded definable sets

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Proposition

Let $X \subset \boldsymbol{R}^n$, $Y \subset \boldsymbol{R}^m$ be definable sets and $f : X \longrightarrow Y$ a definable map. If X is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).

Facts on definable maps

Theorem (Intermediate value property)

For every definable function $f(x)$ *defined on* $[a, b]$ *with* $f(a) \neq f(b)$*,* $f([a, b]_R)$ *contains* $[f(a), f(b)]_R$ *if* $f(a) < f(b)$ *or* $[f(b), f(a)]_R$ *if* $f(b) < f(a)$*.*

Definition

(1) A definable subset $\boldsymbol{G}$ *of* $\boldsymbol{R}^{\boldsymbol{n}}$ *is a definable group if* $\boldsymbol{G}$ *is a group and the group operations* $\boldsymbol{G} \times \boldsymbol{G} \longrightarrow \boldsymbol{G}$ *and* $\boldsymbol{G} \longrightarrow \boldsymbol{G}$ *are definable.*
(2) A definable group $\boldsymbol{G}$ *is a definably compact definable group if* $\boldsymbol{G}$ *is definably compact.*

Definable groups and definable G sets

Definition

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(2) A definable group G is a definably compact definable group if G is definably compact.

Definition

Let G be a definably compact definable group. A group homomorphism from G to some $O_n(R)$ is a representation if it is definable, where $O_n(R)$ means the nth orthogonal group of R. A representation space of G is R^n with the orthogonal action induced from a representation of G. A definable G set means a G invariant definable subset of some representation space of G.

- Let $X \subset R^n, Z \subset R^m$ be definable sets and $f : X \to Z$ a definable map. We say that f is a definable homeomorphism if there exists a definable map $h : Z \to X$ such that $f \circ h = id_Z$ and $h \circ f = id_X$. We call f definably proper if for every definably compact subset C of Z, $f^{-1}(C)$ is definably compact.

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  Let $X \subset R^n, Z \subset R^m$ be definable open sets and $f : X \to Z$ a definable map. We say that f is a definable C^r map if f is of class C^r.
  A definable C^r map is a definable C^r diffeomorphism if f is a C^r diffeomorphism.

Definition

(1) Let r be a non-negative integer or ∞. A Hausdorff space X is an *n-dimensional definable C^r manifold* if there exist a **finite** open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X, **finite** open sets $\{V_\lambda\}_{\lambda \in \Lambda}$ of R^n, and **finite** homeomorphisms $\{\phi_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that for any λ, ν with $U_\lambda \cap U_\nu \neq \emptyset$, $\phi_\lambda(U_\lambda \cap U_\nu)$ is definable and $\phi_\nu \circ \phi_\lambda^{-1} : \phi_\lambda(U_\lambda \cap U_\nu) \to \phi_\nu(U_\lambda \cap U_\nu)$ is a definable C^r diffeomorphism.

- This pair $(U_\lambda, \phi_\lambda)$ of sets and homeomorphisms is called a *definable C^r coordinate system*.
  In the rest of this presentation, r means a positive integer unless otherwise stated.

Definition

A definable C^r manifold G is a definable C^r group if G is a group and the group operations $G \times G \longrightarrow G$ and $G \longrightarrow G$ are definable C^r maps.

Facts on definable groups

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Theorem

(1) Let G be a definable group. For any positive integer r, G admits a unique definable C^r group structure up to definable C^r group isomorphism.
(2) If $\mathcal{N}$ is an o-minimal expansion of the standard structure of $\mathbb{R}$ and it admits the C^∞ cell decomposition, then we can take $r = \infty$ in (1).

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Corollary

If $R = \mathbb{R}$ and G is a compact Lie group of positive dimension, then $\chi(G) = 0$.

Theorem

(1) (Definable triangulation). Let $S \subset R^n$ *be a definable set and* $S_1, \dots, S_k$ *definable subsets of* S*. Then there exist a finite simplicial complex* K *in* R^n *and a definable map* $\phi : S \to R^n$ *such that* ϕ *maps* S *and each* S_i *definably homeomorphically onto a union of open simplexes of* K*. If* S *is definably compact, then we can take* $K = \phi(S)$*.*

(2) (Piecewise definable trivialization). Let X *and* Z *be definable sets and* $f : X \to Z$ *a definable map. Then there exist a finite partition* $\{T_i\}_{i=1}^k$ *of* Z *into definable sets and definable homeomorphisms* $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(z_i)$ *such that* $f|f^{-1}(T_i) = p_i \circ \phi_i$*,* $(1 \leq i \leq k)$*, where* $z_i \in T_i$ *and* $p_i : T_i \times f^{-1}(z_i) \to T_i$ *denotes the projection.*

(3) (Existence of definable quotient). Let G *be a definably compact definable group and* X *a definable* G *set. Then the orbit space* X/G *exists as a definable set and the orbit map* $\pi : X \to X/G$ *is surjective, definable and definably proper.*

Theorem

(Zeros of definable maps). Let A be a definable closed subset of R^n. Then there exists a definable C^r function $f : R^n \to R$ such that $A = f^{-1}(0)$.

Definition

Let G be a definable C^r group. A pair (X, ϕ) consisting a definable C^r manifold and a group action ϕ is a definable C^rG manifold if $\phi : G \times X \longrightarrow X$ is a definable C^r map.

- We simply write X instead of (X, ϕ).

Tangent bundles

Definition

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- Let Y be a definable C^rG submanifold of an l-dimensional representation space Ω of G. For any $y \in Y$, let $T_y(Y)$ denote the tangent space of Y at y. We define the tangent bundle $T(Y) \subset R^{2l}$ as the union $\cup_{y \in Y}\{y\} \times T_y(Y)$.

Normal bundles

- Let Y be a definable $C^r G$ submanifold of an l-dimensional representation space Ω of G. For any $y \in Y$, let $N_y(Y)$ be the orthogonal complement of the tangent space $T_y(Y)$ of Y at y with respect to the usual inner product on Ω.

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As in kawakubo's book, the above R^{2l} is a representation space Ξ of G such that $\Omega \times \{0\}$ is G invariant and $T(Y)$ and $N(Y)$ are definable $C^{r-1}G$ submanifolds of Ξ.

Definable $C^{r-1}G$ tubular neighborhoods

Theorem

Let Y be a definably compact definable C^rG submanifold of a representation space Ω of G without boundary. Then there exist a G invariant definable open neighborhood V of X in Ω and a definable $C^{r-1}G$ submersion $\theta : V \to X$ such that $\theta|X = id_X$.

Submersive definable $C^{r-1}G$ maps

- A definable C^r map $f : X \longrightarrow Y$ is a submersive definable C^r map if for any $x \in X$, the differential $(df)_x$ of f at x is surjective.

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Proposition

Let X (resp. Y) be a definable $C^r G$ submanifold of a representation space Ω (resp. Ξ) such that Y is definably compact and without boundary, and $f : X \to Y$ a definable $C^r G$ map. Let D be the open unit disk of Ξ. Then there exists a definable $C^{r-1}G$ map $F : X \times D \to Y$ such that $F(x, 0) = f(x)$ and for fixed $x \in X$ the map $F_x : D \to Y$ defined by $F_x(d) = F(x, d)$ is a submersive definable $C^{r-1}G$ map.

Definable version of Sard's theorem

- Let $f : X \longrightarrow Y$ be a definable C^r map. A point $x \in X$ is a critical point of f if all first derivatives of f at x are 0. If x is a critical point of f, then $f(x)$ is a critical value of f.

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Theorem (Berarducci and Otero 2001)

Let $X \subset R^n, Y \subset R^m$ be definable C^1 manifolds and $f : X \to Y$ a definable C^1 map. Then the set of critical values of f has dimension of less than $\dim Y$.

Transverse condition

- Let X, Y be definable C^r submanifolds of R^n and Z a definable C^r submanifold of Y. A definable C^r map $f : X \to Y$ is transverse to Z if for each $x \in X$ with $f(x) \in Z$, $(df)_x(T_x X) + T_{f(x)} Z = T_{f(x)} Y$.
  A definable $C^r G$ manifold is affine if it is definably $C^r G$ diffeomorphic to a definable $C^r G$ submanifold of some representation space of G.

Our results 1

Theorem (2010)

Let G be a definably compact definable C^r group. Let X, Y, D be affine definable C^rG manifolds such that Y and D are without boundary, and $F : X \times D \to Y$ a definable C^rG map. Suppose that Z is a definable C^rG submanifold of Y without boundary. If F and $F|\partial(X \times D)$ are transverse to Z, then for all $d \in D$ outside of a G invariant definable set of dimension $< \dim D$, f_d and $f_d|\partial X$ are transverse to Z, where $f_d : X \to Y$ is the map defined by $f_d(x) = F(x, d)$.

Our results 2

- Let G be a definably compact definable C^r group, X, Y definable C^rG manifolds and and $f, h : X \to Y$ definable C^rG maps. We say that f is definably C^rG homotopic to h if there exists a definable C^rG map $F : X \times [0,1]_R \to Y$ such that $f(x) = F(x,0)$ and $h(x) = F(x,1)$ for all $x \in X$, where the G action on $[0,1]_R = \{t \in R | 0 \leq t \leq 1\}$ is trivial.

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- Let G be a definably compact definable C^r group, X, Y definable $C^r G$ manifolds and and $f, h : X \to Y$ definable $C^r G$ maps. We say that f is definably $C^r G$ homotopic to h if there exists a definable $C^r G$ map $F : X \times [0,1]_R \to Y$ such that $f(x) = F(x,0)$ and $h(x) = F(x,1)$ for all $x \in X$, where the G action on $[0,1]_R = \{t \in R | 0 \le t \le 1\}$ is trivial.

Theorem (2010)

Let G be a definably compact definable C^r group and X, Y affine definable $C^r G$ manifolds such that Y is definably compact and without boundary. Then for every definable $C^r G$ map $f : X \to Y$ and for every definable $C^r G$ submanifold Z of Y, there exists a definable $C^{r-1} G$ map $h : X \to Y$ such that h is definably $C^{r-1} G$ homotopic to f and h and $h | \partial X$ are transverse to Z.

Our results 3

Theorem (2010)

Let $R = \mathbb{R}$ and G a compact definable C^r group. Let X, Y be affine definable C^rG manifolds such that Y is compact and without boundary. Let $f : X \longrightarrow Y$ be a definable C^rG map and Z a definable C^rG submanifold of Y without boundary. If $f|\partial X$ is transverse to Y and ∂X is compact, then there exists a definable $C^{r-1}G$ map $h : X \longrightarrow Y$ such that h is definably $C^{r-1}G$ homotopic to f, $f = h$ on ∂X and h is transverse to Z.

Thank you very much.