Independent partitions and indiscernibility

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Outline

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- Formulas are denoted by φ, ψ,...
- $m, n, k, ...$ are natural numbers.

Simple Theory

A simple theory is characterized as a theory in which the length of dividing sequence of types is bounded $(< \infty)$.

Low Theory

A low theory is characterized by the following property: For each formula $\varphi(x, y)$ there is a number $n_\varphi \in \omega$ such that whenever $\{\varphi(x, a_i) : i < m\}$ satisfies

1. $\{\varphi(x, a_i) : i < m\}$ is consistent, and
2. $\varphi(x, a_i)$ divides over $A_i = \{a_j : j < i\}$ $(i < m)$,

then $m \leq n_\varphi$.

Non-Low Simple Theory

Casanovas constructed a simple nonlow theory $T_1 = Th(M, P, P_1, P_2, ..., Q, R)$.

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3. $P = \bigcup_{i \in \omega} P_i \cup G$, where G is a random graph.
4. Q is the set of all sequences $(A_1, A_2, ..., A_\omega)$, where A_n is an n-elment subset of P_n and for some $a \in G$, $A_\omega \subset G$ is the set of all $g \in G$ directly connected to a.

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5. $R \subset P \times Q$.
6. $R(a, (A_1, A_2, ..., A_\omega))$ if (i) $a \in P_n$ and $a \in A_n$ $(\exists n \in \omega)$ or (ii) $a \notin \bigcup_n P_n$ and $a \in A_\omega$.

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This theory T_1 is not supersimple. $R(x, y)$ defines infinitely many mutually independent partitions in the following sense: If we enumerate P_n as $P_n = \{a_{nm} : m \in \omega\}$, then

- for each $\eta \in \omega^\omega$, $\{R(a_{n\eta(n)}, y) : n = 1, 2, ...\}$ is consistent, and
- for each $n = 1, 2, ...$, $\{R(a_{nm}, y) : m \in \omega\}$ is $(n + 1)$-inconsistent.

Non-Low Supersimple Theory

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However, for each $k \in \omega$, we can find a formula $\varphi(x, y)$ and parameter sets $A_i = \{a_{ij} : j \in \omega\}$ $(i < k)$ defining k independent partitions.

$D_{\mathrm{inp}}(*, *)$

Definition

$D_{\mathrm{inp}}(\Sigma(x), \varphi(x, y))$ is the first cardinal κ such that there are no κ-many independent partitions $\{\varphi(x, a_{ij}) : j \in \omega\}$ $(i < \kappa)$ of Σ.

Remark

- *For* T_1, $D_{\mathrm{inp}}(x = x, R(y, x)) = \omega_1$.
- *For* T_2, *for some* $\varphi(x, y)$, $D_{\mathrm{inp}}(x = x, \varphi(x, y)) = \omega$.

So it is natural to ask whether there is a simple nonlow theory T such that

$$D_{\mathrm{inp}}(x = x, \varphi(x, y)) < \omega,$$

for any φ.

Ranks

First we recall definitions of basic ranks.
Let $\Sigma(x)$ be a set of formulas and $\varphi(x, y)$ a formula.
Let $k \in \omega$.

Definition

$D(\Sigma(x), \varphi(x, y), k)$

1
- $D(\Sigma(x), \varphi(x, y), k) \geq 0$ if $\Sigma(x)$ is consistent.
- $D(\Sigma(x), \varphi(x, y), k) \geq n + 1$ if there is an indiscernible sequence $\{b_i : i \in \omega\}$ over $\mathbf{dom}(\Sigma)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi(x, y), k) \geq n$ for all $i \in \omega$, and $\{\varphi(x, b_i) : i \in \omega\}$ is k-inconsistent.

Definition

2

- $D(\Sigma(x), \varphi(x, y)) \geq 0$ if $\Sigma(x)$ is consistent.
- For a limit ordinal δ, $D(\Sigma(x), \varphi(x, y)) \geq \delta$ if $D(\Sigma(x), \varphi(x, y)) \geq \alpha$ for all $\alpha < \delta$.
- $D(\Sigma(x), \varphi(x, y)) \geq \alpha + 1$ if there is an indiscernible sequence $\{b_i : i \in \omega\}$ over $\mathbf{dom}(\Sigma)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi(x, y)) \geq \alpha$ $(i \in \omega)$, and $\{\varphi(x, b_i) : i \in \omega\}$ is inconsistent.

Fact

1. $D(\Sigma(x), \varphi(x, y), k) \geq n$ if there is a tree $A = \{a_\nu : \nu \in \omega^{\leq n}\}$ such that (1) $\Sigma(x) \cup \{\varphi(x, a_{\eta|i}) : 1 \leq i \leq n\}$ is consistent $(\forall \eta \in \omega^n)$, and (2) $\{\varphi(x, a_{\nu^\frown i}) : i \in \omega\}$ is k-inconsistent $(\forall \nu \in \omega^{<n})$.

Fact

2. $D(\Sigma(x), \varphi(x, y)) \geq n$ *if there is a tree* $A = \{a_\nu : \nu \in \omega^{\leq n}\}$ *and numbers* $k_0, ..., k_{n-1}$ *such that (1)* $\Sigma(x) \cup \{\varphi(x, a_{\eta|i}) : 1 \leq i \leq n\}$ *is consistent* $(\forall \eta \in \omega^n)$*, and (2)* $\{\varphi(x, a_{\nu^\frown i}) : i \in \omega\}$ *is* $k_{\mathrm{lh}(\nu)}$*-inconsistent* $(\forall \nu \in \omega^{<n})$*.*

Main Result

Theorem

Suppose that the size of independent partitions is bounded in T. Then the following are equivalent:

1. T *is simple.*
2. T *is low.*

Proposition

Suppose $D_{\mathrm{inp}}(x = x, \varphi(x, y)) = k - 1 < \omega$ *and* $D(x = x, \varphi(x, y)) \geq \omega$. *Then* T *is not simple.*

Proof.

Fix $m \in \omega$.

- By $D(x = x, \varphi(x, y)) \geq \omega$, there is a set $A = \{a_\nu : \nu \in \omega^{\leq m}\}$ witnessing $D(x = x, \varphi(x, y)) \geq m$.

Proof.

Fix $m \in \omega$.

- By $D(x = x, \varphi(x, y)) \geq \omega$, there is a set $A = \{a_\nu : \nu \in \omega^{\leq m}\}$ witnessing $D(x = x, \varphi(x, y)) \geq m$.
- We have
  1. $\{\varphi(x, a_{\eta|i}) : 1 \leq i \leq m\}$ is consistent $(\forall \eta \in \omega^m)$,
  2. $\{\varphi(x, a_{\nu^\frown i}) : i \in \omega\}$ is $k_{\mathrm{lh}(\nu)}$-inconsistent $(\forall \nu \in \omega^{<m})$.

- We can assume that A is an indiscernible tree.

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- For $\nu \in \omega^m$, let ν^* be the sequence

$$\nu(0), 0^k, \nu(1), 0^k, \ldots, \nu(\mathrm{lh}(\nu) - 1), 0^k.$$

- For $\nu = \nu_0 \frown m$, let

$$a^*_\nu = a_{\nu_0{}^* \frown m \frown 0}, a_{\nu_0{}^* \frown m \frown 0^2}, \ldots, a_{\nu_0{}^* \frown m \frown 0^k}.$$

- Let $\varphi^*(x, y_1, ..., y_k)$ be the formula $\varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_k)$.
- **Claim A** $\{\varphi^*(x, a^*_{\nu_0 \frown m}) : m \in \omega\}$ is k-inconsistent.

- Let $\varphi^*(x, y_1, ..., y_k)$ be the formula $\varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_k)$.
- **Claim A** $\{\varphi^*(x, a^*_{\nu_0 ^\frown m}) : m \in \omega\}$ is k-inconsistent.

  Suppose this is not the case. Then there is $F = \{i_1, ..., i_k\} \subset \omega$ such that

$$\{\varphi^*(x, a^*_{\nu_0 ^\frown i_1}), ..., \varphi^*(x, a^*_{\nu_0 ^\frown i_k})\}$$

  is consistent.

- By the definition of φ^*, in particular, the following set is consistent.

$$\{\varphi(x, a_{\nu_0^* \frown i_1 \frown 0}), \ldots, \varphi(x, a_{\nu_0^* \frown i_k \frown 0^k})\}$$

- By the definition of φ^*, in particular, the following set is consistent.

$$\{\varphi(x, a_{\nu_0^* \frown i_1 \frown 0}), \ldots, \varphi(x, a_{\nu_0^* \frown i_k \frown 0^k})\}$$

- For each ν of length k, let Γ_ν be the set:

$$\{\varphi(x, a_{\nu_0^* \frown i_1 \frown \nu(1)}), \ldots, \varphi(x, a_{\nu_0^* \frown i_k \frown 0^{k-1} \frown \nu(k)})\}.$$

- Then each Γ_γ is consistent, by the indiscernibility of A.

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- On the other hand, by our choice of the tree A, for each $l = 0, ..., k-1$, the set

$$\{\varphi(x, a^*{}_{\nu_0 \frown i_2 \frown 0^l \frown i}) : i \in \omega\}$$

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is inconsistent ($k_{\mathrm{lh}(\nu_0)+(1+l)}$-inconsistent).
- This yields $D_{\mathrm{inp}}(x = x, \varphi(x, z)) \geq k$, a contradiction. (End of Proof of Claim)

- By Claim A, the set $\{\varphi^*(x, a^*{}_\nu) : \nu \in \omega^m\}$ witnesses $D(x = x, \varphi^*, k) \geq m$.

- By Claim A, the set $\{\varphi^*(x, a^*{}_\nu) : \nu \in \omega^m\}$ witnesses $D(x = x, \varphi^*, k) \geq m$.
- Since m is arbitrary, we conclude $D(x = x, \varphi^*, k) = \infty$, which means that T is not simple.