Unsaturated Generic Structures II

Hirotaka Kikyo

Graduate School of System Informatics, Kobe University

December 1, 2010
Model Theory Meeting at RIMS

Outline

Acknowledgement

This work is in progress with Koichiro Ikeda.

Predimension Function and Generic Structures

For a hyper-graph structure A, let

$$\delta(A) = \delta_\alpha(A) = |A| - \alpha e(A).$$

Here,
α is an real number such that $0 < \alpha \leq 1$,
$e(A) =$ the number of hyperedges in A.
$\delta_\alpha(A)$ is called a predimension function.

For a hyper-graph structure A, let

$$\delta(A) = \delta_\alpha(A) = |A| - \alpha e(A).$$

Here,
α is an real number such that $0 < \alpha \leq 1$,
$e(A) =$ the number of hyperedges in A.
$\delta_\alpha(A)$ is called a predimension function.

Suppose $A \subseteq_{\text{fin}} B$ (substructure = induced subgraph).
$A \leq B$ (A is B a strong substructure) if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta(A) \leq \delta(X).$$

For a hyper-graph structure A, let

$$\delta(A) = \delta_\alpha(A) = |A| - \alpha e(A).$$

Here,
α is an real number such that $0 < \alpha \leq 1$,
$e(A) =$ the number of hyperedges in A.
$\delta_\alpha(A)$ is called a predimension function.

Suppose $A \subseteq_{\text{fin}} B$ (substructure = induced subgraph).
$A \leq B$ (A is B a strong substructure) if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta(A) \leq \delta(X).$$

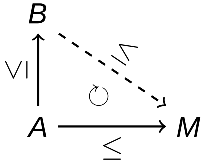
With this notation,

$$\mathbf{K}_\alpha = \{A : \text{finite} \,|\, A \geq \emptyset\}.$$

Generic Structures

Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$.
A countable hypergraph M is a generic structure of $\mathbf{K}$ if

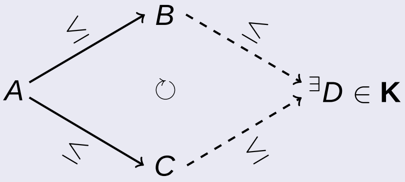
- $A \subseteq_{\text{fin}} M \Rightarrow$ there exists B such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq M$;
- $A \subseteq_{\text{fin}} M \Rightarrow A \in \mathbf{K}$;
- for any $A, B \in \mathbf{K}$,



Fact

Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$.
If $\emptyset \in \mathbf{K}$, $A \subset B \in \mathbf{K} \Rightarrow A \in \mathbf{K}$
and K has the AP, defined below, $\mathbf{K}$ has a generic structure.

For any $A, B, C \in \mathbf{K}$,



Question 1

Is there any *ab initio* generic structure which is not saturated?
Is there any *ab initio* generic structure which is superstable but not ω-stable?

Question 1

Is there any *ab initio* generic structure which is not saturated?
Is there any *ab initio* generic structure which is superstable but not ω-stable?

Theorem (Koichiro Ikeda)

For the predimension function with $\alpha = 1$,
there is an unsaturated generic structure which is superstable but not ω-stable.

Question 1

Is there any *ab initio* generic structure which is not saturated?
Is there any *ab initio* generic structure which is superstable but not ω-stable?

Theorem (Koichiro Ikeda)

For the predimension function with $\alpha = 1$,
there is an unsaturated generic structure which is superstable but not ω-stable.

Question 2

For the predimension functions with α such that $0 < \alpha < 1$,
Is there any *ab initio* generic structure which is not saturated?
Is there any *ab initio* generic structure which is superstable but not ω-stable?

Definition (special s-component)

Suppose $1 \leq s < 2$. A triple (E, a, b) with $a, b \in E$ is a special s-component if
for any substructure X of E such that $1 < |X| < |E|$,

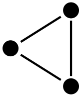
1. $\delta(X) > 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) > s$ if $a, b \in X$, and
3. $\delta(E) = s$.

Components

Definition (special s-component)

Suppose $1 \leq s < 2$. A triple (E, a, b) with $a, b \in E$ is a special s-component if
for any substructure X of E such that $1 < |X| < |E|$,

1. $\delta(X) > 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) > s$ if $a, b \in X$, and
3. $\delta(E) = s$.

When $\delta(X) = |X| - e(X)$,  is a 1-component, and  is a 0-component.

Definition (special s-component)

Suppose $1 \leq s < 2$. A triple (E, a, b) with $a, b \in E$ is a special s-component if
for any substructure X of E such that $1 < |X| < |E|$,

1. $\delta(X) > 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) > s$ if $a, b \in X$, and
3. $\delta(E) = s$.

Proposition (K.)

If $\delta(X) = |X| - \alpha e(X)$ with a rational number α satisfying $0 < \alpha < 1$ then there is a special 1-component in $\mathbf{K}_\alpha$.
If A is a special 1-component then there is another special 1-component B such that $|B| > |A|$. Note that A cannnot be embedded in B and B cannot be ebedded in A.

Let D be a finite set of finite hypergraphs. Suppose that there are hypergraphs $A_1, A_2, \ldots, A_n$ and points $a_{i-1}, a_i \in A_i$ such that (A_i, a_{i-1}, a_i) is isomorphic to some element of D for each i, and

$$X = A_1 \oplus_{a_1} A_2 \oplus_{a_2} \cdots \oplus_{a_{n-1}} A_n.$$

With such X, if we can write

$$C = X/(a_0 = a_n)$$

then we call C a D-cycle. n is called the length of the D-cycle.

Suppose $0 < \alpha < 1$. Choose minimal 1-components (A, a, b), (B, c, d) such that they cannot be embedded mutually.
Let $\mathbf{K}_0$ be the class of $\{(A, a, b),\ (B, c, d)\}$-cycles with length greater than $e(A)$ and $e(B)$. Let $\mathbf{K}_1 := \{A \mid A \subset {}^{\exists}B \in K_0\}$ and $\mathbf{K}_2$ be the smallest class with the *thrifty* amalgamation property containing $\mathbf{K}_1$.

Thrifty Amalgamation

A class **K** has the thrifty amalgamation property if
$A, B, C \in \mathbf{K}$, $A \leq B$, $A \leq C$, B is minimal over $A \Rightarrow$
B is strongly embedded in C over A,
or
$B \oplus_A C \in \mathbf{K}$.

Theorem (Ikeda, K.)

The generic structure of $\mathbf{K}_2$ is unsaturated, superstable, but not ω-stable.

Remark

In the case $\alpha = 1$, Ikeda constructed ab initio generic structure with a strictly superstable theory. In this case, a minimal 1-component is unique.

We can replace each edge of Ikeda's "Jelly fishes" by a 1-component for $\alpha < 1$, but this approach does not work well.