Twisted derivatives with Alexander pairs for quandles

Atsushi Ishii and Kanako Oshiro

Abstract
Fox defined free derivatives for groups, which can be used to define an invariant of a group equipped with a group representation such as the (twisted) Alexander invariants from a knot group. In this paper, we define twisted derivatives for quandles, which can be used to define an invariant of a quandle equipped with a quandle representation. The twisted Alexander invariants and the quandle cocycle invariants can be found in our framework with suitable setting of augmented Alexander pairs. As an application, we demonstrate that our invariant detects 5-move equivalence classes of some knots.

MSC: 57M27, 57M25.
Keywords: Fox derivative; twisted Alexander invariant; quandle cocycle invariant, n-move equivalence.

1 Introduction
The Alexander polynomial [1] is a classical link invariant that is defined as a generator of the elementary ideal of the Alexander matrix of a link. This construction is generalized with a group representation to the twisted Alexander invariant, which was introduced by Lin [19] and Wada [24]. The twisted Alexander invariant with a trivial group representation coincides with the Alexander polynomial. The behavior of the Alexander polynomial in terms of topological properties such as the genus and fiberedness of knots can be extended to the twisted Alexander invariant (e.g. [4, 10, 11, 12]). The (twisted) Alexander matrices are obtained from the link group by using free derivatives, which was introduced by Fox [8]. In general, by using free derivatives, we can obtain an invariant of a group equipped with a group representation.

In this paper, we introduce $f$-derivatives for quandles and define invariants of a quandle equipped with a quandle representation, where a quandle [16, 20] is an algebra whose axioms correspond to the Reidemeister moves on link diagrams. We also call an $f$-derivative a twisted derivatives for quandles. In general, it is not easy to distinguish two quandles, although the link quandle is an important invariant of a link. Counting the number of quandle homomorphisms from the link quandle to a quandle gives an elementary combinatorial invariant, which we call the quandle coloring invariant. By using $f$-derivatives, we are able to extract more information from the link quandle. An (augmented) Alexander pair is a dynamical cocycle [2] corresponding to a linear (or affine) extension of a quandle, and we can choose an (augmented) Alexander pair $f$ for $f$-derivatives.
By setting suitable Alexander pairs $f$, the twisted Alexander invariants can be obtained in our framework.

The quandle cocycle invariants [3] are also appear in our framework. The quandle cocycle invariant is a weight-sum invariant on quandle colorings, where the weight at a crossing is given using a quandle cocycle. A quandle 2-cocycle is a dynamical cocycle corresponding to an abelian extension of a quandle. The quandle coloring invariant is recoverable from the quandle cocycle invariant through the use of the trivial quandle cocycle and can be used to investigate the properties of knots, surface-knots (e.g. [3, 18, 22]), and handlebody-knots (e.g. [13, 14]), where a surface-knot is a closed surface embedded in a 4-sphere and a handlebody-knot is a handlebody embedded in a 3-sphere.

An $n$-move is a simple move for links, which may reduce a link to a trivial one in some cases. A 2-move is identical with a crossing change and is an unknotting operation. The Montesinos–Nakanish 3-move conjecture states that a 3-move is an unknotting operation, and Dąbkowski and Przytycki [5] showed that the conjecture is not true. Equivalence classes under $n$-moves are also studied (e.g. [6, 23]). In this paper, we introduce a quandle $IQ_n(L)$ which is invariant under $n$-moves and calculate our invariant for the quandle $IQ_n(L)$. In particular, we demonstrate that some knots are not 5-move equivalent with our invariant.

This paper is organized as follows. In Section 2, we introduce an (augmented) Alexander pair and see that it is related to an extension of a quandle. We also give some examples of (augmented) Alexander pairs. In Section 3, we recall the definition of a quandle presentation and see that two finite presentations represent the same quandle if they are related by a finite sequence of Tietze transformations. In Section 4, we introduce $f$-derivatives for quandles and show that they are well-defined. In Section 5, we introduce $f$-twisted Alexander matrices and see that they give invariants of (link) quandles, whose invariance is proven in Section 10. In Section 6, we show that the (twisted) Alexander matrices are recoverable as $f$-twisted Alexander matrices for some Alexander pairs $f$. In Section 7, we see a relationship between our invariant and the quandle cocycle invariant. As an application of our invariant, we demonstrate that some knots are not 5-move equivalent in Section 8. In Section 9, we introduce the notion of cohomologous for (augmented) Alexander pairs and show that cohomologous (augmented) Alexander pairs induce the same invariant. In Section 11, we introduce the notion of an augmented matrix, which can be used to remove some restrictions of the results given in this paper.

2 Alexander pairs and twisted 2-cocycles

In [2], Andruskiewitsch and Graña presented a method for constructing an extension of a quandle (or rack) through the use of a dynamical cocycle of a quandle (or rack). We provide their definition below with the notation changed to match our conventions.

A quandle [16, 20] is a set $Q$ equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

(Q1) For any $a \in Q$, $a \triangleleft a = a$.

(Q2) For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
(Q3) For any \(a, b, c \in Q\), \((a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)\).

We denote \((a^n) : Q \to Q\) by \(a^n\) for \(n \in \mathbb{Z}\).

For quandles \((X_1, \triangleleft_1)\) and \((X_2, \triangleleft_2)\), a quandle homomorphism \(f : X_1 \to X_2\) is defined to be a map \(f : X_1 \to X_2\) satisfying \(f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)\) for any \(a, b \in X_1\). We denote by \(\text{Hom}(Q_1, Q_2)\) the set of quandle homomorphisms from \(Q_1\) to \(Q_2\). We call a bijective quandle homomorphism a quandle isomorphism.

A quandle homomorphism \(\rho : X \to Q\) is also called a quandle representation of \(X\) to \(Q\). A quandle representation is trivial if it is a constant map. Let \(\rho_1 : X_1 \to Q\) and \(\rho_2 : X_2 \to Q\) be quandle representations. We say \((X_1, \rho_1)\) and \((X_2, \rho_2)\) are isomorphic if there exists a quandle isomorphism \(f : X_1 \to X_2\) such that \(\rho_1 = \rho_2 \circ f\).

For a positive integer \(n\), we denote by \(\mathbb{Z}_n\) the cyclic group \(\mathbb{Z}/n\mathbb{Z}\) of order \(n\). We define a binary operation \(<\) on \(\mathbb{Z}_n\) by \(a < b = 2b - a\). Then, \((\mathbb{Z}_n, <)\) is a quandle. We call it the dihedral quandle of order \(n\) and denote it by \(R_n\).

Let \(G\) be a group and \(n\) an integer. We define a binary operation \(<\) on \(G\) by \(a < b = b^{-n}ab^n\). Then, \((G, <)\) is a quandle. We call it the \(n\)-fold conjugation quandle of \(G\) and denote it by \(\text{Conj}_n G\). The 1-fold conjugation quandle of \(G\) is called the conjugation quandle of \(G\) and denoted by \(\text{Conj} G\).

On a ring \(R\) (or an \(R\)-module \(M\)), we denote by \(0\) the zero map and denote by \(1\) the constant map that sends all elements of the domain to the multiplicative identity 1 of \(R\). We denote by \(R^*\) the group of units of \(R\).

**Definition 2.1.** Let \((Q, \triangleleft)\) be a quandle. Let \(R\) be a ring and \(M\) a left \(R\)-module.

1. The pair \((f_1, f_2)\) of \(f_1, f_2 : Q \times Q \to R\) is an Alexander pair if \(f_1\) and \(f_2\) satisfy the following conditions:
   - For any \(a \in Q\), \(f_1(a, a) + f_2(a, a) = 1\).
   - For any \(a, b \in Q\), \(f_1(a, b)\) is invertible.
   - For any \(a, b, c \in Q\),
     \[
     f_1(a \triangleleft b, c)f_1(a, b) = f_1(a \triangleleft c, b \triangleleft c)f_1(a, c),
     
     f_1(a < b, c)f_2(a, b) = f_2(a < c, b < c)f_1(b, c),
     
     f_2(a < b, c) = f_1(a < c, b < c)f_2(a, c) + f_2(a < c, b < c)f_2(b, c).
     
   \]

2. The map \(\phi : Q \times Q \to M\) is an \((f_1, f_2)\)-twisted 2-cocycle if \(\phi\) satisfies the following conditions:
   - For any \(a \in Q\), \(\phi(a, a) = 0\).
   - For any \(a, b, c \in Q\),
     \[
     \phi(a < b, c) + f_1(a < b, c)\phi(a, b) = \phi(a < c, b < c) + f_1(a < c, b < c)\phi(a, c) + f_2(a < c, b < c)\phi(b, c).
     \]

We note that \(f_1, f_2\) and \(\phi\) correspond, respectively, to \(\eta, \tau\) and \(\kappa\) in [2], where \(\phi\) is called a generalized quandle 2-cocycle. We call \((1, 0)\) the trivial Alexander pair and 0 the trivial \((f_1, f_2)\)-twisted 2-cocycle. The notion of a \((1, 0)\)-twisted 2-cocycle coincides with that of a quandle 2-cocycle. For more detail, we refer
the reader to [3]. We call \((f_1, f_2; \phi)\) an augmented Alexander pair if \((f_1, f_2)\) is an Alexander pair and \(\phi\) is an \((f_1, f_2)\)-twisted 2-cocycle.

An extension of a quandle \((Q, \cdot)\) is a quandle \((\tilde{Q}, \tilde{\cdot})\) that has a surjective homomorphism \(f : (\tilde{Q}, \tilde{\cdot}) \to (Q, \cdot)\) such that the cardinality of \(f^{-1}(a)\) coincides with that of \(f^{-1}(b)\) for any \(a, b \in Q\). Augmented Alexander pairs correspond to extensions of a quandle as shown in the following proposition.

**Proposition 2.2** (c.f. [2]). Let \((Q, \cdot)\) be a quandle. Let \(R\) be a ring and \(M\) a left \(R\)-module. Let \(f_1, f_2 : Q \times Q \to R\) and \(\phi : Q \times Q \to M\) be maps.

1. If \(\phi\) is a quandle 2-cocycle, then \(Q \times M\) is a quandle with the binary operation \(* : (Q \times M) \times (Q \times M) \to Q \times M\) defined by:
   \[
   (a, x) * (b, y) = (a \cdot b, x + \phi(a, b)).
   \]

2. If \((f_1, f_2)\) is an Alexander pair, then \(Q \times M\) is a quandle with the binary operation \(* : (Q \times M) \times (Q \times M) \to Q \times M\) defined by:
   \[
   (a, x) * (b, y) = (a \cdot b, f_1(a, b)x + f_2(a, b)y).
   \]

3. If \((f_1, f_2; \phi)\) is an augmented Alexander pair, then \(Q \times M\) is a quandle with the binary operation \(* : (Q \times M) \times (Q \times M) \to Q \times M\) defined by:
   \[
   (a, x) * (b, y) = (a \cdot b, f_1(a, b)x + f_2(a, b)y + \phi(a, b)).
   \]

The quandles \(Q \times M\) in Proposition 2.2 are extensions of \(Q\), since the projection \(pr_Q : Q \times M \to Q\) satisfies the defining condition of an extension. The quandle \(Q \times M\) of Proposition 2.2 (1) is called an abelian extension of \(Q\). We call the quandles \(Q \times M\) of Proposition 2.2 (2) and (3) a linear extension and affine extension of \(Q\), respectively.

We give some examples of Alexander pairs \((f_1, f_2)\) and \((f_1, f_2)\)-twisted 2-cocycles \(\phi\).

**Example 2.3.** Let \(Q\) be a quandle and \(R[t^{\pm 1}]\) the Laurent polynomial ring over a commutative ring \(R\). The following maps \(f_1, f_2 : Q \times Q \to R[t^{\pm 1}]\) give an Alexander pair:

\[
 f_1(a, b) = t, \quad f_2(a, b) = 1 - t.
\]

In Section 6, we will show that this pair is related to the Alexander polynomial.

**Example 2.4.** Let \(G\) be a group and \(R[G]\) the group ring of \(G\) over a commutative ring \(R\). Let \(Q := \text{Conj}_n G\). Then, the following maps \(f_1, f_2 : Q \times Q \to R[G][t^{\pm 1}]\) give Alexander pairs:

1. \(f_1(a, b) = b^{-n}a^n, f_2(a, b) = 0,\)
2. \(f_1(a, b) = tb^{-n}, f_2(a, b) = 1 - tb^{-n},\)
3. \(f_1(a, b) = tb^{-n}, f_2(a, b) = b^{-n}a^n - tb^{-n}.\)

We note that Alexander pair (2) is related to a \(G\)-family of quandles [15]. In Section 6, we will show that Alexander pair (3) is related to the twisted Alexander invariant. Let \(Q := \text{Conj}_{mn} G\). Then, the following maps \(f_1, f_2 : Q \times Q \to R[G][t^{\pm 1}]\) give an Alexander pair:
Example 2.5. Let \( Q \) be a quandle and \( R \) a ring. Let \( f : Q \to \text{Conj}_{-1}R^\times \) be a quandle homomorphism. The following maps \( f_1, f_2 : Q \times Q \to R \) give Alexander pairs:

\[
\begin{align*}
(1) & \quad f_1(a, b) = f(b)f(a)^{-1}, \quad f_2(a, b) = 0, \\
(2) & \quad f_1(a, b) = f(b), \quad f_2(a, b) = 1 - f(b), \\
(3) & \quad f_1(a, b) = f(b), \quad f_2(a, b) = f(b)f(a)^{-1} - f(b).
\end{align*}
\]

Let \( f : Q \to \text{Conj}_{-m}R^\times \) be a quandle homomorphism. Then, the following maps \( f_1, f_2 : Q \times Q \to R \) give an Alexander pair:

\[
\begin{align*}
(4) & \quad f_1(a, b) = f(b)^m, \quad f_2(a, b) = f(b)^m(f(a)^{-1} - 1) \sum_{k=0}^{m-1} f(b)^{-k},
\end{align*}
\]

which is a generalization of Alexander pair (3), since they coincide when \( m = 1 \).

Example 2.6. Let \( Q \) be the dihedral quandle \( R_3 \) of order 3, \( R \) the quotient ring \( \mathbb{Z}_3 \), and \( M \) the polynomial ring \( \mathbb{Z}_3[t_1, t_2] \). Then, the following maps \( f_1, f_2 : Q \times Q \to R \) and \( \phi : Q \times Q \to M \) give an augmented Alexander pair:

\[
f_1(a, b) = 2, \quad f_2(a, b) = 2, \quad \phi(a, b) = (t_1a + t_2b)(a - b)^2.
\]

Let \( \rho : X \to Q \) be a quandle representation. For maps \( f_1, f_2 : Q \times Q \to R \) and \( \phi : Q \times Q \to M \), we define \( f_1^\rho := f_1 \circ (\rho \times \rho) \), \( f_2^\rho := f_2 \circ (\rho \times \rho) \) and \( \phi^\rho := \phi \circ (\rho \times \rho) \).

Proposition 2.7. If \( (f_1, f_2) \) is an Alexander pair of maps \( f_1, f_2 : Q \times Q \to R \), then \( (f_1^\rho, f_2^\rho) \) is an Alexander pair of maps \( f_1^\rho, f_2^\rho : X \times X \to R \). If \( \phi : Q \times Q \to M \) is an \( (f_1, f_2) \)-twisted 2-cocycle, then \( \phi^\rho : X \times X \to M \) is an \( (f_1^\rho, f_2^\rho) \)-twisted 2-cocycle.

Proof. We can verify this proposition by direct calculation.

\[\square\]

3 Quandle presentations

In this section, we recall the definition of a quandle presentation and Tietze transformations for quandle presentations. See also [7, 17] for more details.

Let \( S \) be a set. We denote by \( F_{\text{Grp}}(S) \) the free group on \( S \). For \( (a, x), (b, y) \in S \times F_{\text{Grp}}(S) \), we write \( (a, x) \sim (b, y) \) if \( a = b \) and \( x = a^n y \) for some \( n \in \mathbb{Z} \). Then \( \sim \) is an equivalence relation on \( S \times F_{\text{Grp}}(S) \). Set \( F_{\text{Quad}}(S) := S \times F_{\text{Grp}}(S)/\sim \). We define a binary operation \( \triangleleft \) on \( F_{\text{Quad}}(S) \) by \( [(a, x)] \triangleleft [(b, y)] := [(a, xy^{-1}b y)] \). Then, \( (F_{\text{Quad}}(S), \triangleleft) \) is a quandle, which we call the free quandle on \( S \). For convenience, we denote \( [(a, 1)] \) by \( a \), where \( 1 \) is the identity element of \( F_{\text{Grp}}(S) \). Then it holds that \( [(a, b_1 \ldots b_n)] = ((a \triangleleft b_1) \triangleleft b_2) \ldots \triangleleft b_n \) for \( a, b_1, \ldots, b_n \in S \) and \( \varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \), since \( [(a, x)] \triangleleft [(b, 1)] = [(a, xb)] \).
Let $W_{\text{Qnd}}(S)$ be the set of appropriate sequences of elements of $S$, $\circ, \circ^{-1}$, parentheses "(" and ")", which we call \textit{quandle words}. More precisely, we define $W_{\text{Qnd}}(S) := \bigcup_{n=0}^{\infty} W_{\text{Qnd}}(S; n)$ with $W_{\text{Qnd}}(S; 0) := S$ and

$$W_{\text{Qnd}}(S; n + 1) := W_{\text{Qnd}}(S; n) \cup \{ a \circ b | a, b \in W_{\text{Qnd}}(S; n), \varepsilon \in \{ \pm 1 \} \},$$

where we put parentheses in appropriate places. For example,

$$W_{\text{Qnd}}(\{ a \}; 1) = \{ a, a \circ a, a \circ^{-1} a \},$$

$$W_{\text{Qnd}}(\{ a \}; 2) = \{ a, a \circ a, a \circ^{-1} a, a \circ (a \circ a), a \circ (a \circ^{-1} a), (a \circ a) \circ a, (a \circ a) \circ (a \circ^{-1} a), (a \circ^{-1} a) \circ (a \circ a), (a \circ^{-1} a) \circ (a \circ^{-1} a), a \circ^{-1} (a \circ^{-1} a), (a \circ a) \circ^{-1} a, (a \circ a) \circ^{-1} (a \circ a), (a \circ a) \circ^{-1} a, (a \circ^{-1} a) \circ^{-1} a, (a \circ^{-1} a) \circ^{-1} (a \circ^{-1} a) \}. $$

**Lemma 3.1.** For $w_1, w_2 \in W_{\text{Qnd}}(S)$, $w_1$ and $w_2$ represent the same element in $F_{\text{Qnd}}(S)$ if and only if $w_1$ and $w_2$ are related by the following transformations:

- $\cdots (\cdots (a \circ^1 a) \circ^2 a_1 \cdots) \circ^m a_n \leftrightarrow (\cdots (a \circ^2 a_1) \cdots) \circ^m a_n$
- $\cdots (\cdots (a \circ b \circ^1 b) \circ^2 a_1 \cdots) \circ^m a_n \leftrightarrow (\cdots (a \circ^1 a_1) \cdots) \circ^m a_n$
- $\cdots (\cdots (a \circ b \circ^1 c) \circ^2 a_1 \cdots) \circ^m a_n \leftrightarrow (\cdots (a \circ^1 c) \circ^2 (b \circ^1 c)) \cdots) \circ^m a_n$

for $a, b, c, a_1, \ldots, a_n \in W_{\text{Qnd}}(S)$ and $\varepsilon, \delta, \varepsilon_1, \ldots, \varepsilon_n \in \{ \pm 1 \}$.

**Proof.** It is easy to show the “if” part. We show the “only if” part. By using the transformations, $w_1$ and $w_2$ are respectively transformed into

$$\cdots (\cdots (a \circ^1 a_1) \circ^2 a_2 \cdots) \circ^m a_m \quad \text{and} \quad \cdots (\cdots (b \circ^1 b_1) \circ^2 b_2 \cdots) \circ^m b_n$$

for some $a, a_1, \ldots, a_m, b, b_1, \ldots, b_n \in S$ and $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \in \{ \pm 1 \}$ such that $a \neq a_1$ and $b \neq b_1$ and that $a \circ^1 \cdots \circ^m a_m$ and $b \circ^1 \cdots \circ^m b_n$ are irreducible, that is, $a_i = a_{i+1}$ and $b_i = b_{i+1}$ respectively imply $\varepsilon_i = \varepsilon_{i+1}$ and $\delta_i = \delta_{i+1}$. Since $w_1$ and $w_2$ represent the same element in $F_{\text{Qnd}}(S)$, we have $[(a, a_1, \ldots, a_m)] = [(b, b_1, \ldots, b_n)]$, which implies $a = b$, $m = n$, $a_i = b_i$, $\varepsilon_i = \delta_i$ for any $i$. This completes the proof. \hfill $\square$

Let $(Q, \circ)$ be a quandle, and $P$ a subset of $Q \times Q$. We write $a \sim_P b$ if $(a, b) \in P$ for $a, b \in Q$. A \textit{quandle congruence relation} on $Q$ is an equivalence relation on $Q$ satisfying the following condition:

If $a_1 \sim_P a_2$ and $b_1 \sim_P b_2$, then $a_1 \sim_{\pm 1} b_1 \sim_P a_2 \sim_{\pm 1} b_2$.

Then $Q/\sim_P$ is a quandle with the binary operation $\circ : Q/\sim_P \times Q/\sim_P \to Q/\sim_P$ defined by $[a] \circ [b] = [a \circ b]$. For $R \subseteq F_{\text{Qnd}}(S) \times F_{\text{Qnd}}(S)$, we define $N_{\text{Qnd}}(R)$ to be the minimal quandle congruence relation including $R$. For $A \subseteq F_{\text{Qnd}}(S)$, we define

$$N_{\text{Qnd}}(A) := A \cup \{ (x, x) | x \in F_{\text{Qnd}}(S) \} \cup \{ (b, a) | (a, b) \in A \} \cup \{ (a, c) | (a, b), (b, c) \in A \} \cup \{ (a_1 \circ^1 b_1, a_2 \circ^1 b_2) | (a_1, a_2), (b_1, b_2) \in A, \varepsilon \in \{ \pm 1 \} \}. $$

6
The following transformations (T1), (T2) are called the Tietze transformations on quandle presentations:

(T1) $\langle S \mid R \rangle \leftrightarrow \langle S \mid R \cup \{r\} \rangle$ ($r \in N_{\text{qnd}}(R)$),

(T2) $\langle S \mid R \rangle \leftrightarrow \langle S \cup \{y\} \mid R \cup \{(y, w_y)\} \rangle$ ($y \notin F_{\text{qnd}}(S)$, $w_y \in F_{\text{qnd}}(S)$).

It is known that two finite presentations $\langle S \mid R \rangle$ and $\langle S' \mid R' \rangle$ represent the same quandle if and only if they can be transformed into each other by a finite sequence of Tietze transformations.
For the transformation (T1), the identity from $F_{\text{qnd}}(S)$ to $F_{\text{qnd}}(S)$ induces the isomorphism $\varphi_1: (S|R) \to (S|R \cup \{r\})$ that sends $[x]$ to $[x]$, which gives a one-to-one correspondence between $\text{Hom}(\langle S|R\rangle, Q)$ and $\text{Hom}(\langle S|R \cup \{r\}, Q)$.

For $\rho \in \text{Hom}(\langle S|R\rangle, Q)$, we use the same symbol $\rho$ to represent $\rho \circ \varphi_1$. For the transformation (T2), the inclusion from $F_{\text{qnd}}(S)$ to $F_{\text{qnd}}(S)$ induces the isomorphism $\varphi_2: (S|R) \to \langle S \cup \{y\} | R \cup \{y, w_y\} \rangle$ that sends $[x]$ to $[x]$, which gives a one-to-one correspondence between $\text{Hom}(\langle S|R\rangle, Q)$ and $\text{Hom}(\langle S \cup \{y\} | R \cup \{y, w_y\}, Q)$.

It is easy to see that $(\langle S|R\rangle, \rho)$ and $(\langle S'|R'\rangle, \rho')$ are isomorphic if and only if they can be transformed into each other by a finite sequence of Tietze transformations with quandle representations.

The transformation (T1) can be decomposed into the following transformations:

(T1) $(\langle S|R\rangle, \rho) \leftrightarrow (\langle S|R \cup \{(x, x)\}, \rho) \ (x \in F_{\text{qnd}}(S))$.

(T1-1) $(\langle S|R\rangle, \rho) \leftrightarrow (\langle S|R \cup \{(x, x)\}, \rho) \ (x \in F_{\text{qnd}}(S))$.

(T1-2) $(\langle S|R\rangle, \rho) \leftrightarrow (\langle S|R \cup \{(a, b)\}, \rho) \ (a \in F_{\text{qnd}}(S))$.

(T1-3) $(\langle S|R\rangle, \rho) \leftrightarrow (\langle S|R \cup \{(a, b), (b, a)\}, \rho)$.

(T1-4) $(\langle S|R\rangle, \rho) \leftrightarrow (\langle S|R \cup \{(a_1, a_2), (b_1, b_2)\}, \rho)$.

(T1-5) $(\langle S|R\rangle, \rho) \leftrightarrow (\langle S|R \cup \{(a_1, a_2), (b_1, b_2)\}, \rho)$.

Lemma 3.3. Let $(\langle S|R\rangle)$ and $(\langle S'|R'\rangle)$ be finite presentations of quandles. Let $\rho: (\langle S|R\rangle) \to Q$ and $\rho': (\langle S'|R'\rangle) \to Q$ be quandle representations. Then $(\langle S|R\rangle, \rho)$ and $(\langle S'|R'\rangle, \rho')$ are isomorphic if and only if they can be transformed into each other by a finite sequence of the transformations (T1-1)–(T1-5) and (T2).

Proof. It is sufficient to show that, for $r \in N_{\text{qnd}}(S)$, $(\langle S|R\rangle, \rho)$ and $(\langle S|R \cup \{r\}, \rho)$ can be transformed into each other by a finite sequence of the transformations (T1-1)–(T1-5). There is an integer $n \geq 1$ such that $r \in N_{\text{qnd}}(S) \subset N_{\text{qnd}}(R_i)$. Let $R_n := \{r\}$. For $i \in \{1, \ldots, n-1\}$, we choose a finite set $R_i \subset N_{\text{qnd}}(R_i)$ so that $R_{i+1} \subset N_{\text{qnd}}(R_i)$. Then we have

$$(\langle S|R\rangle, \rho) \leftrightarrow \cdots \leftrightarrow (\langle S| R \cup R_1 \cup R_2 \cup \cdots \cup R_n\rangle, \rho),$$

$$(\langle S|R \cup \{r\}, \rho) \leftrightarrow \cdots \leftrightarrow (\langle S| R \cup \{r \cup R_1\}, \rho),$$

where each “$\leftrightarrow$” stands for one of the transformations (T1-1)–(T1-5). Since $R_n = \{r\}$, $(\langle S|R\rangle, \rho)$ and $(\langle S|R \cup \{r\}, \rho')$ are transformed into each other by a finite sequence of the transformations (T1-1)–(T1-5).
4 \( f \)-derivatives for quandles

Let \( S = \{x_1, \ldots, x_n\} \) be a finite set, and let \( Q = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle \) be a finitely presented quandle. Let \( F_{\text{Qnd}}(S) \) be the free quandle on \( S \), and \( \text{pr} : F_{\text{Qnd}}(S) \to Q \) the canonical projection. We often omit “\( \text{pr} \)” to represent \( \text{pr}(a) \) as \( a \). Let \((f_1, f_2)\) be an Alexander pair of maps \( f_1, f_2 : Q \times Q \to R \), and \( \phi : Q \times Q \to M \) an \((f_1, f_2)\)-twisted 2-cocycle. Put \( f = (f_1, f_2) \) and \( \bar{f} = (f_1, f_2; \phi) \).

**Definition 4.1.**  
(1) The \( f \)-derivative with respect to \( x_j \) for \( j \in \{1, \ldots, n\} \) is a map \( \frac{\partial f}{\partial x_j} : F_{\text{Qnd}}(S) \to R \) satisfying

\[
\frac{\partial f}{\partial x_j}(a \triangle b) = f_1(a, b) \frac{\partial f}{\partial x_j}(a) + f_2(a, b) \frac{\partial f}{\partial x_j}(b), \quad \frac{\partial f}{\partial x_j}(x_i) = \delta_{ij}
\]

for any \( a, b \in F_{\text{Qnd}}(S) \) and \( i \in \{1, \ldots, n\} \), where \( \delta_{ij} \) is the Kronecker delta.

(2) The \( \bar{f} \)-derivative with respect to \( x_\infty \) is a map \( \frac{\partial \bar{f}}{\partial x_\infty} : F_{\text{Qnd}}(S) \to M \) satisfying

\[
\frac{\partial \bar{f}}{\partial x_\infty}(a \triangle b) = f_1(a, b) \frac{\partial \bar{f}}{\partial x_\infty}(a) + f_2(a, b) \frac{\partial \bar{f}}{\partial x_\infty}(b) + \phi(a, b), \quad \frac{\partial \bar{f}}{\partial x_\infty}(x_i) = 0
\]

for any \( a, b \in F_{\text{Qnd}}(S) \) and \( i \in \{1, \ldots, n\} \), where \( x_\infty \) is a symbol. For \( j \in \{1, \ldots, n\} \), we define the \( \bar{f} \)-derivative with respect to \( x_j \) to be the \( f \)-derivative with respect to \( x_j \). The \((f_1, f_2; \phi)\)-derivative is also called the \( \phi \)-augmented \((f_1, f_2)\)-derivative.

We note that the derivatives are also called **twisted derivatives**.

**Example 4.2.** For \( a, b, c \in F_{\text{Qnd}}(S) \) and \( j \in \{1, \ldots, n\} \), we have

\[
\frac{\partial f}{\partial x_j}((a \triangle b) \triangle c) = f_1(a \triangle b, c) \frac{\partial f}{\partial x_j}(a \triangle b) + f_2(a \triangle b, c) \frac{\partial f}{\partial x_j}(c)
\]

\[
= f_1(a \triangle b, c) f_1(a, b) \frac{\partial f}{\partial x_j}(a) + f_1(a \triangle b, c) f_2(a, b) \frac{\partial f}{\partial x_j}(b) + f_2(a \triangle b, c) \frac{\partial f}{\partial x_j}(c)
\]

\[
= f_1(a \triangle c, b \triangle c) f_1(a, c) \frac{\partial f}{\partial x_j}(a) + f_2(a \triangle c, b \triangle c) f_2(a, c) \frac{\partial f}{\partial x_j}(b)
\]

\[
+ f_1(a \triangle c, b \triangle c) f_2(a, c) \frac{\partial f}{\partial x_j}(c) + f_2(a \triangle c, b \triangle c) f_2(b, c) \frac{\partial f}{\partial x_j}(c)
\]

\[
= f_1(a \triangle c, b \triangle c) \frac{\partial f}{\partial x_j}(a \triangle c) + f_2(a \triangle c, b \triangle c) \frac{\partial f}{\partial x_j}(b \triangle c)
\]

\[
= \frac{\partial f}{\partial x_j}((a \triangle c) \triangle (b \triangle c)).
\]

**Lemma 4.3.** For \( a, b \in F_{\text{Qnd}}(S) \) and \( j \in \{1, \ldots, n\} \), we have

\[
\frac{\partial f}{\partial x_j}(a \triangle^{-1} b) = f_1(a \triangle^{-1} b, b)^{-1} \frac{\partial f}{\partial x_j}(a) - f_1(a \triangle^{-1} b, b)f_2(a \triangle^{-1} b, b) \frac{\partial f}{\partial x_j}(b).
\]
For any \( a, b \in F_{\text{Qnd}}(S) \), we have

\[
\frac{\partial f}{\partial x_\infty}(a \triangle^{-1} b) = f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b, b) \frac{\partial f}{\partial x_\infty}(b)
\]

Proof. The first equality follows from

\[
\frac{\partial f}{\partial x_j}(a) = f_1(a \triangle^{-1} b) + f_2(a \triangle^{-1} b) \frac{\partial f}{\partial x_j}(b).
\]

In the same way, we have the second equality.

Remark 4.4. When \( M = R \), we can unite the formulas of \( \frac{\partial f}{\partial x_j} \) for \( j \in \{1, \ldots, n, \infty\} \) as

\[
\frac{\partial f}{\partial x_j}(a \triangle b) = f_1(a \triangle b) \frac{\partial f}{\partial x_j}(a) + f_2(a \triangle b) \frac{\partial f}{\partial x_j}(b) + \phi(a \triangle b) \frac{\partial f}{\partial x_j}(x_\infty),
\]

\[
\frac{\partial f}{\partial x_j}(a \triangle^{-1} b) = f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) \frac{\partial f}{\partial x_j}(b),
\]

\[
\frac{\partial f}{\partial x_j}(x_i) = \delta_{ij}.
\]

Theorem 4.5. (1) For \( j \in \{1, \ldots, n\} \), the \( f \)-derivative \( \frac{\partial f}{\partial x_j} : F_{\text{Qnd}}(S) \to R \) is well-defined.

(2) The \( \tilde{f} \)-derivative \( \frac{\partial \tilde{f}}{\partial x_j} : F_{\text{Qnd}}(S) \to M \) is well-defined.

Proof. We prove (1). We temporarily regard the \( f \)-derivative \( \frac{\partial f}{\partial x_j} \) as the map \( \frac{\partial f}{\partial x_j} : W_{\text{Qnd}}(S) \to R \) defined by using the following equalities inductively:

\[
\frac{\partial f}{\partial x_j}(a \triangle b) = f_1(a \triangle b) \frac{\partial f}{\partial x_j}(a) + f_2(a \triangle b) \frac{\partial f}{\partial x_j}(b),
\]

\[
\frac{\partial f}{\partial x_j}(a \triangle^{-1} b) = f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) - f_1(a \triangle^{-1} b) \frac{\partial f}{\partial x_j}(b),
\]

\[
\frac{\partial f}{\partial x_j}(x_i) = \delta_{ij}
\]

for \( a, b \in W_{\text{Qnd}}(S) \). By Lemma 3.1, the well-definedness of the \( f \)-derivative
follows from the following equalities:

\[ \frac{\partial f}{\partial x_j}(\cdots ((a \triangleleft^1 a) \triangleleft^2 a_1) \cdots) \triangleleft^n a_n) = \frac{\partial f}{\partial x_j}(\cdots (a \triangleleft^1 a_1) \cdots) \triangleleft^n a_n), \tag{2} \]

\[ \frac{\partial f}{\partial x_j}(\cdots (((a \triangleleft^1 b) \triangleleft^{-1} b) \triangleleft^2 a_1) \cdots) \triangleleft^n a_n) = \frac{\partial f}{\partial x_j}(\cdots (a \triangleleft^1 a_1) \cdots) \triangleleft^n a_n), \tag{3} \]

\[ \frac{\partial f}{\partial x_j}(\cdots (((a \triangleleft^1 c) \triangleleft^2 c \triangleleft^{n+1} a_1) \cdots) \triangleleft^n a_n) \]

\[ = \frac{\partial f}{\partial x_j}(\cdots (((a \triangleleft^1 c) \triangleleft^2 (b \triangleleft^c c) \triangleleft^{n+1} a_1) \cdots) \triangleleft^n a_n) \tag{4} \]

for \( a, b, c, a_1, \ldots, a_n \in W_{\triangleleft}(S) \) and \( \varepsilon, \delta, \varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \). For \( a_1, a_2, b_1, b_2 \in W_{\triangleleft}(S) \), if

\[ \frac{\partial f}{\partial x_j}(a_1) = \frac{\partial f}{\partial x_j}(a_2), \quad \frac{\partial f}{\partial x_j}(b_1) = \frac{\partial f}{\partial x_j}(b_2) \]

and \( a_1 = a_2, b_1 = b_2 \) in \( F_{\triangleleft}(S) \), then we have

\[ \frac{\partial f}{\partial x_j}(a_1 \triangleleft b_1) = f_1(a_1, b_1) \frac{\partial f}{\partial x_j}(a_1) + f_2(a_1, b_1) \frac{\partial f}{\partial x_j}(b_1) \]

\[ = f_1(a_2, b_2) \frac{\partial f}{\partial x_j}(a_2) + f_2(a_2, b_2) \frac{\partial f}{\partial x_j}(b_2) = \frac{\partial f}{\partial x_j}(a_2 \triangleleft b_2) \]

and

\[ \frac{\partial f}{\partial x_j}(a_1 \triangleleft^{-1} b_1) \]

\[ = f_1(a_1 \triangleleft^{-1} b_1, b_1) \frac{\partial f}{\partial x_j}(a_1) + f_1(a_1 \triangleleft^{-1} b_1, b_1) \frac{\partial f}{\partial x_j}(b_1) \]

\[ = f_1(a_2 \triangleleft^{-1} b_2, b_2) \frac{\partial f}{\partial x_j}(a_2) + f_1(a_2 \triangleleft^{-1} b_2, b_2) \frac{\partial f}{\partial x_j}(b_2) \]

\[ = \frac{\partial f}{\partial x_j}(a_2 \triangleleft^{-1} b_2). \]

Hence the equalities \( (2) \)–\( (4) \) follow from

\[ \frac{\partial f}{\partial x_j}(a \triangleleft a) = \frac{\partial f}{\partial x_j}(a), \]

\[ \frac{\partial f}{\partial x_j}((a \triangleleft^c b) \triangleleft^{-c} b) = \frac{\partial f}{\partial x_j}(a), \]

\[ \frac{\partial f}{\partial x_j}((a \triangleleft^c b) \triangleleft^c c) = \frac{\partial f}{\partial x_j}((a \triangleleft^c b) \triangleleft^c (b \triangleleft^c c)) \]

for \( a, b, c \in W_{\triangleleft}(S) \) and \( \varepsilon, \delta \in \{\pm 1\} \). It is left to the reader to verify these equalities (see Example 4.2). This completes the proof of (1).

In a similar manner, we can prove (2). \( \square \)

11
5 \textit{f}-twisted Alexander matrices

Let \( R \) be a ring. We denote by \( M(m, n; R) \) the set of \( m \times n \) matrices over \( R \). We say that two matrices \( A_1 \) and \( A_2 \) over \( R \) are equivalent \((A_1 \sim A_2)\) if they are related by a finite sequence of the following transformations:

- \((a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) \leftrightarrow (a_1, \ldots, a_i + a_j r, \ldots, a_j, \ldots, a_n) \quad (r \in R),\)

\[
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_i \\
  \vdots \\
  a_j \\
  \vdots \\
  a_n
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_i + r a_j \\
  \vdots \\
  a_j \\
  \vdots \\
  a_n
\end{pmatrix}
\quad (r \in R),
\]

- \(A \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\).

**Proposition 5.1.** We have the following equivalences:

1. \((a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) \sim (a_1, \ldots, a_i, -a_i, \ldots, a_n)\).

2. \((a_1, \ldots, a_i, \ldots, a_n) \sim (a_1, \ldots, a_i u, \ldots, a_n) \quad (u \in R^\times)\),

3. \(a_i \\
  \vdots \\
  a_j \\
  \vdots \\
  a_n
\sim
a_i \\
  \vdots \\
  a_j \\
  \vdots \\
  a_n
\)

4. \(a_i \\
  \vdots \\
  a_i u \\
  \vdots \\
  a_n
\sim
u a_i \\
  \vdots \\
  a_n
\quad (u \in R^\times)\).

**Proof.** The equivalence (1) follows from

\[
(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) \sim (a_1, \ldots, a_i + a_j, \ldots, a_j, \ldots, a_n)
\sim (a_1, \ldots, a_i + a_j, \ldots, -a_i, \ldots, a_n)
\sim (a_1, \ldots, a_j, \ldots, -a_i, \ldots, a_n).
\]

In the same way, we have the equivalence (3). The equivalence (4) follows from

\[
\begin{pmatrix}
  A_1 \\
  a_i \\
  A_2
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 \\
  a_i \\
  u a_i
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 \\
  a_i u \\
  A_i
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 \\
  a_i u \\
  A_i
\end{pmatrix}.
\]

We have the equivalence (2) as

\[
\begin{pmatrix}
  A_1 & a_i & A_2
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 & a_i & A_2 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 & a_i & A_2 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 & a_i u & A_2 & a_i
\end{pmatrix}
\sim
\begin{pmatrix}
  A_1 & a_i u & A_2 & a_i
\end{pmatrix}.
\]

\(\square\)
Let $R$ be a commutative ring, and let $A \in M(m, n; R)$. A $k$-	extit{minor} of $A$ is the determinant of a $k \times k$ submatrix of $A$. The $d$-	extit{th elementary ideal} $E_d(A)$ of $A$ is the ideal of $R$ generated by all $(n-d)$-minors of $A$ if $n - m \leq d < n$, and

$$E_d(A) = \begin{cases} 0 & \text{if } d < n - m, \\ R & \text{if } n \leq d. \end{cases}$$

Suppose that $R$ is a GCD domain. Then, the $d$-	extit{th Alexander polynomial} $\Delta_d(A)$ of $A$ is the greatest common divisor of all $(n-d)$-minors of $A$ if $n - m \leq d < n$, and

$$\Delta_d(A) = \begin{cases} 0 & \text{if } d < n - m, \\ 1 & \text{if } n \leq d. \end{cases}$$

We remark that $\Delta_d(A)$ coincides with the greatest common divisor of generators of $E_d(A)$ and is determined up to unit multiple. If $A \sim B$, then $E_d(A) = E_d(B)$ and $\Delta_d(A) \doteq \Delta_d(B)$, where the symbol $\doteq$ indicates equality up to a unit factor. See [9] for more details.

\textbf{Remark 5.2.} Let $R$ be a commutative ring. We may regard a matrix in $M(m, n; M(k, k; R))$ as a matrix in $M(km, kn; R)$. We call such matrices flat matrices, and emphasize that equivalent matrices are equivalent as flat matrices. The twisted Alexander invariant is defined through this process.

For a relator $r = (r_1, r_2)$, we define

$$\frac{\partial f}{\partial x_j}(r) := \frac{\partial f}{\partial x_j}(r_1) - \frac{\partial f}{\partial x_j}(r_2).$$

For $f = (f_1, f_2)$ and $\bar{f} = (f_1, f_2; \phi)$, we set

$$f \circ \rho := (f_1^\rho, f_2^\rho), \quad \bar{f} \circ \rho := (f_1^\rho, f_2^\rho; \phi^\rho).$$

\textbf{Definition 5.3.} Let $Q$ be a quandle. Let $R$ be a ring and $M$ a left $R$-module. Let $(f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$, and $\phi : Q \times Q \to M$ an $(f_1, f_2)$-twisted 2-cocycle. Suppose that $M = R$. Let $X = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finitely presented quandle, and $\rho : X \to Q$ a quandle representation. Put $f = (f_1, f_2)$ and $\bar{f} = (f_1, f_2; \phi)$.

- The $f$-	extit{twisted Alexander matrix} of $(X, \rho)$ (with respect to the presentation $\langle x \mid r \rangle$) is

$$A(X, \rho; f_1, f_2) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(r_1) & \cdots & \frac{\partial f_1}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(r_m) & \cdots & \frac{\partial f_m}{\partial x_n}(r_m) \end{pmatrix}.$$

- The $\bar{f}$-	extit{twisted Alexander matrix} of $(X, \rho)$ (with respect to the presentation $\langle x \mid r \rangle$) is

$$A(X, \rho; f_1, f_2; \phi) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(r_1) & \cdots & \frac{\partial f_1}{\partial x_n}(r_1) & \frac{\partial f_1}{\partial x_n}(\phi r_1) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(r_m) & \cdots & \frac{\partial f_m}{\partial x_n}(r_m) & \frac{\partial f_m}{\partial x_n}(\phi r_m) \end{pmatrix}.$$

The $(f_1, f_2; \phi)$-twisted Alexander matrix is also called the $\phi$-	extit{augmented} $(f_1, f_2)$-twisted Alexander matrix.
The following theorem shows that the equivalence class of \((\phi\text{-augmented})\) \((f_1, f_2)\)-twisted Alexander matrix does not depend on the choice of a presentation \((x \mid r)\).

**Theorem 5.4.** Let \(X = \langle x \mid r \rangle\) and \(X' = \langle x' \mid r' \rangle\) be finitely presented quandles, and let \(\rho : X \to Q\) and \(\rho' : X' \to Q\) be quandle representations. If \((X, \rho) \cong (X', \rho')\), then we have
\[
A(X, \rho; f_1, f_2) \sim A(X', \rho'; f_1, f_2), \\
A(X, \rho; f_1, f_2, \phi) \sim A(X', \rho'; f_1, f_2, \phi).
\]

Furthermore, we have
\[
E_d(A(X, \rho; f_1, f_2)) = E_d(A(X', \rho'; f_1, f_2)), \\
E_d(A(X, \rho; f_1, f_2, \phi)) = E_d(A(X', \rho'; f_1, f_2, \phi))
\]
if \(R\) is a commutative ring, and we have
\[
\Delta_1(A(X, \rho; f_1, f_2)) = \Delta_1(A(X', \rho'; f_1, f_2)), \\
\Delta_1(A(X, \rho; f_1, f_2, \phi)) = \Delta_1(A(X', \rho'; f_1, f_2, \phi))
\]
if \(R\) is a GCD domain.

We postpone the proof of Theorem 5.4 to Section 10.

**Remark 5.5.** A quandle is not always finitely presented. Theorem 5.4 holds for finitely generated presentations with the following corrections. Two presentations \((S \mid R)\) and \((S' \mid R')\) represent the same quandle if and only if they can be transformed into each other by a finite sequence of the following transformations:

- \(\langle S \mid R \rangle \leftrightarrow \langle S \mid R \cup R' \rangle \) \((R' \subset N_{\text{Qnd}}(R))\),
- \(\langle S \mid R \rangle \leftrightarrow \langle S \cup \{y\} \mid R \cup \{(y, w_y)\} \) \((y \notin F_{\text{Qnd}}(S), w_y \in F_{\text{Qnd}}(S))\).

We allow \(A(X, \rho; f_1, f_2)\) and \(A(X, \rho; f_1, f_2, \phi)\) to be row-infinite matrices. We define an equivalence relation \(\sim\) on the set of row-infinite matrices by the following transformations:

- \((a_{ij})_{i \in I, j \in J} \leftrightarrow (a_{ij} + \delta_{ij} a_{it} r_t)_{i \in I, j \in J}\) \((s \neq t, r_t \in R)\),
- \((a_{ij})_{i \in I, j \in J} \leftrightarrow (a_{ij} + \delta_{i} \sum_{k \in I} r_{ik} a_{kj})_{i \in I, j \in J}\) \((I, r_{ik} \in R)\),
- \((a_{ij})_{i \in I, j \in J} \leftrightarrow (\delta_{i} a_{ij})_{i \in I, j \in J}\),
- \((a_{ij})_{i \in I, j \in J} \leftrightarrow (\delta_{i} a_{ij} + \delta_{j} a_{ij} + \delta_{i,j} a_{ij})_{i \in I, j \in J}\).

where \(\delta_{s \in S} = \begin{cases} 1 & (s \in S) \\ 0 & (s \notin S). \end{cases}\) We leave the details of the proof to the reader.

Let \(Q\) be a quandle, and \(R\) a ring. Let \(L\) be an oriented link, and \(Q(L)\) the fundamental quandle of \(L\). Let \(\rho : Q(L) \to Q\) be a quandle representation. Let \((f_1, f_2)\) be an Alexander pair of maps \(f_1, f_2 : Q \times Q \to R\), and \(\phi : Q \times Q \to R\)
an \((f_1, f_2)\)-twisted 2-cocycle. We define the \(d\)th elementary ideals of \((L, \rho)\) with respect to \((f_1, f_2)\) and \((f_1, f_2, \phi)\) by
\[
E_d(L, \rho; f_1, f_2) := E_d(A(Q(L), \rho; f_1, f_2)), \\
E_d(L, \rho; f_1, f_2, \phi) := E_d(A(Q(L), \rho; f_1, f_2, \phi)),
\]
respectively. We define the \(d\)th Alexander invariants of \((L, \rho)\) with respect to \((f_1, f_2)\) and \((f_1, f_2, \phi)\) by
\[
\Delta_d(L, \rho; f_1, f_2) := \Delta_d(A(Q(L), \rho; f_1, f_2)), \\
\Delta_d(L, \rho; f_1, f_2, \phi) := \Delta_d(A(Q(L), \rho; f_1, f_2, \phi)),
\]
respectively.

## 6 The (twisted) Alexander matrices

We recall the definition of the (twisted) Alexander matrix and see that it can be realized as an \((f_1, f_2)\)-twisted Alexander matrix for some Alexander pair \((f_1, f_2)\).

Let \(L\) be an oriented link, and \(D\) a diagram of \(L\). Let
\[
Q(L) = \langle x_1, \ldots, x_n \mid u_1 \triangleleft v_1 = w_1, \ldots, u_m \triangleleft v_m = w_m \rangle,
\]
\[
G(L) = \langle x_1, \ldots, x_n \mid v_1^{-1}u_1v_1w_1^{-1}, \ldots, v_m^{-1}u_mv_mw_m^{-1} \rangle
\]
be the Wirtinger presentations of the fundamental quandle \(Q(L)\) and the fundamental group \(G(L)\) with respect to \(D\), respectively. See Example 3.2. Let \(R\) be a commutative ring. Set \(G := GL(k; R)\). Let \(\rho : G(L) \to G\) be a group representation. The induced quandle representation of \(\rho\) is a quandle homomorphism from \(Q(L)\) to \(\text{Conj}G\) which sends \(x_i\) to \(\rho(x_i)\), and we denote it by the same symbol \(\rho : Q(L) \to \text{Conj}G\). Let \(F(x)\) be the free group on the generating set \(\{x_1, \ldots, x_n\}\). The Fox derivative [8] with respect to \(x_i\) is the \(R\)-homomorphism
\[
\frac{\partial_{\text{Grp}}}{\partial x_i} : R[F(x)] \to R[F(x)]
\]
satisfying
\[
\frac{\partial_{\text{Grp}}}{\partial x_i} (pq) = \frac{\partial_{\text{Grp}}}{\partial x_i} (p) + p \frac{\partial_{\text{Grp}}}{\partial x_i} (q) \quad \text{and} \quad \frac{\partial_{\text{Grp}}}{\partial x_i} (x_i) = \delta_{ij}.
\]
Let \(\text{pr} : F(x) \to G(L)\) be the canonical projection, and \(\alpha : G(L) \to \langle t \rangle\) the abelianization sending every meridian to \(t^{-1}\). We denote the linear extensions of \(\text{pr}, \alpha\) and \(\rho\) by the same symbols \(\text{pr} : R[F(x)] \to R[G(L)]\), \(\alpha : R[G(L)] \to R[t^{\pm 1}]\), and \(\rho : R[G(L)] \to M(k, k; R)\), respectively. The Alexander matrix of \(L\) is the \(m \times n\) matrix
\[
\left( (\alpha \circ \text{pr}) \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i^{-1}u_iw_i^{-1}) \right) \right),
\]
and the twisted Alexander matrix of \((L, \rho)\) is
\[
\left( (\rho \circ \alpha \circ \text{pr}) \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i^{-1}u_iw_i^{-1}) \right) \right),
\]
which we regard as a \( km \times km \) matrix over \( R[t^\pm 1] \).

Under the following proposition, the (twisted) Alexander invariants can be obtained in our framework.

**Proposition 6.1.** (1) Let \((f_1, f_2)\) be the Alexander pair in Example 2.3, that is, \(f_1(a, b) = t\) and \(f_2(a, b) = 1 - t\). Let \(\rho : Q(L) \to Q\) be an arbitrary quandle representation. Then \(A(Q(L), \rho; f_1, f_2)\) coincides with the Alexander matrix of \(L\).

(2) Set \(G := GL(k; R)\) and \(Q := \text{Conj}G\). Let \((f_1, f_2)\) be the Alexander pair in Example 2.4 (3) with \(n = 1\), that is, \(f_1(a, b) = tb^{-1}\) and \(f_2(a, b) = b^{-1}a - tb^{-1}\). Let \(\rho : G(L) \to G\) be a group representation and \(\rho : Q(L) \to Q\) its induced quandle representation. Then \(A(Q(L), \rho; f_1, f_2)\) coincides with the twisted Alexander matrix of \((L, \rho)\).

**Proof.** We prove (2). Put \(f = (f_1, f_2)\). From

\[
\text{pr} \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i^{-1} u_i v_i w_i^{-1}) \right)
\]

\[
= \text{pr} \left( -v_i^{-1} \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) + v_i^{-1} u_i \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - v_i^{-1} u_i v_i w_i^{-1} \frac{\partial_{\text{Grp}}}{\partial x_j} (w_i) \right)
\]

\[
= v_i^{-1} \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) + (v_i^{-1} u_i - v_i^{-1}) \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (w_i),
\]

we have

\[
((\rho \otimes \alpha) \circ \text{pr}) \left( \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i^{-1} u_i v_i w_i^{-1}) \right)
\]

\[
= t \rho(v_i)^{-1} \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) + (\rho(v_i)^{-1} \rho(u_i) - t \rho(v_i)^{-1}) \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (w_i)
\]

\[
= f_1'(u_i, v_i) \frac{\partial_{\text{Grp}}}{\partial x_j} (u_i) + f_2'(u_i, v_i) \frac{\partial_{\text{Grp}}}{\partial x_j} (v_i) - \frac{\partial_{\text{Grp}}}{\partial x_j} (w_i)
\]

\[
= \frac{\partial_{f^2}}{\partial x_j} (u_i \cdot v_i = w_i).
\]

Therefore, \(A(Q(L), \rho; f_1, f_2)\) coincides with the twisted Alexander matrix of \((L, \rho)\).

In the same way, we can prove (1). \(\square\)

## 7 Quandle cocycle invariants

We recall the definition of a quandle cocycle invariant introduced in [3] and see how we find a quandle cocycle invariant in our framework.

Let \(L\) be an oriented link, and let \(D\) be a diagram of \(L\) with \(n\) crossings \(c_1, \ldots, c_n\). We denote by \(\text{sgn}(c_i)\) the sign of a crossing \(c_i\). A normal orientation is often used to represent an orientation of a link on its diagram. The normal orientation is obtained by rotating the usual orientation counterclockwise by \(\pi/2\) on the diagram. We denote by \(u_i, w_i\) and \(v_i\) the under-arcs and over-arc, respectively, of a crossing \(c_i\) such that the normal orientation of \(v_i\) points from \(u_i\) to \(w_i\) (see the left picture of Figure 2). Let \(Q\) be a quandle and \(\phi\) a quandle
2-cocycle, which is a (1,0)-twisted 2-cocycle. The quandle cocycle invariant \( \Phi_\phi(L) \) of \( L \) is the multiset
\[
\Phi_\phi(L) = \{ \Phi_\phi(L, \rho) \mid \rho \in \text{Hom}(Q(L), Q) \},
\]
where
\[
\Phi_\phi(L, \rho) := \sum_{i=1}^{n} \text{sgn}(c_i) \phi(\rho(u_i), \rho(v_i)).
\]
We call \( \Phi_\phi(L, \rho) \) the quandle cocycle invariant of \( (L; \rho) \).

**Proposition 7.1.** Let \( Q \) be a quandle, and \( R \) a commutative ring. Let \( \phi : Q \times Q \to R \) be a quandle 2-cocycle. Let \( K \) be an oriented knot, and \( \rho : Q(K) \to Q \) a quandle representation. Then we have
\[
\Phi_\phi(K, \rho) \equiv \Delta_1(K, \rho; 1, 0, \phi).
\]

**Proof.** Let \( D \) be a diagram of \( L \) with \( n \) crossings \( c_1, \ldots, c_n \) and \( n \) arcs \( x_1, \ldots, x_n \). We label the crossings and arcs so that \( x_i \) starts from \( c_i \) and the terminal point of \( x_i \) is \( c_{i+1} \) for any \( i = 1, \ldots, n \), where \( c_{n+1} \) stands for \( c_1 \). See the right picture of Figure 2. Let \( Q(L) = \langle x_1, \ldots, x_n \mid u_1 \triangle v_1 = w_1, \ldots, u_n \triangle v_n = w_n \rangle \) be the Wirtinger presentation of the fundamental quandle \( Q(L) \). Put \( \tilde{f} = (1, 0; \phi) \). Since we have
\[
\frac{\partial f_{\phi \rho}}{\partial x_j}(u_i \triangle v_i - w_i)
= f_1'(u_i, v_i) \frac{\partial f_{\phi \rho}}{\partial x_j}(u_i) + f_2'(u_i, v_i) \frac{\partial f_{\phi \rho}}{\partial x_j}(v_i) + \phi'(u_i, v_i) \frac{\partial f_{\phi \rho}}{\partial x_j}(x_\infty) - \frac{\partial f_{\phi \rho}}{\partial x_j}(w_i)
= \frac{\partial f_{\phi \rho}}{\partial x_j}(u_i) - \frac{\partial f_{\phi \rho}}{\partial x_j}(w_i) + \phi'(u_i, v_i) \frac{\partial f_{\phi \rho}}{\partial x_j}(x_\infty),
\]
the matrix \( A(D, \rho; 1, 0, \phi) \) is
\[
\begin{pmatrix}
-\text{sgn}(c_1) & \text{sgn}(c_2) & \text{sgn}(c_3) & \cdots & \text{sgn}(c_n) \\
\text{sgn}(c_2) & -\text{sgn}(c_2) & \text{sgn}(c_3) & \cdots & \text{sgn}(c_n) \\
\text{sgn}(c_3) & \text{sgn}(c_3) & -\text{sgn}(c_3) & \cdots & \text{sgn}(c_n) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\text{sgn}(c_n) & \text{sgn}(c_n) & \text{sgn}(c_n) & \cdots & -\text{sgn}(c_n)
\end{pmatrix}
\begin{pmatrix}
\phi'(u_1, v_1) \\
\phi'(u_2, v_2) \\
\phi'(u_3, v_3) \\
\vdots \\
\phi'(u_n, v_n)
\end{pmatrix}.
\]

17
where we note that \( x_i \) is \( u_{i+1} \) or \( w_{i+1} \). We then have

\[
A(D, \rho; 1, 0, \phi) \sim \begin{pmatrix}
-1 & 1 & \text{sgn}(c_1) \phi^\rho(u_1, v_1) \\
1 & -1 & \text{sgn}(c_2) \phi^\rho(u_2, v_2) \\
\vdots & \vdots & \vdots \\
1 & -1 & \text{sgn}(c_{n-1}) \phi^\rho(u_{n-1}, v_{n-1}) \\
1 & -1 & \text{sgn}(c_n) \phi^\rho(u_n, v_n)
\end{pmatrix}
\]

\[
\sim \begin{pmatrix}
-1 & 1 & \text{sgn}(c_1) \phi^\rho(u_1, v_1) \\
1 & -1 & \text{sgn}(c_2) \phi^\rho(u_2, v_2) \\
\vdots & \vdots & \vdots \\
1 & -1 & \text{sgn}(c_{n-1}) \phi^\rho(u_{n-1}, v_{n-1}) \\
0 & 0 & \Phi_\phi(K, \rho)
\end{pmatrix}
\]

\[
\sim \begin{pmatrix}
1 & 0 \\
1 & 0 \\
\vdots & \vdots \\
1 & 0 \\
0 & 0 & \Phi_\phi(K, \rho)
\end{pmatrix}
\]

It follows that \( \Phi_\phi(K, \rho) \cong \Delta_1(K, \rho; 1, 0, \phi) \). \( \square \)

### 8 The \( n \)-move equivalence relation

In this section, we introduce a quandle \( IQ_n(L) \) which is invariant under \( n \)-moves and calculate our invariants for the quandle \( IQ_n(L) \). In particular, we demonstrate that our invariant detects 5-move equivalence classes of some knots.

An \( n \)-move is a local move on links as illustrated in Figure 3. Two links are \( n \)-move equivalent if they are related by a finite sequence of \( n \)-moves. We denote by \# \text{Hom}(Q(L), R_n) \) the number of quandle representations of \( Q(L) \) to \( R_n \). It is well-known that \# \text{Hom}(Q(L), R_n) \) is invariant under \( n \)-moves. A quandle representation of \( Q(L) \) to \( R_n \) can be regarded as a quandle representation of \( IQ(L) \) to \( R_n \) by the bijection from \text{Hom}(IQ(L), R_n) \) to \text{Hom}(Q(L), R_n) \) which sends \( \rho \) to \( \rho \circ \text{pr} \), where \( \text{pr} : Q(L) \rightarrow IQ(L) \) is the canonical projection.

Let \( X \) be a quandle. For \( a, b \in X \) and \( n \in \mathbb{Z}_{\geq 0} \), we define \( w(a, b; n) \in X \) by

\[
w(a, b; 0) = b \quad \text{and} \quad w(a, b; i) = w(b, a; i - 1) \ast b.
\]

We set \( R(X, n) := \{ (w(a, b; n), b) \mid a, b \in X \} \). For an unoriented link \( L \), we define \( IQ_n(L) := IQ(L)/\sim_{\text{Qua}}(R(IQ(L), n)) \).

**Proposition 8.1.** If two unoriented links \( L_1 \) and \( L_2 \) are \( n \)-move equivalent, then we have \( IQ_n(L_1) \cong IQ_n(L_2) \).
We note that if two knots are not $5$-move equivalent. In this example, Example 8.2.

Let $K_1 := 4_1 \# 4_1$ and $K_2 := 10_{155}$. We demonstrate that these two knots are not $5$-move equivalent. In this example, $a \bowtie b_1 \cdots b_n$ stands for $(\cdots (a \bowtie b_1) \cdots) \bowtie b_n$. We show that $IQ_5(K_1) \ncong IQ_5(K_2)$, where

$$IQ_5(K_1) \cong \left\{ x, y, z \mid y \bowtie xyzxyz = z \bowtie xyzxyz, \\
y \bowtie xyzxyz = x \bowtie xyzxyz, \\
z \bowtie xyzxyz = z \bowtie xyzxyz, \\
um \bowtie v = u \bowtie v^{-1} \ (u, v \in \{x, y, z\}), \\
w(u, v; 5) = v \ (u, v \in F_{Qnd}(\{x, y, z\}) \right\}$$

$$IQ_5(K_2) \cong \left\{ x, y, z \mid y \bowtie xyzxyz = x \bowtie xyzxyz, \\
y \bowtie xyzxyz = y \bowtie xyzxyz, \\
z \bowtie xyzxyz = z \bowtie xyzxyz, \\
um \bowtie v = u \bowtie v^{-1} \ (u, v \in \{x, y, z\}), \\
w(u, v; 5) = v \ (u, v \in F_{Qnd}(\{x, y, z\}) \right\}$$

We note that $\# Hom(IQ(K_i), R_3) = \# Hom(Q(K_i), R_3) = 125$ for $i = 1, 2$. We define $f_1, f_2 : R_5 \times R_5 \to \Z_5[t^\pm 1]/(t^2 + 1)$ by $f_i(a, b) = -1$ and $f_2(a, b) = \alpha_{a-b}$, where

$$\alpha_x = \begin{cases} 
2 & (x = 0), \\
t & (x = \pm 1), \\
-t - 1 & (x = \pm 2)
\end{cases}$$
for \( x \in \mathbb{Z}_5 \). Then \( f = (f_1, f_2) \) is an Alexander pair.

For any knot \( K \) and any quandle representation \( \rho : IQ_5(K) \to R_5 \), by direct calculation, we have

\[
\frac{\partial f_\rho}{\partial w}(u \triangle v, v) = \frac{\partial f_\rho}{\partial w}(u \triangle v^{-1}, v),
\]

\[
\frac{\partial f_\rho}{\partial w}(w(u, v; 5)) = \frac{\partial f_\rho}{\partial w}(u \triangle v v u v u v) = \frac{\partial f_\rho}{\partial w}(v).
\]

Let \( \rho : IQ_5(K_1) \to R_5 \) be a quandle representation. Putting \( a := \rho(x) \), \( b := \rho(y) \) and \( c := \rho(z) \), we have

\[
\frac{\partial f_\rho}{\partial w}(y \triangle x y x) - \frac{\partial f_\rho}{\partial w}(z \triangle y x y x y y z)
= (\alpha_{a-b} + \alpha_{a-2b} - \alpha_{a-4b+2c} - \alpha_{a-2b+c}) \frac{\partial f_\rho}{\partial w}(x)
+ (\alpha_{4a-2b-2c} + \alpha_{a-2c} + \alpha_{a+2b-4c} + \alpha_{b-c} + \alpha_{a-b}) \frac{\partial f_\rho}{\partial w}(y)
+ (-\alpha_{a-2b+c} - \alpha_{a-4b+2c} - 1) \frac{\partial f_\rho}{\partial w}(z)
= 0.
\]

Then we have \( E_2(A(IQ_5(K_1), \rho; f_1, f_2)) = (0) \) for any quandle representation \( \rho : IQ_5(K_1) \to R_5 \), since we have

\[
A(IQ_5(K_1), \rho; f_1, f_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim (0 \ 0 \ 0).
\]

In a similar manner, we have \( E_2(A(IQ_5(K_2), \rho; f_1, f_2)) = (t + 3) \) from

\[
A(IQ_5(K_2), \rho; f_1, f_2) = \begin{pmatrix} -3t + 1 & 3t - 1 & 0 \\ -t + 2 & t - 2 & 0 \\ t - 2 & -t + 2 & 0 \end{pmatrix} \sim (t - 2 \ 0 \ 0)
\]

for the quandle representation \( \rho : IQ_5(K_2) \to R_5 \) defined by \( \rho(x) = \rho(y) = 0 \) and \( \rho(z) = 1 \). Therefore, the knots \( K_1 \) and \( K_2 \) are not 5-move equivalent.

We note that Jones polynomials with \( t = e^{\pi i/5} \) is invariant under 5-moves, refer to [21]. We can also see that the knots \( 4_1 \# 4_1 \) and \( 10_{155} \) are not 5-move equivalent by using Jones polynomials.

## 9 Cohomologous (augmented) Alexander pairs

In [2], the notion of cohomologous was introduced for dynamical cocycles. In this section, we introduce the notion for (augmented) Alexander pairs and show that cohomologous (augmented) Alexander pairs give the same invariant.

**Definition 9.1.** (1) Two Alexander pairs \((f_1, f_2)\) and \((g_1, g_2)\) are cohomolo-
gous if there exists a map \( h : Q \to R^\times \) satisfying the following conditions:

- For any \( a, b \in Q \), \( h(a \triangle b) f_1(a, b) = g_1(a, b) h(a) \).
• For any \( a, b \in Q \), \( h(a \triangleleft b)f_2(a, b) = g_2(a, b)h(b) \).

We then write \((f_1, f_2) \sim_h (g_1, g_2)\) to specify \( h \).

(2) Two augmented Alexander pairs \((f_1, f_2; \phi)\) and \((g_1, g_2; \psi)\) are cohomologous if there exist maps \( h : Q \to R^* \) and \( \eta : Q \to M \) satisfying the following conditions:

• \((f_1, f_2) \sim_h (g_1, g_2)\)
• For any \( a, b \in Q \),
  \[ h(a \triangleleft b)\phi(a, b) + \eta(a \triangleleft b) = g_1(a, b)\eta(a) + g_2(a, b)\eta(b) + \psi(a, b). \]

To specify \( h \) and \( \eta \), we then write \((f_1, f_2; \phi) \sim_{h, \eta} (g_1, g_2; \psi)\).

This definition can be understood with extensions of a quandle as shown in the following proposition.

**Proposition 9.2.** Let \((Q, \triangleleft)\) be a quandle. Let \( R \) be a ring and \( M \) a left \( R \)-module.

(1) If two Alexander pairs \((f_1, f_2)\) and \((g_1, g_2)\) are cohomologous, then \((Q \times M, \star)\) and \((Q \times M, \star)\) are isomorphic, where the binary operations \( \star \) and \( \ast \) are defined by

\[
\begin{align*}
(a, x) \ast (b, y) &= (a \triangleleft b, f_1(a, b)x + f_2(a, b)y), \\
(a, x) \star (b, y) &= (a \triangleleft b, g_1(a, b)x + g_2(a, b)y).
\end{align*}
\]

(2) If two augmented Alexander pairs \((f_1, f_2; \phi)\) and \((g_1, g_2; \psi)\) are cohomologous, then \((Q \times M, \star)\) and \((Q \times M, \ast)\) are isomorphic, where the binary operations \( \star \) and \( \ast \) are defined by

\[
\begin{align*}
(a, x) \ast (b, y) &= (a \triangleleft b, f_1(a, b)x + f_2(a, b)y + \phi(a, b)), \\
(a, x) \star (b, y) &= (a \triangleleft b, g_1(a, b)x + g_2(a, b)y + \psi(a, b)).
\end{align*}
\]

**Proof.** (1) If \((f_1, f_2) \sim_h (g_1, g_2)\), then \( \varphi : (Q \times M, \ast) \to (Q \times M, \star) \) defined by \( \varphi(a, x) = (a, h(a)x) \) is an quandle isomorphism.

(2) If \((f_1, f_2; \phi) \sim_{h, \eta} (g_1, g_2; \psi)\), then \( \varphi : (Q \times M, \star) \to (Q \times M, \ast) \) defined by \( \varphi(a, x) = (a, h(a)x + \eta(a)) \) is an quandle isomorphism. \( \square \)

We denote by \( \text{diag}(a_1, \ldots, a_n) \) the diagonal matrix whose diagonal entries are \( a_1, \ldots, a_n \).

**Theorem 9.3.** Let \( Q \) be a quandle. Let \( R \) be a ring and \( M \) a left \( R \)-module. Let \((f_1, f_2)\) and \((g_1, g_2)\) be Alexander pairs of maps \( f_1, f_2, g_1, g_2 : Q \times Q \to R \), and \( \phi, \psi : Q \times Q \to M \) an \((f_1, f_2)\)-twisted 2-cocycle and \((g_1, g_2)\)-twisted 2-cocycle, respectively. Suppose that \( M = R \). Let \( X = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle \) be a finitely presented quandle, and \( \rho : X \to Q \) a quandle representation.

(1) If \((f_1, f_2)\) and \((g_1, g_2)\) are cohomologous, then we have

\[ A(X, \rho; f_1, f_2) \sim A(X, \rho; g_1, g_2). \]
(2) If \((f_1, f_2; \phi)\) and \((g_1, g_2; \psi)\) are cohomologous, then we have
\[ A(X, \rho; f_1, f_2, \phi) \sim A(X, \rho; g_1, g_2, \psi). \]

Proof. (1) Suppose \((f_1, f_2) \sim_h (g_1, g_2)\). We assume that
\[ X = \langle x_1, \ldots, x_n | u_1 \triangleleft v_1 = w_1, \ldots, u_m \triangleleft v_m = w_m \rangle \]
for some \(u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_m \in \{ x_1, \ldots, x_n \}\), where we note that any finitely presented quandle can be presented in this form. Since we have
\[
\begin{align*}
& h(\rho(u_i)) \left( f_1^\rho(u_i, v_i) \frac{\partial f_2^\rho}{\partial x_j}(u_i) + f_2^\rho(u_i, v_i) \frac{\partial f_2^\rho}{\partial x_j}(v_i) - \frac{\partial f_2^\rho}{\partial x_j}(w_i) \right) \\
= & g_1^\rho(u_i, v_i) h(\rho(u_i)) \frac{\partial f_1^\rho}{\partial x_j}(u_i) + g_2^\rho(u_i, v_i) h(\rho(v_i)) \frac{\partial f_2^\rho}{\partial x_j}(v_i) - h(\rho(w_i)) \frac{\partial f_2^\rho}{\partial x_j}(w_i) \\
= & \left( g_1^\rho(u_i, v_i) \frac{\partial f_1^\rho}{\partial x_j}(u_i) + g_2^\rho(u_i, v_i) \frac{\partial f_2^\rho}{\partial x_j}(v_i) - \frac{\partial f_2^\rho}{\partial x_j}(w_i) \right) h(\rho(x_j)),
\end{align*}
\]
we have
\[
\text{diag}(h(\rho(w_1)), \ldots, h(\rho(w_m))) A(X, \rho; f_1, f_2) = A(X, \rho; g_1, g_2) \text{diag}(h(\rho(x_1)), \ldots, h(\rho(x_n))),
\]
which implies \(A(X, \rho; f_1, f_2) \sim A(X, \rho; g_1, g_2)\).

(2) Suppose \((f_1, f_2; \psi) \sim_{h, \eta} (g_1, g_2; \psi)\). In a similar manner as (1), we have
\[
\text{diag}(h(\rho(w_1)), \ldots, h(\rho(w_m))) A(X, \rho; f_1, f_2, \phi) = A(X, \rho; g_1, g_2, \psi)
\]
which implies \(A(X, \rho; f_1, f_2, \phi) \sim A(X, \rho; g_1, g_2, \psi)\).

Let \((f_1, f_2)\) be an Alexander pair and \(\phi\) an \((f_1, f_2)\)-twisted 2-cocycle. Fix \(a \in Q\). We define \(f_i \triangleleft a\) and \(\phi \triangleleft a\) by
\[
(f_i \triangleleft a)(x, y) = f_i(x \triangleleft a, y \triangleleft a), \quad (\phi \triangleleft a)(x, y) = \phi(x \triangleleft a, y \triangleleft a)
\]
for \(i = 1, 2\). Then, \((f_1 \triangleleft a, f_2 \triangleleft a)\) is an Alexander pair, and \(\phi \triangleleft a\) is an \((f_1 \triangleleft a, f_2 \triangleleft a)\)-twisted 2-cocycle. Defining \(h(x) := f_1(x, a)\) and \(\eta(x) := \phi(x, a)\), we have
\[
(f_1, f_2) \sim_h (f_1 \triangleleft a, f_2 \triangleleft a), \quad (f_1, f_2; \phi) \sim_{h, \eta} (f_1 \triangleleft a, f_2 \triangleleft a; \phi \triangleleft a).
\]
For \(\rho : X \to Q\), we define \(\rho \triangleleft a : X \to Q\) by \((\rho \triangleleft a)(x) = \rho(x) \triangleleft a\). Then we have
\[
A(X, \rho \triangleleft a; f_1, f_2) = A(X, \rho; f_1, f_2) = A(X, \rho; f_1, f_2, \phi) = A(X, \rho; f_1, f_2, \phi) \sim A(X, \rho \triangleleft a; f_1, f_2, \phi).
\]
We then have the following corollary from Theorem 9.3.

**Corollary 9.4.** We have the equivalences
\[
\begin{align*}
& A(X, \rho; f_1, f_2) \sim A(X, \rho \triangleleft a; f_1, f_2), \\
& A(X, \rho; f_1, f_2, \phi) \sim A(X, \rho \triangleleft a; f_1, f_2, \phi).
\end{align*}
\]
10 Proof of Theorem 5.4

We show \( A(X, \rho; f_1, f_2) \sim A(X', \rho'; f_1, f_2) \). It is sufficient to show that the equivalence class of an \((f_1, f_2)\)-twisted Alexander matrix is invariant under the transformations (T1-1)-(T1-5) and (T2). We set \( A := A(\langle x \mid r \rangle, \rho; f_1, f_2) \) and \( A' := A(\langle x' \mid r' \rangle, \rho; f_1, f_2) \). We denote by \( a_{ij} \) the \( i \)-th row vector of \( A \) and denote by \( a_{ij} \) the \((i, j)\)-entry of \( A \).

For the presentations

\[
\langle x \mid r \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle, \\
\langle x' \mid r' \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, x = x \rangle \quad (x \in F_{Qd}(a)),
\]
we have

\[
A = \begin{pmatrix}
\frac{\partial f_{op}}{\partial x_1}(r_1) & \cdots & \frac{\partial f_{op}}{\partial x_n}(r_1) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_{op}}{\partial x_1}(r_m) & \cdots & \frac{\partial f_{op}}{\partial x_n}(r_m)
\end{pmatrix} \sim \begin{pmatrix} A \end{pmatrix} = A',
\]

since we have

\[
\frac{\partial f_{op}}{\partial x_j}(x) = \frac{\partial f_{op}}{\partial x_j}(x) = 0.
\]

For the presentations

\[
\langle x \mid r \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a = b \rangle, \\
\langle x' \mid r' \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a = b, b = a \rangle,
\]
we have

\[
A = \begin{pmatrix} a_1 \\
\vdots \\
a_m+1 \\
0 \end{pmatrix} \sim \begin{pmatrix} a_1 \\
\vdots \\
a_{m+1}+1 \\
a_{m+1} \end{pmatrix} = A',
\]

since we have

\[
\frac{\partial f_{op}}{\partial x_j}(b) = -\frac{\partial f_{op}}{\partial x_j}(a) = -\left( \frac{\partial f_{op}}{\partial x_j}(a) - \frac{\partial f_{op}}{\partial x_j}(b) \right) = -a_{m+1,j}.
\]

For the presentations

\[
\langle x \mid r \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a = b, b = c \rangle, \\
\langle x' \mid r' \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a = b, b = c, a = c \rangle,
\]
we have

\[
A = \begin{pmatrix} a_1 \\
\vdots \\
a_{m+1}+2 \\
a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 \\
\vdots \\
a_{m+1}+2 \\
a_{m+2} + a_{m+2} \end{pmatrix} = A',
\]

since we have

\[
\frac{\partial f_{op}}{\partial x_j}(a) = \frac{\partial f_{op}}{\partial x_j}(c) = \frac{\partial f_{op}}{\partial x_j}(a) - \frac{\partial f_{op}}{\partial x_j}(b) + \frac{\partial f_{op}}{\partial x_j}(b) - \frac{\partial f_{op}}{\partial x_j}(c) = a_{m+1,j} + a_{m+2,j}.
\]
For the presentations
\[ \langle x \mid r \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2 \rangle, \]
\[ \langle x' \mid r' \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2, a_1 \triangleleft b_1 = a_2 \triangleleft b_2 \rangle, \]
we have
\[
A = \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & \ddots & \vdots \\ a_{m+1} & \cdots & a_{m+1} \\ a_{m+2} & \cdots & a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & \ddots & \vdots \\ a_{m+1} & \cdots & a_{m+1} \\ a_{m+2} & \cdots & a_{m+2} \end{pmatrix} = A',
\]
since we have
\[
\frac{\partial f_{\varphi}}{\partial x_j}(a_1 \triangleleft b_1) - \frac{\partial f_{\varphi}}{\partial x_j}(a_2 \triangleleft b_2)
\]
\[= f''_1(a_1, b_1) \frac{\partial^2 f_{\varphi}}{\partial^2 x_j}(a_1) + f''_2(a_1, b_1) \frac{\partial^2 f_{\varphi}}{\partial^2 x_j}(b_1) + f''_2(a_2, b_2) \frac{\partial^2 f_{\varphi}}{\partial^2 x_j}(a_2) - f''_2(a_2, b_2) \frac{\partial^2 f_{\varphi}}{\partial^2 x_j}(b_2)
\]
\[= f''_1(a_1, b_1) + f''_2(a_1, b_1) a_{m+1} \sim f''_1(a_1, b_1) a_{m+2}.
\]
In the same way, for the presentations
\[ \langle x \mid r \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2 \rangle, \]
\[ \langle x' \mid r' \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2, a_1 \triangleleft^{-1} b_1 = a_2 \triangleleft^{-1} b_2 \rangle, \]
we have
\[
A = \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & \ddots & \vdots \\ a_{m+1} & \cdots & a_{m+1} \\ a_{m+2} & \cdots & a_{m+2} \end{pmatrix} \sim \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & \ddots & \vdots \\ a_{m+1} & \cdots & a_{m+1} \\ a_{m+2} & \cdots & a_{m+2} \end{pmatrix} = A',
\]
where \(a_{m+3} = f_1(a_1 \triangleleft^{-1} b_1, b_1)^{-1} a_{m+1} - f_1(a_1 \triangleleft^{-1} b_1, b_1)^{-1} f_2(a_1 \triangleleft^{-1} b_1, b_1) a_{m+2}.
\]
For the presentations
\[ \langle x \mid r \rangle = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, \rangle, \]
\[ \langle x' \mid r' \rangle = \langle x_1, \ldots, x_n, y \mid r_1, \ldots, r_m, y = w \rangle \quad (y \notin F_{\text{Qnd}}(x), \ w \in F_{\text{Qnd}}(x)), \]
we have
\[
A = \begin{pmatrix} \frac{\partial f_{\varphi}}{\partial x_1}(r_1) & \cdots & \frac{\partial f_{\varphi}}{\partial x_1}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{\varphi}}{\partial x_1}(r_m) & \cdots & \frac{\partial f_{\varphi}}{\partial x_1}(r_m) \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = A',
\]
where \(-\frac{\partial f_{\varphi}}{\partial x}(w)\) is the row vector \(-\frac{\partial f_{\varphi}}{\partial x}(w), \ldots, -\frac{\partial f_{\varphi}}{\partial x}(w)\), since we have
\[
\frac{\partial f_{\varphi}}{\partial y}(r_i) = 0, \quad \frac{\partial f_{\varphi}}{\partial x_j}(y) - \frac{\partial f_{\varphi}}{\partial x_j}(w) = -\frac{\partial f_{\varphi}}{\partial x_j}(w), \quad \frac{\partial f_{\varphi}}{\partial y}(y) - \frac{\partial f_{\varphi}}{\partial y}(w) = 1.
\]
In a similar manner, we can prove \(A(X, \rho; f_1, f_2, \phi) \sim A(X', \rho'; f_1, f_2, \phi)\).
11 Augmented matrices

In Definition 5.3, Proposition 7.1 and Theorem 9.3, we assumed that $M = R$. We can remove this restriction by using augmented matrices, which we introduce in this section.

Let $R$ be a ring and $M$ a left $R$-module. An $m \times (n + 1)$ $M$-augmented matrix over $R$ is a matrix $(A \mathrel{\|} v)$ such that $A \in M(m, n; R)$ and $v \in M^n$. We set $M(m, n + 1; R, M) := \{(A \mathrel{\|} v) \mid A \in M(m, n; R), v \in M^n\}$. For example,

$$
\begin{pmatrix}
  a_{11} & \cdots & a_{1n} & v_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & \cdots & a_{mn} & v_m
\end{pmatrix} \in M(m, n + 1; R, M),
$$

where $a_{ij} \in R$ and $v_i \in M$. We denote this matrix by $(a_{ij} \mathrel{\|} v_i)$. We call $v$ the augmented vector of $(A \mathrel{\|} v)$.

We say that two $M$-augmented matrices $(A_1 \mathrel{\|} v_1)$ and $(A_2 \mathrel{\|} v_2)$ over $R$ are equivalent $(A_1 \mathrel{\|} v_1) \sim (A_2 \mathrel{\|} v_2)$ if they are related by a finite sequence of the following transformations:

- $(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \mathrel{\|} v) \leftrightarrow (a_1, \ldots, a_i + a_j r, \ldots, a_j, \ldots, a_n \mathrel{\|} v) \ (r \in R),$
- $(a_1, \ldots, a_i, \ldots, a_n \mathrel{\|} v) \leftrightarrow (a_1, \ldots, a_j, \ldots, a_n \mathrel{\|} v + a_j r) \ (r \in M),$
- $(a_1 \mathrel{\|} v_1) \leftrightarrow \begin{pmatrix}
  a_1 \\
  \vdots \\
  a_i \\
  \vdots \\
  a_j \\
  \vdots \\
  a_n
\end{pmatrix} \begin{pmatrix}
  v_1 \\
  \vdots \\
  v_i \\
  \vdots \\
  v_j \\
  \vdots \\
  v_n
\end{pmatrix} + \begin{pmatrix}
  v_i \\
  \vdots \\
  v_i + rv_j \\
  \vdots \\
  v_j \\
  \vdots \\
  v_n
\end{pmatrix} \ (r \in R),$
- $(A \mathrel{\|} v) \leftrightarrow \begin{pmatrix}
  A \\
  0
\end{pmatrix} \mathrel{\|} \begin{pmatrix}
  v \\
  0
\end{pmatrix},$
- $(A \mathrel{\|} v) \leftrightarrow \begin{pmatrix}
  A \\
  0
\end{pmatrix} \mathrel{\|} \begin{pmatrix}
  v \\
  1
\end{pmatrix}.$

We note that the equivalence $(A_1 \mathrel{\|} v_1) \sim (A_2 \mathrel{\|} v_2)$ implies $A_1 \sim A_2$. We also note that $A_1 \sim A_2$ if and only if $(A_1 \mathrel{\|} 0) \sim (A_2 \mathrel{\|} 0)$.

Let $R$ be a commutative ring $R$, and $M$ a left $R$-module. The determinant of $(A \mathrel{\|} v) \in M(n, n; R, M)$ is defined by

$$
\det \begin{pmatrix}
  a_{11} & \cdots & a_{1,n-1} & v_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & \cdots & a_{m,n-1} & v_n
\end{pmatrix} = \sum_{\sigma \in S_n} \text{sgn} (\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n-1)n-1} v_{\sigma(n)},
$$

which is an element of $M$. A $k \times (l + 1)$ $M$-augmented submatrix of $(A \mathrel{\|} v) \in M(m, n + 1; R, M)$ is an $M$-augmented matrix obtained by removing $m - k$ rows and $n - l$ columns from the $M$-augmented matrix $(A \mathrel{\|} v)$, where we note that the augmented vector is not removed. A $k$-minor of $(A \mathrel{\|} v)$ is the determinant of a $k \times k$ $M$-augmented submatrix of $(A \mathrel{\|} v)$. For $(A \mathrel{\|} v) \in M(m, n + 1; R, M)$,
we define $E_d((A \mid v))$ to be the submodule of $M$ generated by the elements of the set
\[
\{ v \mid v \text{ is an } (n + 1 - d)\text{-minor of } (A \mid v) \} \\
\cup \{ av \mid a \text{ is an } (n + 1 - d)\text{-minor of } A, v \in M \}.
\]

If $(A \mid v) \sim (B \mid w)$, then $E_d((A \mid v)) = E_d((B \mid w))$. We leave the proof to the reader.

Let $Q$ be a quandle. Let $R$ be a ring and $M$ a left $R$-module. Let $(f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$, and $\phi : Q \times Q \to M$ an $(f_1, f_2)$-twisted $2$-cocycle. Let $X = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finitely presented quandle, and $\rho : X \to Q$ a quandle representation. Put $f = (f_1, f_2; \phi)$. The $f$-twisted Alexander matrix of $(X, \rho)$ (with respect to the presentation $(x \mid r)$) is
\[
A(X, \rho; f_1, f_2, \phi) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} (r_1) & \cdots & \frac{\partial f_1}{\partial x_n} (r_1) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} (r_m) & \cdots & \frac{\partial f_m}{\partial x_n} (r_m)
\end{pmatrix}.
\]

As Theorem 5.4, if $(X, \rho) \cong (X', \rho')$, then we have
\[
A(X, \rho; f_1, f_2, \phi) \sim A(X', \rho'; f_1, f_2, \phi),
\]
and $E_d(A(X, \rho; f_1, f_2, \phi)) = E_d(A(X', \rho'; f_1, f_2, \phi))$.

Acknowledgments

The authors would like to thank Tomo Murao for carefully reading and giving helpful comments on this manuscript. The first author was supported by JSPS KAKENHI Grant Number 18K03292. The second author was supported by JSPS KAKENHI Grant Number 16K17600.

References


