NORMALIZED QUANDLE TWISTED ALEXANDER INVARIANTS

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Abstract. We introduce a quandle version of the normalized (twisted) Alexander polynomial, which is an invariant of a pair of an oriented link and a quandle representation. The invariant can be constructed by fixing each Alexander pair, and we find various invariants in our framework, which include the quandle cocycle invariant and the normalized (twisted) Alexander polynomial of a knot. In this paper, we develop the theory of normalization with row and column relations of matrices. The theory works for several row and column relations, although the twisted Alexander polynomial is defined with one column relation. We give a formula of our invariant for the mirror image of an oriented link, which explains why the Alexander polynomial fails to detect the chirality of knots and why the quandle cocycle invariant effectively detects it from a unified point of view. We also show that cohomologous Alexander pairs yield the same invariant.

1. Introduction

Invariants derived from a (twisted) Alexander matrix, which include the Alexander ideal [7], the Alexander polynomial [1], and the twisted Alexander polynomial [16, 20], have been studied to reveal topological properties of knots and links (e.g. [4, 14, 18]). We call such invariants Alexander type invariants. The normalized twisted Alexander polynomial was introduced by Kitayama [15] for oriented knots as a twisted version of the Alexander–Conway polynomial [5]. In this paper, we introduce a quandle version of the normalized twisted Alexander polynomial, where a quandle [13, 17] is a generalization of a group whose axioms correspond to the Reidemeister moves for oriented links. The invariant can be constructed by fixing each Alexander pair, and we find various invariants in our framework, where an Alexander pair is a pair of maps corresponding to a linear extension of a quandle [2]. In our framework, the normalized twisted Alexander polynomial is not only recoverable (Proposition 7.4), but also extended to arbitrary links (Definition 7.3). In other words, we succeed in defining the normalized twisted Alexander polynomial for any oriented links. Taniguchi [19] showed that a quandle version of an Alexander type invariant is an essential generalization of a usual Alexander type invariant by proving that the invariant with a suitable Alexander pair can be described with a quandle cocycle invariant [3] for knots. Further, we show that the quandle cocycle invariant is recoverable in our framework (Proposition 5.1).

An Alexander matrix is obtained from a group presentation by using the Fox derivative. Two Alexander matrices A_1, A_2 are equivalent ($A_1 \sim A_2$) if they are obtained from isomorphic link groups. We then have the invariance of Alexander polynomial as $\Delta(A_1) \doteq \Delta(A_2)$, where the symbol $\doteq$ indicates equality up to a unit factor. The twisted Alexander polynomial is defined with a twisted Alexander matrix and one linear relation among the column vectors of the twisted Alexander matrix.

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In [11], we reformulated the process of defining the twisted Alexander polynomial by introducing an equivalence relation for pairs (A, C) of matrices and their column relation matrices. We defined a quandle version of the twisted Alexander polynomial $\Delta(A, C)$ and showed that $(A_1, C_1) \sim (A_2, C_2)$ implies $\Delta(A_1, C_1) \doteq \Delta(A_2, C_2)$. In this paper, we introduce an equivalence relation for triples (B, A, C) of matrices and their row and column relation matrices and show that $(B_1, A_1, C_1) \sim (B_2, A_2, C_2)$ implies $\Delta(B_1, A_1, C_1) = \Delta(B_2, A_2, C_2)$ (Theorem 2.12). Applying Δ to the triple obtained from a link diagram with a quandle representation, we can define a quandle version of the normalized twisted Alexander polynomial (Theorem 4.7).

We give a formula of our invariant for the mirror image of an oriented link (Proposition 6.2). From the formula, we see that whether Alexander type invariants detect the chirality of links is greatly affected by the parity of $k(r + m)$, where k is the dimension of matrices used in a quandle representation, r is the number of the components of a link, and m is the number of row relations. We then understand why the Alexander polynomial fails to detect the chirality of knots and why the quandle cocycle invariant effectively detects it from a unified point of view. Furthermore, by using the formula, we demonstrate that the granny knot is chiral, which implies that the knot is not equivalent to the square knot (Example 6.3). We also show that cohomologous Alexander pairs yield the same invariant (Proposition 8.3). In particular, we see that the invariant with the Alexander pair corresponding to the linear extension obtained from a G-family of quandles [9] coincides with the twisted Alexander polynomial (Example 8.4).

This paper is organized as follows. In Section 2, we introduce an equivalence relation on triples of matrices and their row and column relation matrices, and show that $\Delta(B, A, C)$ is an invariant of the equivalence class. In Section 3, we recall the definitions of a quandle, an Alexander pair and relation maps with some examples. In Section 4, we introduce the quandle version of the normalized twisted Alexander polynomial. In Section 5, we show that the quandle cocycle invariant is recoverable in our framework. In Section 6, we evaluate our invariant for the trivial knot and the mirror image of a link. In Section 7, we show that the Alexander–Conway polynomial and the normalized twisted Alexander polynomial are recoverable in our framework. In Section 8, we show that cohomologous Alexander pairs yield the same invariant. In Section 9, we show the invariance of our invariant.

2. The Alexander invariant of triple matrices

In this section, we introduce an equivalence relation on triples of matrices and their row and column relation matrices, and show that $\Delta(B, A, C)$ is an invariant of the equivalence class.

Let R be a unital ring. We denote by E_n the $n \times n$ identity matrix. We denote by $\boldsymbol{e}_i$ the unit column vector whose components are all 0, except the ith component that equals 1. We then have $E_n = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$. We denote by $M(m, n; R)$ the set of $m \times n$ matrices over R and denote by $GL(n; R)$ the set of $n \times n$ invertible matrices over R. For matrices A, B over R, we define

$$A \oplus B := \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

For $A = (a_{ij}) \in M(m, n; R)$, $\boldsymbol{i} = (i_1, \dots, i_s)$ and $\boldsymbol{j} = (j_1, \dots, j_t)$, we define

$$A_{\boldsymbol{i},\boldsymbol{j}} := \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_t} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_s j_1} & a_{i_s j_2} & \cdots & a_{i_s j_t} \end{pmatrix}.$$

For example,

$$A_{(3,2),(1,4)} = \begin{pmatrix} a_{31} & a_{34} \\ a_{21} & a_{24} \end{pmatrix}$$

for $A = (a_{ij}) \in M(4,4;R)$. We further note that

$$A_{\boldsymbol{i},\boldsymbol{j}} = (\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_s})^T A (\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_t}),$$

where we denote by B^T the transpose of a matrix B.

Let S_n be the symmetric group on $\{1, \ldots, n\}$. We denote by $\operatorname{sgn}\sigma$ the sign of $\sigma \in S_n$. Put $\overline{n} := (1, \ldots, n)$. For $\sigma \in S_n$, we set

$$\sigma(i_1, \ldots, i_s) := (\sigma(i_1), \ldots, \sigma(i_s)),$$
$$(i_1, \ldots, i_s) + k := (i_1 + k, \ldots, i_s + k).$$

For $\sigma \in S_m$ and $\tau \in S_n$, we define $\sigma \oplus \tau \in S_{m+n}$ by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq m, \\ \tau(i-m) + m & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

For $\sigma \in S_n$, we define $P_\sigma := (\boldsymbol{e}_{\sigma(1)}, \ldots, \boldsymbol{e}_{\sigma(n)}) \in GL(n;R)$, which is the permutation matrix associated with σ. Then, $P_\sigma P_\tau = P_{\sigma\tau}$, $P_\sigma^{-1} = P_\sigma^T$ and $P_{\sigma\oplus\tau} = P_\sigma \oplus P_\tau$.

Let $A \in M(d+m, d+n; R)$, where $d, m, n > 0$. We call $B \in M(m, d+m; R)$ a *row relation matrix* of A if $BA = O$. A row relation matrix $B \in M(m, d+m; R)$ is *regular* if $B_{\overline{m},\sigma(\overline{m})}$ is invertible for some $\sigma \in S_{d+m}$. We call $C \in M(d+n, n; R)$ a *column relation matrix* of A if $AC = O$. A column relation matrix $C \in M(d+n, n; R)$ is *regular* if $C_{\tau(\overline{n}),\overline{n}}$ is invertible for some $\tau \in S_{d+n}$.

Definition 2.1. Let R be a commutative ring. Let $A \in M(d+m, d+n; R)$. Let $B \in M(m, d+m; R)$ be a regular row relation matrix of A, and let $C \in M(d+n, n; R)$ be a regular column relation matrix of A. We choose $\sigma \in S_{d+m}$ and $\tau \in S_{d+n}$ so that $B_{\overline{m},\sigma(\overline{m})}$ and $C_{\tau(\overline{n}),\overline{n}}$ are invertible. We then define

$$\Delta(B, A, C) := \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det B_{\overline{m},\sigma(\overline{m})} \det C_{\tau(\overline{n}),\overline{n}}}.$$

We allow m or n to be zero; when $m = 0$ (resp. $n = 0$), we set $M(m, d+m; R) = \{\emptyset\}$ (resp. $M(d+n, n; R) = \{\emptyset\}$), where we call $\emptyset$ an *empty matrix* and regard it as a regular relation matrix of A. We then define

$$\Delta(\emptyset, A, C) := \frac{\operatorname{sgn}\tau \det A_{\overline{d},\tau(\overline{d}+n)}}{\det C_{\tau(\overline{n}),\overline{n}}}, \qquad \Delta(B, A, \emptyset) := \frac{\operatorname{sgn}\sigma \det A_{\sigma(\overline{d}+m),\overline{d}}}{\det B_{\overline{m},\sigma(\overline{m})}},$$
$$\Delta(\emptyset, A, \emptyset) := \det A.$$

The following proposition implies that $\Delta(B, A, C)$ is independent of the choices of σ and τ.

Proposition 2.2. *Let R be a commutative ring. Let $A \in M(d+m, d+n; R)$. Let $B \in M(m, d+m; R)$ be a regular row relation matrix of A. Let $\sigma, \sigma' \in S_{d+m}$ such that $B_{\overline{m},\sigma(\overline{m})}$ and $B_{\overline{m},\sigma'(\overline{m})}$ are invertible. Let $C \in M(d+n, n; R)$ be a regular column relation matrix of A. Let $\tau, \tau' \in S_{d+n}$ such that $C_{\tau(\overline{n}),\overline{n}}$ and $C_{\tau'(\overline{n}),\overline{n}}$ are invertible. Then we have*

$$\frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det B_{\overline{m},\sigma(\overline{m})} \det C_{\tau(\overline{n}),\overline{n}}} = \frac{\operatorname{sgn}\sigma' \operatorname{sgn}\tau' \det A_{\sigma'(\overline{d}+m),\tau'(\overline{d}+n)}}{\det B_{\overline{m},\sigma'(\overline{m})} \det C_{\tau'(\overline{n}),\overline{n}}}.$$

Proof. We choose $\sigma_1, \sigma_1' \in S_m$, $\sigma_2, \sigma_2' \in S_d$, $\tau_1, \tau_1' \in S_n$ and $\tau_2, \tau_2' \in S_d$ so that

$$B_{\overline{m},\sigma(\overline{d+m})} P_{\sigma_1 \oplus \sigma_2} = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \end{pmatrix},$$
$$B_{\overline{m},\sigma'(\overline{d+m})} P_{\sigma_1' \oplus \sigma_2'} = \begin{pmatrix} B_1 & B_3 & B_2 & B_4 \end{pmatrix},$$

$$P^{-1}_{\sigma_1\oplus\sigma_2}A_{\sigma(\overline{d+m}),\tau(\overline{d+n})}P_{\tau_1\oplus\tau_2}=\begin{pmatrix}A_{11}&A_{12}&A_{13}&A_{14}\\A_{21}&A_{22}&A_{23}&A_{24}\\A_{31}&A_{32}&A_{33}&A_{34}\\A_{41}&A_{42}&A_{43}&A_{44}\end{pmatrix},$$
$$P^{-1}_{\sigma'_1\oplus\sigma'_2}A_{\sigma'(\overline{d+m}),\tau'(\overline{d+n})}P_{\tau'_1\oplus\tau'_2}=\begin{pmatrix}A_{11}&A_{13}&A_{12}&A_{14}\\A_{31}&A_{33}&A_{32}&A_{34}\\A_{21}&A_{23}&A_{22}&A_{24}\\A_{41}&A_{43}&A_{42}&A_{44}\end{pmatrix}$$

and

$$P^{-1}_{\tau_1\oplus\tau_2}C_{\tau(\overline{d+n}),\overline{n}}=\begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix},\qquad P^{-1}_{\tau'_1\oplus\tau'_2}C_{\tau'(\overline{d+n}),\overline{n}}=\begin{pmatrix}C_1\\C_3\\C_2\\C_4\end{pmatrix}.$$

for some $B_i\in M(m,m_i;R)$, $A_{ij}\in M(m_i,n_j;R)$ and $C_i\in M(n_i,n;R)$, where $m_2=m_3$, $n_2=n_3$, $m_1+m_2=m$, $n_1+n_2=n$ and $m_3+m_4=n_3+n_4=d$. By the equalities

$$B_{\overline{m},\sigma(\overline{d+m})}=BP_\sigma,\qquad A_{\sigma(\overline{d+m}),\tau(\overline{d+n})}=P_\sigma^{-1}AP_\tau,\qquad C_{\tau(\overline{d+n}),\overline{n}}=P_\tau^{-1}C,$$

we have $B_1A_{1j}+\cdots+B_4A_{4j}=O$ and $A_{i1}C_1+\cdots+A_{i4}C_4=O$. We then have

$$\begin{aligned}\begin{vmatrix}E_{n_1}&O&O\\O&A_{33}&A_{34}\\O&A_{43}&A_{44}\end{vmatrix}\begin{vmatrix}C_1&O\\C_3&O\\O&E_{n_4}\end{vmatrix}&=\begin{vmatrix}C_1&O\\A_{33}C_3&A_{34}\\A_{43}C_3&A_{44}\end{vmatrix}\\&=\begin{vmatrix}C_1&O\\-A_{31}C_1-A_{32}C_2-A_{34}C_4&A_{34}\\-A_{41}C_1-A_{42}C_2-A_{44}C_4&A_{44}\end{vmatrix}\\&=\begin{vmatrix}C_1&O\\-A_{32}C_2&A_{34}\\-A_{42}C_2&A_{44}\end{vmatrix}\\&=(-1)^{n_2}\begin{vmatrix}C_1&O\\A_{32}C_2&A_{34}\\A_{42}C_2&A_{44}\end{vmatrix}\\&=(-1)^{n_2}\begin{vmatrix}E_{n_1}&O&O\\O&A_{32}&A_{34}\\O&A_{42}&A_{44}\end{vmatrix}\begin{vmatrix}C_1&O\\C_2&O\\O&E_{n_4}\end{vmatrix},\end{aligned}$$

which implies

$$\begin{vmatrix}A_{33}&A_{34}\\A_{43}&A_{44}\end{vmatrix}\begin{vmatrix}C_1\\C_3\end{vmatrix}=(-1)^{n_2}\begin{vmatrix}A_{32}&A_{34}\\A_{42}&A_{44}\end{vmatrix}\begin{vmatrix}C_1\\C_2\end{vmatrix}.$$

In a similar manner, we have

$$\begin{vmatrix}B_1&B_3\end{vmatrix}\begin{vmatrix}A_{32}&A_{34}\\A_{42}&A_{44}\end{vmatrix}=(-1)^{m_2}\begin{vmatrix}B_1&B_2\end{vmatrix}\begin{vmatrix}A_{22}&A_{24}\\A_{42}&A_{44}\end{vmatrix}.$$

Hence we have

$$\begin{aligned}\frac{\operatorname{sgn}\sigma\operatorname{sgn}\tau\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det B_{\overline{m},\sigma(\overline{m})}\det C_{\tau(\overline{n}),\overline{n}}} &= \frac{\operatorname{sgn}\sigma\operatorname{sgn}\tau\operatorname{sgn}\sigma_2\operatorname{sgn}\tau_2\begin{vmatrix}A_{33} & A_{34}\\ A_{43} & A_{44}\end{vmatrix}}{\operatorname{sgn}\sigma_1\begin{vmatrix}B_1 & B_2\end{vmatrix}\operatorname{sgn}\tau_1\begin{vmatrix}C_1\\ C_2\end{vmatrix}}\\ &= \frac{\operatorname{sgn}\sigma'\operatorname{sgn}\tau'\operatorname{sgn}\sigma_2'\operatorname{sgn}\tau_2'\begin{vmatrix}A_{22} & A_{24}\\ A_{42} & A_{44}\end{vmatrix}}{\operatorname{sgn}\sigma_1'\begin{vmatrix}B_1 & B_3\end{vmatrix}\operatorname{sgn}\tau_1'\begin{vmatrix}C_1\\ C_3\end{vmatrix}}\\ &= \frac{\operatorname{sgn}\sigma'\operatorname{sgn}\tau'\det A_{\sigma'(\overline{d}+m),\tau'(\overline{d}+n)}}{\det B_{\overline{m},\sigma'(\overline{m})}\det C_{\tau'(\overline{n}),\overline{n}}}.\end{aligned}$$

where we remark that

$$\begin{aligned}\operatorname{sgn}\sigma'\operatorname{sgn}\sigma_1'\operatorname{sgn}\sigma_2' &= (-1)^{m_2}\operatorname{sgn}\sigma\operatorname{sgn}\sigma_1\operatorname{sgn}\sigma_2,\\ \operatorname{sgn}\tau'\operatorname{sgn}\tau_1'\operatorname{sgn}\tau_2' &= (-1)^{n_2}\operatorname{sgn}\tau\operatorname{sgn}\tau_1\operatorname{sgn}\tau_2.\end{aligned}$$

□

Let R be a unital ring. We denote by $R^\times$ the group of units of R. We define $P_{ij}, E_{ij}(r), E_i(u) \in GL(n;R)$ by

$$\begin{aligned}P_{ij} &= (\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{i-1},\boldsymbol{e}_j,\boldsymbol{e}_{i+1},\ldots,\boldsymbol{e}_{j-1},\boldsymbol{e}_i,\boldsymbol{e}_{j+1},\ldots,\boldsymbol{e}_n),\\ E_{ij}(r) &= (\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{j-1},\boldsymbol{e}_j+r\boldsymbol{e}_i,\boldsymbol{e}_{j+1},\ldots,\boldsymbol{e}_n)\ (i\neq j),\\ E_i(u) &= (\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{i-1},u\boldsymbol{e}_i,\boldsymbol{e}_{i+1},\ldots,\boldsymbol{e}_n)\end{aligned}$$

for $r \in R$ and $u \in R^\times$. We note that $P_{ij}^{-1} = P_{ij}$, $E_{ij}(r)^{-1} = E_{ij}(-r)$ and $E_i(u)^{-1} = E_i(u^{-1})$.

Definition 2.3. For matrices A and A' over a unital ring R and their relation matrices B, B', C and C', we write $(B,A,C) \sim (B',A',C')$ if they are related by a finite sequence of the following transformations:

- $(B,A,C) \leftrightarrow (BE_{ij}(r)^{-1}, E_{ij}(r)A, C)$ $(r \in R)$,
- $(B,A,C) \leftrightarrow (B, AE_{ij}(r), E_{ij}(r)^{-1}C)$ $(r \in R)$,
- $(B,A,C) \leftrightarrow (BE_i(u), E_i(u)^{-1}AE_j(u), E_j(u)^{-1}C)$ $(u \in R^\times)$,
- $(B,A,C) \leftrightarrow \left(\begin{pmatrix}B & \mathbf{0}\end{pmatrix}, \begin{pmatrix}A & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}, \begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}\right)$.

When $B = \emptyset$ (resp. $C = \emptyset$), we replace the first (resp. third) matrices in the above transformations with $\emptyset$.

Remark 2.4. When $(B,A,C) \sim (B',A',C')$, B (resp. C) is a row (resp. column) relation matrix of A if and only if B' (resp. C') is a row (resp. column) relation matrix of A'. When R is a field, B (resp. C) is regular if and only if B' (resp. C') is regular. We remark that $\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)$ is regular, while $\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)$ is not regular as matrices over $M(2,2;\mathbb{Q})$.

We may regard a matrix in $M(m,n;M(k,k;R))$ as a matrix in $M(km,kn;R)$. We call such a matrix a *flat matrix*. We denote by $\overline{A}$ the flat matrix of a matrix A.

Remark 2.5. Suppose that R is a Euclidean domain. For $S \in M(k,k;R)$ and $U \in GL(k;R)$, the flat matrices $\overline{E_{st}(S)}$ and $\overline{E_s(U)}$ can be represented as products of $E_{ij}(r)$'s and $E_i(u)$'s, where $r \in R$ and $u \in R^\times$. Therefore,

$$(B,A,C) \sim (B',A',C')$$

implies

$$(\overline{B}, \overline{A}, \overline{C}) \sim (\overline{B'}, \overline{A'}, \overline{C'}).$$

Proposition 2.6. (1) *For $u \in R^\times$, we have*

$$\begin{aligned}(B, A, C) &\sim (BE_i(u)^{-1}E_j(u), E_j(u)^{-1}E_i(u)A, C)\\ &\sim (B, AE_i(u)E_j(u)^{-1}, E_j(u)E_i(u)^{-1}C).\end{aligned}$$

(2) *For $a \in R$ and $u_1, u_2 \in R^\times$, we have*

$$\begin{aligned}\left(\begin{pmatrix}B & \mathbf{0} & \mathbf{0}\end{pmatrix}, A \oplus (a) \oplus (u_1u_2), \begin{pmatrix}C\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\right) &\sim \left(\begin{pmatrix}B & \mathbf{0}\end{pmatrix}, A \oplus (u_1au_2), \begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}\right)\\ &\sim \left(\begin{pmatrix}B & \mathbf{0}\end{pmatrix}, A \oplus (u_2au_1), \begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}\right).\end{aligned}$$

Proof. (1) We have

$$\begin{aligned}(B, A, C) &\sim (BE_i(u)^{-1}, E_i(u)AE_j(u)^{-1}, E_j(u)C)\\ &\sim (BE_i(u)^{-1}E_j(u), E_j(u)^{-1}E_i(u)A, C).\end{aligned}$$

In the same way, we have

$$(B, A, C) \sim (B, AE_i(u)E_j(u)^{-1}, E_j(u)E_i(u)^{-1}C).$$

(2) We have the equivalences, since the left-hand side is equivalent to

$$\left(B', A \oplus (u_1au_2) \oplus (1), \begin{pmatrix}C\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\right) \quad \text{and} \quad \left(B', A \oplus (u_2au_1) \oplus (1), \begin{pmatrix}C\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\right),$$

where $B' = \begin{pmatrix}B & \mathbf{0} & \mathbf{0}\end{pmatrix}$.

□

It is easy to see that the third transformation of Definition 2.3 can be replaced with the pair of the following transformations:

- $(B, A, C) \leftrightarrow (BE_i(u)^{-1}E_j(u), E_j(u)^{-1}E_i(u)A, C)$ $(u \in R^\times)$,
- $(B, A, C) \leftrightarrow (B, AE_i(u)E_j(u)^{-1}, E_j(u)E_i(u)^{-1}C)$ $(u \in R^\times)$.

Proposition 2.7. *We have the following.*

(1) $(B, A, C) \sim (BP_{ij}E_j(-1), E_j(-1)P_{ij}A, C)$.
(2) $(B, A, C) \sim (B, AP_{ij}E_j(-1), E_j(-1)P_{ij}C)$.
(3) $(B, A, C) \sim (BP_{ij}, P_{ij}AP_{kl}, P_{kl}C)$.

Proof. (1) We have

$$\begin{aligned}(B, A, C) &\sim (BE_{ij}(1)^{-1}, E_{ij}(1)A, C)\\ &\sim (BE_{ij}(1)^{-1}E_{ji}(-1)^{-1}, E_{ji}(-1)E_{ij}(1)A, C)\\ &\sim (BE_{ij}(1)^{-1}E_{ji}(-1)^{-1}E_{ij}(1)^{-1}, E_{ij}(1)E_{ji}(-1)E_{ij}(1)A, C)\\ &= (BP_{ij}E_j(-1), E_j(-1)P_{ij}A, C).\end{aligned}$$

(2) In the same way as (1), we have the equivalence.
(3) From (1) and (2), we have

$$\begin{aligned}(B, A, C) &\sim (BP_{ij}E_j(-1), E_j(-1)P_{ij}A, C)\\ &\sim (BP_{ij}E_j(-1), E_j(-1)P_{ij}AP_{kl}E_l(-1), E_l(-1)P_{kl}C)\\ &\sim (BP_{ij}, P_{ij}AP_{kl}, P_{kl}C).\end{aligned}$$

□

By $i \in (n_1, \ldots, n_s)$, we mean $i \in \{n_1, \ldots, n_s\}$. Let R be a commutative ring. Let $A \in M(d+m, d+n; R)$. Let $B \in M(m, d+m; R)$ be a row relation matrix of A, and let $C \in M(d+n, n; R)$ be a column relation matrix of A. We then have the following lemmas:

Lemma 2.8. *Suppose that R is a field. For regular relation matrices B and C of A, we have*

$$\Delta(B, A, C) = \Delta(BE_{ij}(r)^{-1}, E_{ij}(r)A, C).$$

Proof. We choose $\tau \in S_{d+n}$ so that $C_{\tau(\overline{n}),\overline{n}}$ is invertible. We denote by $\boldsymbol{b}_i$ the ith column vector of B and by $\boldsymbol{a}_i$ the ith row vector of A. It is sufficient show

$$\frac{\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det B_{\overline{m},\sigma(\overline{m})}} = \frac{\det(E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det(BE_{ij}(r)^{-1})_{\overline{m},\sigma(\overline{m})}} \tag{1}$$

for some $\sigma \in S_{d+m}$.

When $\boldsymbol{b}_i = \boldsymbol{b}_j = 0$, we can choose $\sigma \in S_{d+m}$ so that

$$B_{\overline{m},\sigma(\overline{m})} \in GL(m; R) \qquad \text{and} \qquad B_{\overline{m},\sigma(\overline{d}+m)} = \begin{pmatrix} \boldsymbol{b}_i & \boldsymbol{b}_j & B_2 \end{pmatrix}$$

for some $B_2 \in M(m, d-2; R)$. We then have

$$B_{\overline{m},\sigma(\overline{m})} = (BE_{ij}(r)^{-1})_{\overline{m},\sigma(\overline{m})},$$

$$A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} = \begin{pmatrix} \boldsymbol{a}_i \\ \boldsymbol{a}_j \\ A_2 \end{pmatrix}, \qquad (E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} = \begin{pmatrix} \boldsymbol{a}_i + r\boldsymbol{a}_j \\ \boldsymbol{a}_j \\ A_2 \end{pmatrix}$$

for some $A_2 \in M(d-2, d; R)$. The equality (1) follows from $\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} = \det(E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}$.

When $\boldsymbol{b}_i = \boldsymbol{0}$ and $\boldsymbol{b}_j \neq \boldsymbol{0}$, we can choose $\sigma \in S_{d+m}$ so that

$$B_{\overline{m},\sigma(\overline{m})} = \begin{pmatrix} \boldsymbol{b}_j & B_1 \end{pmatrix} \in GL(m; R) \qquad \text{and} \qquad B_{\overline{m},\sigma(\overline{d}+m)} = \begin{pmatrix} \boldsymbol{b}_i & B_2 \end{pmatrix}$$

for some $B_1 \in M(m, m-1; R)$ and $B_2 \in M(m, d-1; R)$. We then have

$$(BE_{ij}(r)^{-1})_{\overline{m},\sigma(\overline{m})} = \begin{pmatrix} \boldsymbol{b}_j - r\boldsymbol{b}_i & B_1 \end{pmatrix} = B_{\overline{m},\sigma(\overline{m})}$$

and

$$A_{\sigma(\overline{m}),\tau(\overline{d}+n)} = \begin{pmatrix} \boldsymbol{a}_j \\ A_1 \end{pmatrix}, \qquad (E_{ij}(r)A)_{\sigma(\overline{m}),\tau(\overline{d}+n)} = \begin{pmatrix} \boldsymbol{a}_j \\ A_1 \end{pmatrix},$$

$$A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} = \begin{pmatrix} \boldsymbol{a}_i \\ A_2 \end{pmatrix}, \qquad (E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} = \begin{pmatrix} \boldsymbol{a}_i + r\boldsymbol{a}_j \\ A_2 \end{pmatrix}$$

for some $A_1 \in M(m-1, d; R)$ and $A_2 \in M(d-1, d; R)$. From

$$\boldsymbol{b}_j\boldsymbol{a}_j + B_1A_1 + \boldsymbol{b}_i\boldsymbol{a}_i + B_2A_2 = BA = O,$$

we have

$$\begin{pmatrix} \boldsymbol{b}_j & B_1 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_j \\ A_1 \end{pmatrix} = -\boldsymbol{b}_i\boldsymbol{a}_i - B_2A_2 = -B_2A_2.$$

Since $\begin{pmatrix} \boldsymbol{b}_j & B_1 \end{pmatrix}$ is invertible, we have

$$\begin{pmatrix} \boldsymbol{a}_j \\ A_1 \end{pmatrix} = -\begin{pmatrix} \boldsymbol{b}_j & B_1 \end{pmatrix}^{-1} B_2A_2,$$

which implies that $\boldsymbol{a}_j$ is a linear combination of row vectors of A_2. We then have $\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} = \det(E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}$, which implies the equality (1).

When $\boldsymbol{b}_i \neq \boldsymbol{0}$ and $\operatorname{rank}(\boldsymbol{b}_i, \boldsymbol{b}_j) = 1$, we can choose $\sigma \in S_{d+m}$ so that

$$B_{\overline{m},\sigma(\overline{m})} = \begin{pmatrix} \boldsymbol{b}_i & B_1 \end{pmatrix} \in GL(m; R) \qquad \text{and} \qquad B_{\overline{m},\sigma(\overline{d}+m)} = \begin{pmatrix} \boldsymbol{b}_j & B_2 \end{pmatrix}$$

for some $B_1 \in M(m, m-1; R)$ and $B_2 \in M(m, d-1; R)$. We then have

$$\begin{aligned}(BE_{ij}(r)^{-1})_{\overline{m},\sigma(\overline{m})} &= B_{\overline{m},\sigma(\overline{m})},\\ (E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} &= A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)},\end{aligned}$$

which imply the equality (1).

When $\operatorname{rank}(\boldsymbol{b}_i, \boldsymbol{b}_j) = 2$, we can choose $\sigma \in S_{d+m}$ so that

$$B_{\overline{m},\sigma(\overline{m})} = \begin{pmatrix}\boldsymbol{b}_i & \boldsymbol{b}_j & B_1\end{pmatrix} \in GL(m; R)$$

for some $B_1 \in M(m, m-2; R)$. We then have

$$\begin{aligned}(BE_{ij}(r)^{-1})_{\overline{m},\sigma(\overline{m})} &= \begin{pmatrix}\boldsymbol{b}_i & \boldsymbol{b}_j - r\boldsymbol{b}_i & B_1\end{pmatrix},\\ (E_{ij}(r)A)_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} &= A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}.\end{aligned}$$

The equality (1) follows from $\det B_{\overline{m},\sigma(\overline{m})} = \det(BE_{ij}(r)^{-1})_{\overline{m},\sigma(\overline{m})}$. This completes the proof. □

Lemma 2.9. *Suppose that R is a field. For regular relation matrices B and C of A, we have*

$$\Delta(B, A, C) = \Delta(B, AE_{ij}(r), E_{ij}(r)^{-1}C).$$

Proof. This lemma is proved in the same manner as the previous lemma. □

Lemma 2.10. *For regular relation matrices B and C of A, we have*

$$\Delta(B, A, C) = \Delta(BE_i(u), E_i(u)^{-1}AE_j(u), E_j(u)^{-1}C)$$

for $u \in R^\times$.

Proof. We choose $\sigma \in S_{d+m}$ and $\tau \in S_{d+n}$ so that $B_{\overline{m},\sigma(\overline{m})}$ and $C_{\tau(\overline{n}),\overline{n}}$ are invertible. We then have

$$\begin{aligned}&\frac{\det(E_i(u)^{-1}AE_j(u))_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det(BE_i(u))_{\overline{m},\sigma(\overline{m})}\det(E_j(u)^{-1}C)_{\tau(\overline{n}),\overline{n}}}\\ &= \frac{u^{-\delta(i\in\sigma(\overline{d}+m))+\delta(j\in\tau(\overline{d}+n))}\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{u^{\delta(i\in\sigma(\overline{m}))}\det B_{\overline{m},\sigma(\overline{m})}u^{-\delta(j\in\tau(\overline{n}))}\det C_{\tau(\overline{n}),\overline{n}}} = \frac{\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det B_{\overline{m},\sigma(\overline{m})}\det C_{\tau(\overline{n}),\overline{n}}},\end{aligned}$$

where $\delta(x \in S) := \begin{cases}1 & \text{if } x \in S,\\ 0 & \text{otherwise.}\end{cases}$ Hence we have the desired equality. □

Lemma 2.11. *For regular relation matrices B and C of A, we have*

$$\Delta(B, A, C) = \Delta(\begin{pmatrix}B & \mathbf{0}\end{pmatrix}, \begin{pmatrix}A & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}, \begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}).$$

Proof. We choose $\sigma \in S_{d+m}$ and $\tau \in S_{d+n}$ so that $B_{\overline{m},\sigma(\overline{m})}$ and $C_{\tau(\overline{n}),\overline{n}}$ are invertible. We define $\sigma' := \sigma \oplus 1_{S_1} \in S_{d+m+1}$ and $\tau' := \tau \oplus 1_{S_1} \in S_{d+n+1}$. Then, $\begin{pmatrix}B & \mathbf{0}\end{pmatrix}_{\overline{m},\sigma'(\overline{m})}$ and $\begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}_{\tau'(\overline{n}),\overline{n}}$ are invertible, since they coincide with $B_{\overline{m},\sigma(\overline{m})}$ and $C_{\tau(\overline{n}),\overline{n}}$, respectively. We also have

$$\begin{pmatrix}A & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}_{\sigma'(\overline{d+1}+m),\tau'(\overline{d+1}+n)} = \begin{pmatrix}A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)} & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}.$$

We then have

$$\begin{aligned}\Delta(B,A,C) &= \frac{\operatorname{sgn}\sigma\operatorname{sgn}\tau\det A_{\sigma(\overline{d}+m),\tau(\overline{d}+n)}}{\det B_{\overline{m},\sigma(\overline{m})}\det C_{\tau(\overline{n}),\overline{n}}} \\ &= \frac{\operatorname{sgn}\sigma'\operatorname{sgn}\tau'\det\begin{pmatrix}A & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}_{\sigma'(\overline{d+1}+m),\tau'(\overline{d+1}+n)}}{\det\begin{pmatrix}B & \mathbf{0}\end{pmatrix}_{\overline{m},\sigma'(\overline{m})}\det\begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}_{\tau'(\overline{n}),\overline{n}}} \\ &= \Delta(\begin{pmatrix}B & \mathbf{0}\end{pmatrix},\begin{pmatrix}A & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix},\begin{pmatrix}C\\ \mathbf{0}\end{pmatrix}),\end{aligned}$$

where we note that $\operatorname{sgn}\sigma' = \operatorname{sgn}\sigma$ and $\operatorname{sgn}\tau' = \operatorname{sgn}\tau$. □

From the above lemmas, we have the following theorem.

Theorem 2.12. *Suppose that R is a field. Let A be a matrix over R. Let B and C be a regular row relation matrix and a regular column relation matrix of A, respectively. Then $\Delta(B,A,C)$ is an invariant of the equivalence class of (B,A,C). That is, if $(B,A,C) \sim (B',A',C')$, then $\Delta(B,A,C) = \Delta(B',A',C')$.*

From Remark 2.5, we also have the following theorem.

Theorem 2.13. *Suppose that $R = M(k,k;F)$, where F is a field. Let A be a matrix over R. Let B and C be a row relation matrix and a column relation matrix of A, respectively. Suppose that $\overline{B}$ and $\overline{C}$ are regular. Then $\Delta(\overline{B},\overline{A},\overline{C})$ is an invariant of the equivalence class of (B,A,C). That is, if $(B,A,C) \sim (B',A',C')$, then $\Delta(\overline{B},\overline{A},\overline{C}) = \Delta(\overline{B'},\overline{A'},\overline{C'})$.*

From the definition of $\Delta(\emptyset,A,\emptyset)$, we have the following proposition.

Proposition 2.14. *Suppose that R is a commutative ring. Let A be a matrix over R. Then $\Delta(\emptyset,A,\emptyset)$ is an invariant of the equivalence class of $(\emptyset,A,\emptyset)$. That is, if $(\emptyset,A,\emptyset) \sim (\emptyset,A',\emptyset)$, then $\Delta(\emptyset,A,\emptyset) = \Delta(\emptyset,A',\emptyset)$.*

From Remark 2.5, we also have the following proposition.

Proposition 2.15. *Suppose that $R = M(k,k;Z)$, where Z is a Euclidean domain. Let A be a matrix over R. Then $\Delta(\emptyset,\overline{A},\emptyset)$ is an invariant of the equivalence class of $(\emptyset,A,\emptyset)$. That is, if $(\emptyset,A,\emptyset) \sim (\emptyset,A',\emptyset)$, then $\Delta(\emptyset,\overline{A},\emptyset) = \Delta(\emptyset,\overline{A'},\emptyset)$.*

3. Alexander pairs and relation maps

In this section, we recall the definitions of a quandle and a quandle coloring, which is regarded as a quandle homomorphism from the fundamental quandle to a quandle. We also recall the definitions of an Alexander pair and relation maps with some examples.

A *quandle* [13, 17] is a non-empty set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

- For any $a \in Q$, $a \triangleleft a = a$.
- For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
- For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. Let $(Q_1, \triangleleft_1)$ and $(Q_2, \triangleleft_2)$ be quandles. A *quandle homomorphism* from Q_1 to Q_2 is defined to be a map $f : Q_1 \to Q_2$ satisfying $f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)$ for any $a, b \in Q_1$. For a quandle $(Q, \triangleleft)$, a *Q-set* is a non-empty set Y equipped with a map $\triangleleft : Y \times Q \to Y$ satisfying the following axioms:

- For any $a \in Q$, the map $\triangleleft a : Y \to Y$ defined by $\triangleleft a(y) = y \triangleleft a$ is bijective.

- For any $y \in Y$ and $a, b \in Q$, we have $(y \triangleleft a) \triangleleft b = (y \triangleleft b) \triangleleft (a \triangleleft b)$.

Here, we note that we use the same symbol $\triangleleft$ as the binary operation of Q for the map of a Q-set. We denote $(\triangleleft a)^n : Y \to Y$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. Let $(Y_1, \triangleleft_1)$ and $(Y_2, \triangleleft_2)$ be Q-sets. A *Q-set homomorphism* from Y_1 to Y_2 is defined to be a map $f : Y_1 \to Y_2$ satisfying $f(y \triangleleft_1 a) = f(y) \triangleleft_2 a$ for any $y \in Y$ and $a \in Q$. The *associated group* $\operatorname{As} Q$ of a quandle Q is a group defined by the presentation:

$$\langle x \ (x \in Q) \mid x \triangleleft y = y^{-1} x y \ (x, y \in Q) \rangle.$$

Then $\operatorname{As} Q$ is a Q-set with $y \triangleleft a = ya$. We note that a Q-set homomorphism $f : \operatorname{As} Q \to Y$ is determined by the image $f(1)$ of the identity element $1 \in \operatorname{As} Q$.

Throughout this paper, for a positive integer n, we denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n as $\mathbb{Z}_n$. We define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = 2b - a$. Then, $(\mathbb{Z}_n, \triangleleft)$ is a quandle. We call it the *dihedral quandle* of order n and denote it by R_n.

Let G be a group and n an integer. We define a binary operation $\triangleleft$ on G by $a \triangleleft b = b^{-n} a b^n$. Then, $(G, \triangleleft)$ is a quandle. We call it the *n-fold conjugation quandle* of G and denote it by $\operatorname{Conj}_n G$. The 1-fold conjugation quandle of G is called the *conjugation quandle* of G and denoted by $\operatorname{Conj} G$.

Let G be a group. We define a binary operation $\triangleleft$ on G by $a \triangleleft b = ba^{-1}b$. Then, $(G, \triangleleft)$ is a quandle. We call it the *core quandle* of G and denote it by $\operatorname{Core} G$.

Let L be an oriented link. Let $N(L)$ be the regular neighborhood of L, and $E(L)$ the exterior of L. Fix a point p in $E(L)$. The set of homotopy classes of paths from the boundary $\partial N(L)$ of $N(L)$ to the point p forms a quandle structure with the binary operation $\triangleleft$ defined by

$$[\alpha] \triangleleft [\beta] = [\alpha \cdot \beta^{-1} \cdot m_\beta \cdot \beta],$$

where β^{-1} represents the reverse path of β and m_β is the meridian loop on $\partial N(L)$ based at the initial point of β with the orientation such that the linking number of L and m_β is $+1$. We then denote the quandle by $Q(L)$ and call it the *fundamental quandle* of L.

Let D be a diagram of an oriented link L. A normal orientation is often used to represent an orientation of a link on its diagram. The normal orientation is obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $C(D)$ and $\mathcal{A}(D)$ the sets of crossings and arcs of D, respectively. It is known that the fundamental quandle $Q(L)$ is represented by the arcs and crossings as follows. For a crossing c, we denote the relation $u_c \triangleleft v_c = w_c$ by r_c, where v_c is the over-arc of c and u_c, w_c are the under-arcs of c such that the normal orientation of v_c points from u_c to w_c (see the left picture of Figure 1). Then, the fundamental quandle $Q(L)$ is generated by the arcs x $(x \in \mathcal{A}(D))$ and has the relations r_c $(c \in C(D))$; that is, a presentation of $Q(L)$ is given by

$$\langle x \ (x \in \mathcal{A}(D)) \mid r_c \ (c \in C(D)) \rangle. \tag{2}$$

This is called the *Wirtinger presentation* of $Q(L)$ with respect to D. We remark that we obtain a presentation of the fundamental group $G(L) := \pi_1(E(L), p)$ by replacing $u_c \triangleleft v_c = w_c$ by $v_c^{-1} u_c v_c w_c^{-1}$ in (2), which is the Wirtinger presentation of $G(L)$ with respect to D. A *quandle representation* of $Q(L)$ to Q is a quandle homomorphism from $Q(L)$ to Q. For a group representation $\rho : G(L) \to G$, we call the quandle homomorphism $\rho \circ \varphi : Q(L) \to \operatorname{Conj} G$ the *induced quandle representation*, where $\varphi : Q(L) \to G(L)$ is the map which sends $[\alpha]$ to $[\alpha^{-1} \cdot m_\alpha \cdot \alpha]$. For further details, we refer the reader to [6, 13].

Let Q be a quandle. A *Q-coloring* of D is a map $C : \mathcal{A}(D) \to Q$ satisfying the condition

$$C(u_c) \triangleleft C(v_c) = C(w_c)$$

FIGURE 1

for each crossing $c \in C(D)$, where u_c, v_c and w_c are the arcs forming the crossing c as shown in the left picture of Figure 1. A constant map is a type of Q-coloring called a *trivial Q-coloring*. We denote by $\mathrm{Col}_Q(D)$ the set of Q-colorings of D. From the presentation (2), a Q-coloring of D can be regarded as a quandle representation of $Q(L)$ to Q. Let D' be a diagram of L obtained by applying a single Reidemeister move to D. Then, each Q-coloring C of D has a unique Q-coloring C' of D' that coincides with C except in the disk in which the move is applied. This gives a one-to-one correspondence between $\mathrm{Col}_Q(D)$ and $\mathrm{Col}_Q(D')$. Since the two Q-colorings C and C' represent the same quandle representation ρ, we often use ρ instead of C or C'.

We denote by $\mathcal{SA}(D)$ the set of semi-arcs of D, where a semi-arc is a piece of a curve such that the end points of the piece are crossings. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. We denote by $r(\alpha)$ and $r'(\alpha)$ the regions facing a semi-arc α such that the normal orientation of α points from $r(\alpha)$ to $r'(\alpha)$ (see the right picture of Figure 1). Let Y be a Q-set. A *Q_Y-coloring* ρ_Y of D is an extension of a Q-coloring ρ of D that assigns an element of Y to each region of D satisfying the condition

$$\rho_Y(r(\alpha)) \lhd \rho(\alpha) = \rho_Y(r'(\alpha))$$

for each semi-arc $\alpha \in \mathcal{A}(D)$, where the color $\rho(\alpha)$ of a semi-arc α is defined by the color of the arc from which the semi-arc originates. We remark that the colors of the regions are determined by those of the arcs and one region. We denote by r_{out} the outermost region of a link diagram. We denote by $\widetilde{\rho}$ the $Q_{\mathrm{As}\,Q}$-coloring that is the extension of ρ satisfying $\widetilde{\rho}(r_{\mathrm{out}}) = 1$. The *Alexander numbering* $n_A : \mathcal{R}(D) \to \mathbb{Z}$ is a map satisfying $n_A(r_{\mathrm{out}}) = 0$ and $n_A(r'(\alpha)) = n_A(r(\alpha)) + 1$ for any semi-arc α. Let Q be a quandle, and let $Y := \mathbb{Z}$ be the Q-set with $y \lhd a := y + 1$. Then the Alexander numbering gives a Q_Y-coloring.

Let $(Q, \lhd)$ be a quandle. Let R be a unital ring. The pair (f_1, f_2) of maps $f_1, f_2 : Q \times Q \to R$ is an *Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a, a) + f_2(a, a) = 1$.
- For any $a, b \in Q$, $f_1(a, b)$ is invertible.
- For any $a, b, c \in Q$,

$$\begin{aligned} &f_1(a \lhd b, c) f_1(a, b) = f_1(a \lhd c, b \lhd c) f_1(a, c), \\ &f_1(a \lhd b, c) f_2(a, b) = f_2(a \lhd c, b \lhd c) f_1(b, c), \text{ and} \\ &f_2(a \lhd b, c) = f_1(a \lhd c, b \lhd c) f_2(a, c) + f_2(a \lhd c, b \lhd c) f_2(b, c). \end{aligned}$$

Definition 3.1 ([11])**.** Let (f_1, f_2) be an Alexander pair. A *column relation map* $f_{\mathrm{col}} : Q \to R$ is a map satisfying

$$f_{\mathrm{col}}(a \lhd b) = f_1(a, b) f_{\mathrm{col}}(a) + f_2(a, b) f_{\mathrm{col}}(b)$$

for any $a, b \in Q$.

Proposition 3.2 ([11])**.** *For each $c \in Q$, the map $f_{\mathrm{col}} : Q \to R$ defined by $f_{\mathrm{col}}(a) = f_2(a \lhd^{-1} c, c)$ is a column relation map.*

Definition 3.3 ([10])**.** Let (f_1, f_2) be an Alexander pair. Let Y be a Q-set. A *row relation map* $f_{\text{row}} : Y \times Q \to R$ is a map satisfying

$$f_{\text{row}}(y, a) = f_{\text{row}}(y \lhd b, a \lhd b) f_1(a, b), \text{ and}$$
$$f_{\text{row}}(y \lhd a, b) = f_{\text{row}}(y, b) + f_{\text{row}}(y \lhd b, a \lhd b) f_2(a, b)$$

for any $a, b \in Q$ and $y \in Y$.

Proposition 3.4 ([10])**.** *Let Y be the Q-set $Q \times R^\times$ with $(y, z) \lhd a := (y \lhd a, f_1(y, a)z)$. The map $f_{\text{row}} : Y \times Q \to R$ defined by $f_{\text{row}}((y, z), a) = z^{-1} f_1(y, a)^{-1} f_2(y, a)$ is a row relation map.*

Let Y, Y' be a Q-set, and let $\varphi : Y' \to Y$ be a Q-set homomorphism. Let $f_{\text{row}} : Y \times Q \to R$ be a row relation map. Then the map $f'_{\text{row}} : Y' \times Q \to R$ defined by $f'_{\text{row}}(y, a) = f_{\text{row}}(\varphi(y), a)$ is a row relation map. In particular, for $z \in Y$, the map $f'_{\text{row}} : \operatorname{As} Q \times Q \to R$ defined by $f'_{\text{row}}(y, a) = f_{\text{row}}(\varphi_z(y), a)$ is a row relation map, where $\varphi_z : \operatorname{As} Q \to Y$ is the Q-set homomorphism satisfying $\varphi_z(1) = z$. We give some examples of Alexander pairs and relation maps.

Example 3.5. Let Q be a quandle and R a unital ring. Let $f : Q \to \operatorname{Conj} R^\times$ be a quandle homomorphism. Let Y be the Q-set $R^\times$ with $y \lhd a := f(a)^{-1} y$.

(1) The maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = f(b)^{-1}$ and $f_2(a, b) = 1 - f(b)^{-1}$ form an Alexander pair. The map $f_{\text{col}} : Q \to R$ defined by $f_{\text{col}}(a) = 1$ is a column relation map, and the map $f_{\text{row}} : Y \times Q \to R$ defined by $f_{\text{row}}(y, a) = y^{-1}(f(a) - 1)$ is a row relation map.
(2) The maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = f(b)^{-1}$ and $f_2(a, b) = f(b)^{-1} f(a) - f(b)^{-1}$ form an Alexander pair. The map $f_{\text{col}} : Q \to R$ defined by $f_{\text{col}}(a) = f(a) - 1$ is a column relation map, and the map $f_{\text{row}} : Y \times Q \to R$ defined by $f_{\text{row}}(y, a) = y^{-1}$ is a row relation map.

By setting $f(x) = x^n$, we have the following corollary:

Example 3.6. Let G be a group, and let $Q := \operatorname{Conj}_n G$. Let $R[G]$ be the group ring of G over a commutative ring R. Let Y be the Q-set $R[G]^\times$ with $y \lhd a := a^{-n} y$.

(1) The maps $f_1, f_2 : Q \times Q \to R[G]$ defined by $f_1(a, b) = b^{-n}$ and $f_2(a, b) = 1 - b^{-n}$ form an Alexander pair. The map $f_{\text{col}} : Q \to R[G]$ defined by $f_{\text{col}}(a) = 1$ is a column relation map, and the map $f_{\text{row}} : Y \times Q \to R[G]$ defined by $f_{\text{row}}(y, a) = y^{-1}(a^n - 1)$ is a row relation map.
(2) The maps $f_1, f_2 : Q \times Q \to R[G]$ defined by $f_1(a, b) = b^{-n}$ and $f_2(a, b) = b^{-n} a^n - b^{-n}$ form an Alexander pair. The map $f_{\text{col}} : Q \to R[G]$ defined by $f_{\text{col}}(a) = a^n - 1$ is a column relation map, and the map $f_{\text{row}} : Y \times Q \to R[G]$ defined by $f_{\text{row}}(y, a) = y^{-1}$ is a row relation map.

Example 3.7. Let G be a group, and let $Q := \operatorname{Core} G$. Let $R[G]$ be the group ring of G over a commutative ring R. Let Y be the Q-set $Q \times R[G]^\times$ with $(y, z) \lhd a := (ay^{-1}a, -ay^{-1}z)$. The maps $f_1, f_2 : Q \times Q \to R[G]$ defined by $f_1(a, b) = -ba^{-1}$ and $f_2(a, b) = ba^{-1} + 1$ form an Alexander pair. The map $f_{\text{col},c} : Q \to R[G]$ defined by $f_{\text{col},c}(a) = ac + 1$ is a column relation map for $c \in Q$, and the map $f_{\text{row}} : Y \times Q \to R[G]$ defined by $f_{\text{row}}((y, z), a) = -z^{-1}(ya^{-1} + 1)$ is a row relation map.

Example 3.8. Let $Q := R_n$. Let Z be a commutative ring, and let $R := Z[t]/(P)$, where P is a factor of $t^n - 1$ in $Z[t]$. Let Y be the Q-set $Q \times R^\times$ with $(y, z) \lhd a := (2a - y, -t^{a-y} z)$. The maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = -t^{b-a}$ and $f_2(a, b) = t^{b-a} + 1$ form an Alexander pair. The map $f_{\text{col},c} : Q \to R$ defined by $f_{\text{col},c}(a) = t^{a+c} + 1$ is a column relation map for $c \in Q$, and the map $f_{\text{row}} : Y \times Q \to R$ defined by $f_{\text{row}}((y, z), a) = -z^{-1}(t^{y-a} + 1)$ is a row relation map.

FIGURE 2

4. THE NORMALIZED QUANDLE TWISTED ALEXANDER INVARIANT

Hereafter, we assume that link diagrams satisfy the condition that every component has at least one undercrossing, and we label the arc starting from a crossing c_i as x_i (see the left picture of Figure 2). It is easy to see that two diagrams satisfying this condition represent the same link if and only if they are related by a finite sequence of Reidemeister moves on link diagrams that satisfy the condition.

Let L be an oriented link, and let D be a diagram of L with n crossings $c_1, \ldots, c_n$. We note again that x_i denotes the arc starting from a crossing c_i for each i. Then, $C(D) = \{c_1, \ldots, c_n\}$ and $\mathcal{A}(D) = \{x_1, \ldots, x_n\}$. We denote by u_i, w_i and v_i the under-arcs and over-arc, respectively, of a crossing c_i such that the normal orientation of v_i points from u_i to w_i (see the right picture of Figure 2). We denote by $\mathrm{sgn}(c)$ the sign of a crossing c. We define $\mathrm{wr}(D) := \sum_{c \in C(D)} \mathrm{sgn}(c)$.

Let Q be a quandle and R a unital ring. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$, and let $\rho : Q(L) \to Q$ be a quandle representation. The (f_1, f_2)*-twisted Alexander matrix* $A(D, \rho; f_1, f_2)$ *of* (D, ρ) is the $n \times n$ matrix whose (i, j)-entry is

$$\delta(u_i, x_j) f_1(a_i, b_i) + \delta(v_i, x_j) f_2(a_i, b_i) - \delta(w_i, x_j),$$

where $a_i = \rho(u_i)$, $b_i = \rho(v_i)$, and

$$\delta(x, y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

The (f_1, f_2)-twisted Alexander matrix of the diagram depicted in Figure 3 is

$$\begin{pmatrix} -1 & f_2(\rho(x_3), \rho(x_2)) & f_1(\rho(x_3), \rho(x_2)) \\ f_1(\rho(x_1), \rho(x_3)) & -1 & f_2(\rho(x_1), \rho(x_3)) \\ f_2(\rho(x_2), \rho(x_1)) & f_1(\rho(x_2), \rho(x_1)) & -1 \end{pmatrix}.$$

Remark 4.1. Let $\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle$ be the Wirtinger presentation of $Q(L)$ with respect to D, where r_i is the relation $u_i \lhd v_i = w_i$. In [12], for an Alexander pair $f = (f_1, f_2)$, we introduced the notion of an f-derivative $\frac{\partial_f}{\partial x_j}$. Then the (f_1, f_2)-twisted Alexander matrix $A(D, \rho; f_1, f_2)$ coincides with $\left(\frac{\partial_{f \circ \rho^2}}{\partial x_j}(r_i) \right)$, where $f \circ \rho^2 = (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$.

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let D be a diagram of L. Let $D(K_i)$ be the diagram of K_i that is obtained by removing the other components from D. We denote by $\mathcal{A}(D; K_i)$ the set of arcs of D that originate from K_i, and denote by $C(D; K_i)$ the set of crossings of D whose under arcs originate from K_i. We define $\mathrm{wr}(D; K_i) := \sum_{c \in C(D; K_i)} \mathrm{sgn}(c)$. We then have $\mathrm{wr}(D; K_i) = \mathrm{wr}(D(K_i)) + \mathrm{lk}(K_i, L - K_i)$ and $\mathrm{wr}(D) = \sum_{i=1}^{r} \mathrm{wr}(D; K_i)$. We denote by $C_+(D)$ and $C_-(D)$ the sets of positive and negative crossings of D, respectively. We denote by $\#S$ the number of elements of a set S.

Definition 4.2. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$, and let $\rho : Q(L) \to Q$ be a quandle representation. We fix $\omega_1, \ldots, \omega_r \in R^\times$ so

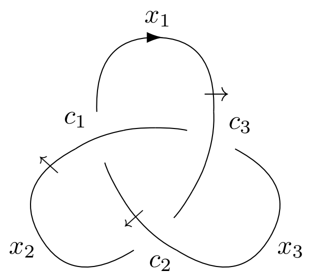

FIGURE 3

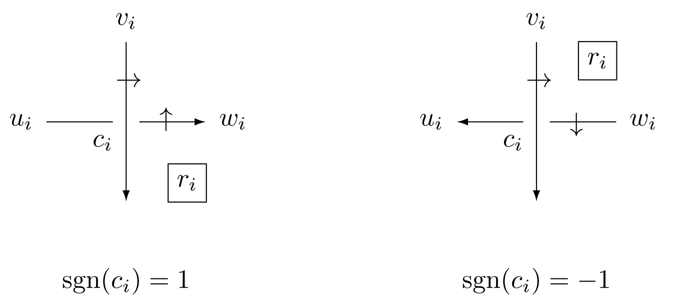

FIGURE 4

that $\omega_i = f_1(\rho(\alpha), \rho(\alpha))$ for some $\alpha \in \mathcal{A}(D; K_i)$. We define the *correction value* $\mathrm{cor}(D, \rho; f_1, f_2)$ of (D, ρ) by

$$\mathrm{cor}(D, \rho; f_1, f_2) = (-1)^{\#C_+(D)} \prod_{i=1}^{r} \omega_i^{\frac{\mathrm{rot}(D(K_i))+\mathrm{wr}(D(K_i))+1}{2}} \prod_{c \in C_-(D)} f_1(\rho(u_c), \rho(v_c)),$$

where $\mathrm{rot}(D(K_i))$ is the rotation number of $D(K_i)$.

We remark that $\mathrm{cor}(D, \rho; f_1, f_2) \in R^\times$ because $\mathrm{rot}(D(K_i)) + \mathrm{wr}(D(K_i)) + 1$ is always even. For the diagram depicted in Figure 3, we have

$$\mathrm{cor}(D, \rho; f_1, f_2) = (-1)^3 \omega_1^{\frac{-2+3+1}{2}} = -\omega_1,$$

as $\#C_+(D) = 3$, $C_-(D) = \emptyset$, $\mathrm{rot}(D) = -2$ and $\mathrm{wr}(D) = 3$.

We define

$$\widetilde{A}(D, \rho; f_1, f_2) := \begin{pmatrix} A(D, \rho; f_1, f_2) & \mathbf{0} \\ \mathbf{0} & \mathrm{cor}(D, \rho; f_1, f_2)^{-1} \end{pmatrix}.$$

We call $\widetilde{A}(D, \rho; f_1, f_2)$ *the normalized* (f_1, f_2)*-twisted Alexander matrix of* (D, ρ).

For column relation maps $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m} : Q \to R$, we define

$$R_{\mathrm{col}}(D, \rho; f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m}) := \begin{pmatrix} f_{\mathrm{col},1}(\rho(x_1)) & \cdots & f_{\mathrm{col},m}(\rho(x_1)) \\ \vdots & \ddots & \vdots \\ f_{\mathrm{col},1}(\rho(x_n)) & \cdots & f_{\mathrm{col},m}(\rho(x_n)) \end{pmatrix}.$$

We denote $R_{\mathrm{col}}(D, \rho; f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m})$ by $R_{\mathrm{col}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})$ for short.

Proposition 4.3 ([11, Proposition 5.1])**.** *For column relation maps* $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m} : Q \to R$*, the matrix* $R_{\mathrm{col}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})$ *is a column relation matrix of* $A(D, \rho; f_1, f_2)$*.*

We define $r_i := r(\alpha(w_i; c_i))$, where $\alpha(w_i; c_i)$ is the semi-arc that originates from the arc w_i and is incident to the crossing c_i (see Figure 4). For row relation maps $f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m} : \mathrm{As}\, Q \times Q \to R$, we define

$$R_{\mathrm{row}}(D, \rho; f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m}) := \begin{pmatrix} \mathrm{sgn}(c_1) f_{\mathrm{row},1}(\widetilde{\rho}(r_1), \rho(w_1)) & \cdots & \mathrm{sgn}(c_n) f_{\mathrm{row},1}(\widetilde{\rho}(r_n), \rho(w_n)) \\ \vdots & \ddots & \vdots \\ \mathrm{sgn}(c_1) f_{\mathrm{row},m}(\widetilde{\rho}(r_1), \rho(w_1)) & \cdots & \mathrm{sgn}(c_n) f_{\mathrm{row},m}(\widetilde{\rho}(r_n), \rho(w_n)) \end{pmatrix}.$$

We denote $R_{\mathrm{row}}(D, \rho; f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m})$ by $R_{\mathrm{row}}(D, \rho; \boldsymbol{f_{\mathrm{row}}})$ for short.

Proposition 4.4 ([10, Theorem 6.1])**.** *The matrix $R_{\mathrm{row}}(D, \rho; \boldsymbol{f_{\mathrm{row}}})$ is a row relation matrix of $A(D, \rho; f_1, f_2)$.*

We set

$$\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f_{\mathrm{row}}}) := \begin{pmatrix} R_{\mathrm{row}}(D, \rho; \boldsymbol{f_{\mathrm{row}}}) & \mathbf{0} \end{pmatrix},$$
$$\widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f_{\mathrm{col}}}) := \begin{pmatrix} R_{\mathrm{col}}(D, \rho; \boldsymbol{f_{\mathrm{col}}}) \\ \mathbf{0} \end{pmatrix}.$$

Remark 4.5. The matrix $\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f_{\mathrm{row}}})$ (resp. $\widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})$) is regular if and only if $R_{\mathrm{row}}(D, \rho; \boldsymbol{f_{\mathrm{row}}})$ (resp. $R_{\mathrm{col}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})$) is regular.

Remark 4.6. By Propositions 2.6 and 2.7 (5), the equivalence class of the triple

$$(\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f_{\mathrm{row}}}), \widetilde{A}(D, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f_{\mathrm{col}}}))$$

does not depend on the choice of the order of crossings and that of $\omega_1, \ldots, \omega_r \in R^\times$, since we have

$$\begin{aligned} f_1(\rho(w_c), \rho(w_c)) &= f_1(\rho(u_c) \triangleleft \rho(v_c), \rho(u_c) \triangleleft \rho(v_c)) \\ &= f_1(\rho(u_c), \rho(v_c)) f_1(\rho(u_c), \rho(u_c)) f_1(\rho(u_c), \rho(v_c))^{-1}. \end{aligned}$$

Theorem 4.7. *Let Q be a quandle and R a unital ring. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. Let $f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m} : \mathrm{As}\, Q \times Q \to R$ be row relation maps, and let $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m} : Q \to R$ be column relation maps. Let D_1, D_2 be diagrams of an oriented link L, and let $\rho : Q(L) \to Q$ be a quandle representation. Then we have*

$$\begin{aligned} &(\widetilde{R_{\mathrm{row}}}(D_1, \rho; \boldsymbol{f_{\mathrm{row}}}), \widetilde{A}(D_1, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D_1, \rho; \boldsymbol{f_{\mathrm{col}}})) \\ &\sim (\widetilde{R_{\mathrm{row}}}(D_2, \rho; \boldsymbol{f_{\mathrm{row}}}), \widetilde{A}(D_2, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D_2, \rho; \boldsymbol{f_{\mathrm{col}}})). \end{aligned}$$

This theorem is proven by verifying the invariance of the triple under each Reidemeister move. We postpone the proof to Section 9.

When R is a field, we define

$$\Delta(L, \rho; f_1, f_2; \boldsymbol{f_{\mathrm{row}}}; \boldsymbol{f_{\mathrm{col}}}) := \Delta(\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f_{\mathrm{row}}}), \widetilde{A}(D, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})).$$

When R is a matrix ring over a field, we define

$$\Delta(L, \rho; f_1, f_2; \boldsymbol{f_{\mathrm{row}}}; \boldsymbol{f_{\mathrm{col}}}) := \Delta\left(\overline{\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f_{\mathrm{row}}})}, \overline{\widetilde{A}(D, \rho; f_1, f_2)}, \overline{\widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})}\right).$$

When R is a commutative ring, we define

$$\Delta(L, \rho; f_1, f_2; \emptyset; \emptyset) := \Delta(\emptyset, \widetilde{A}(D, \rho; f_1, f_2), \emptyset).$$

When R is a matrix ring over a Euclidean domain, we define

$$\Delta(L, \rho; f_1, f_2; \emptyset; \emptyset) := \Delta\left(\emptyset, \overline{\widetilde{A}(D, \rho; f_1, f_2)}, \emptyset\right).$$

From Theorems 2.12, 2.13, 4.7 and Propositions 2.14, 2.15, these are invariants of (L, ρ). We remark that the invariants $\Delta(L, \rho; f_1, f_2; \boldsymbol{f_{\mathrm{row}}}; \boldsymbol{f_{\mathrm{col}}})$ for $R = Z$ and

$R = M(1,1;Z)$ coincide. Hereafter, we regard $M(1,1;Z)$ as the ring Z. When we regard $M(1,1;Z)$ as Z, we identify a matrix $A \in M(m,n;M(1,1;Z))$ with $\overline{A} \in M(m,n;Z)$. For $x \in Z$, $\det x$ stands for x.

We end this section with the following proposition, which is useful in the calculation of $\Delta(L,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{row}};\boldsymbol{f}_{\mathbf{col}})$.

Proposition 4.8. *Let D be a diagram of L with n crossings. Let F be a field. Setting $d := n-m$ and $R = M(k,k;F)$, we have*

$$\Delta(L,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{row}};\boldsymbol{f}_{\mathbf{col}}) = \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det \overline{A(D,\rho;f_1,f_2)}_{\sigma(\overline{kd}+km),\tau(\overline{kd}+km)} \det \operatorname{cor}(D,\rho;f_1,f_2)^{-1}}{\det \overline{R_{\mathrm{row}}(D,\rho;\boldsymbol{f}_{\mathbf{row}})}_{\overline{km},\sigma(\overline{km})} \det \overline{R_{\mathrm{col}}(D,\rho;\boldsymbol{f}_{\mathbf{col}})}_{\tau(\overline{km}),\overline{km}}}$$

for any $\sigma,\tau \in S_{kn}$ such that $\overline{R_{\mathrm{row}}(D,\rho;\boldsymbol{f}_{\mathbf{row}})}_{\overline{km},\sigma(\overline{km})}$ and $\overline{R_{\mathrm{col}}(D,\rho;\boldsymbol{f}_{\mathbf{col}})}_{\tau(\overline{km}),\overline{km}}$ are invertible.

Proof. Setting $\widetilde{\sigma} := \sigma \oplus 1_{S_k}$ and $\widetilde{\tau} := \tau \oplus 1_{S_k}$, we have

$$\begin{aligned}
&\Delta(L,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{row}};\boldsymbol{f}_{\mathbf{col}}) \\
&= \frac{\operatorname{sgn}\widetilde{\sigma} \operatorname{sgn}\widetilde{\tau} \det \overline{\widetilde{A}(D,\rho;f_1,f_2)}_{\widetilde{\sigma}(\overline{kd+k}+km),\widetilde{\tau}(\overline{kd+k}+km)}}{\det \overline{\widetilde{R_{\mathrm{row}}}(D,\rho;\boldsymbol{f}_{\mathbf{row}})}_{\overline{km},\widetilde{\sigma}(\overline{km})} \det \overline{\widetilde{R_{\mathrm{col}}}(D,\rho;\boldsymbol{f}_{\mathbf{col}})}_{\widetilde{\tau}(\overline{km}),\overline{km}}} \\
&= \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det \overline{A(D,\rho;f_1,f_2)}_{\sigma(\overline{kd}+km),\tau(\overline{kd}+km)} \det \operatorname{cor}(D,\rho;f_1,f_2)^{-1}}{\det \overline{R_{\mathrm{row}}(D,\rho;\boldsymbol{f}_{\mathbf{row}})}_{\overline{km},\sigma(\overline{km})} \det \overline{R_{\mathrm{col}}(D,\rho;\boldsymbol{f}_{\mathbf{col}})}_{\tau(\overline{km}),\overline{km}}},
\end{aligned}$$

where we remark that $\operatorname{sgn}\widetilde{\sigma} = \operatorname{sgn}\sigma$ and $\operatorname{sgn}\widetilde{\tau} = \operatorname{sgn}\tau$. □

We note that the equality in Proposition 4.8 implies

$$\Delta(L,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{row}};\boldsymbol{f}_{\mathbf{col}}) = \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det A(D,\rho;f_1,f_2)_{\sigma(\overline{d}+m),\tau(\overline{d}+m)} \operatorname{cor}(D,\rho;f_1,f_2)^{-1}}{\det R_{\mathrm{row}}(D,\rho;\boldsymbol{f}_{\mathbf{row}})_{\overline{m},\sigma(\overline{m})} \det R_{\mathrm{col}}(D,\rho;\boldsymbol{f}_{\mathbf{col}})_{\tau(\overline{m}),\overline{m}}}$$

when $k = 1$.

5. The quandle cocycle invariant

We recall the definition of a quandle cocycle invariant introduced in [3]: Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link and D a diagram of L. Let Q be a quandle and A an abelian group. A quandle 2-cocycle $\phi : Q \times Q \to A$ is a map satisfying $\phi(a,a) = 0$ and $\phi(a \triangleleft b, c) + \phi(a,b) = \phi(a \triangleleft c, b \triangleleft c) + \phi(a,c)$ for $a,b,c \in Q$. The *quandle cocycle invariant* $\Phi(L;\phi)$ of L is the multiset

$$\Phi(L;\phi) = \{\Phi(L,\rho;\phi) \mid \rho \in \mathrm{Col}_Q(D)\},$$

where

$$\Phi(L,\rho;\phi) := \sum_{c \in C(D)} \operatorname{sgn}(c)\phi(\rho(u_c),\rho(v_c)),$$

which we call the *quandle cocycle invariant* of (L,ρ). Here, we recall that u_c, v_c are the arcs around c (see Figure 1). We define

$$\Phi((L,K_i),\rho;\phi) := \sum_{c \in C(D;K_i)} \operatorname{sgn}(c)\phi(\rho(u_c),\rho(v_c)).$$

We then have

$$\Phi(L,\rho;\phi) = \sum_{i=1}^{r} \Phi((L,K_i),\rho;\phi).$$

Proposition 5.1. *Let $(Q, \lhd)$ be a quandle, and let $\phi : Q \times Q \to A$ be a quandle 2-cocycle, where $A = \mathbb{Z}$ or $\mathbb{Z}_p$. Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let $\rho : Q(L) \to Q$ be a quandle representation. Let $\widetilde{Q} := Q \times \{1, \ldots, r\}$ be the quandle with $(a, i) \lhd (b, j) = (a \lhd b, i)$. Set*

$$R := \begin{cases} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}] & \text{if } A = \mathbb{Z}, \\ \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]/(t_1^p - 1, \ldots, t_r^p - 1) & \text{if } A = \mathbb{Z}_p. \end{cases}$$

We define the Alexander pair of maps $f_1, f_2 : \widetilde{Q} \times \widetilde{Q} \to R$ by

$$f_1((a, i), (b, j)) = t_i^{\phi(a,b)}, \qquad f_2((a, i), (b, j)) = 0.$$

Then we have

$$\Delta(L, \rho; f_1, f_2; \emptyset; \emptyset) = \prod_{i=1}^{r} (1 - t_i^{\Phi((L,K_i),\rho;\phi)}).$$

In particular, for a knot K, we have

$$\Delta(K, \rho; f_1, f_2; \emptyset; \emptyset) = 1 - t_1^{\Phi(K,\rho;\phi)}.$$

Proof. Let D be a diagram of L. Set $n_i := \#C(D; K_i)$, which coincides with the number of the arcs of K_i. We define $[i] := n_1 + \cdots + n_i$ and $[0] := 0$. Let $c_{[i-1]+1}, \ldots, c_{[i]}$ be the crossings of $C(D; K_i)$ for $i = 1, \ldots, r$. We assume that the terminal point of x_i is c_{i+1} if $i \notin \{[1], [2], \ldots, [r]\}$, and $c_{[k-1]+1}$ if $i = [k]$. Put $c_{i,j} := c_{[i-1]+j}$, $u_{i,j} := u_{[i-1]+j}$, $v_{i,j} := v_{[i-1]+j}$, and $\varepsilon_{i,j} := \operatorname{sgn}(c_{i,j})$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$. We define $A((D; K_i), \rho; f_1, f_2)$ to be

$$\begin{pmatrix} \phi_{i,1}(-\varepsilon_{i,1}) & & & & & \phi_{i,1}(\varepsilon_{i,1}) \\ \phi_{i,2}(\varepsilon_{i,2}) & \phi_{i,2}(-\varepsilon_{i,2}) & & & & \\ & \phi_{i,3}(\varepsilon_{i,3}) & \phi_{i,3}(-\varepsilon_{i,3}) & & & \\ & & \ddots & \ddots & & \\ & & & & \phi_{i,n_i}(\varepsilon_{i,n_i}) & \phi_{i,n_i}(-\varepsilon_{i,n_i}) \end{pmatrix},$$

where

$$\phi_{i,j}(\varepsilon) := \begin{cases} t_i^{\phi(\rho(u_{i,j}),\rho(v_{i,j}))} & \text{if } \varepsilon = 1, \\ -1 & \text{if } \varepsilon = -1. \end{cases}$$

The determinant of $A((D; K_i), \rho; f_1, f_2)$ is

$$\begin{vmatrix} -1 & & & & \phi_{i,1}(1)^{\varepsilon_{i,1}} \\ \phi_{i,2}(1)^{\varepsilon_{i,2}} & -1 & & & \\ & \phi_{i,3}(1)^{\varepsilon_{i,3}} & -1 & & \\ & & \ddots & \ddots & \\ & & & \phi_{i,n_i}(1)^{\varepsilon_{i,n_i}} & -1 \end{vmatrix} \prod_{c \in C_-(D;K_i)} (-t_i^{\phi(\rho(u_c),\rho(v_c))}),$$

which is

$$(1 - t_i^{\Phi((L,K_i),\rho;\phi)})(-1)^{n_i} \prod_{c \in C_-(D;K_i)} (-t_i^{\phi(\rho(u_c),\rho(v_c))}),$$

where $C_-(D; K_i) = C(D; K_i) \cap C_-(D)$. We define

$$\operatorname{cor}((D; K_i), \rho; f_1, f_2) := (-1)^{\#C_+(D;K_i)} \prod_{c \in C_-(D;K_i)} t_i^{\phi(\rho(u_c),\rho(v_c))},$$

where $C_+(D; K_i) = C(D; K_i) \cap C_+(D)$. We then have

$$\det A((D; K_i), \rho; f_1, f_2) \operatorname{cor}((D; K_i), \rho; f_1, f_2)^{-1} = 1 - t_i^{\Phi((L,K_i),\rho;\phi)}.$$

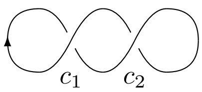

FIGURE 5

Since

$$\begin{aligned} A(D,\rho;f_1,f_2) &= A((D;K_1),\rho;f_1,f_2)\oplus\cdots\oplus A((D;K_r),\rho;f_1,f_2),\\ \operatorname{cor}(D,\rho;f_1,f_2) &= \prod_{i=1}^{r}\operatorname{cor}((D;K_i),\rho;f_1,f_2),\end{aligned}$$

we have

$$\Delta(L,\rho;f_1,f_2;\emptyset;\emptyset)=\prod_{i=1}^{r}(1-t_i^{\Phi((L,K_i),\rho;\phi)}).$$

□

6. Properties

In this section, we determine our invariant for the trivial knot and the mirror image of an oriented link.

For $\sigma\in S_n$, we define $\overline{\sigma}\in S_{kn}$ by

$$\overline{\sigma}((a-1)k+b)=(\sigma(a)-1)k+b$$

for $a=1,\dots,n$ and $b=1,\dots,k$, where we note that $\operatorname{sgn}\overline{\sigma}=(\operatorname{sgn}\sigma)^k$. For example, when $\sigma=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\in S_3$ and $k=2$, we have

$$\overline{\sigma}=\begin{pmatrix}1&2&3&4&5&6\\3&4&1&2&5&6\end{pmatrix}\in S_6 \qquad\text{and}\qquad \operatorname{sgn}\overline{\sigma}=1.$$

Proposition 6.1. *Let Q be a quandle and let $R:=M(k,k;F)$, where F is a field. Let (f_1,f_2) be an Alexander pair of maps $f_1,f_2:Q\times Q\to R$. Let $f_{\text{row}}:\operatorname{As}Q\times Q\to R$ and $f_{\text{col}}:Q\to R$ be row and column relation maps, respectively. Let O be the trivial knot, and let $\rho:Q(O)\to Q$ be a quandle representation. Let $a\in\operatorname{Im}\rho$. If $f_{\text{row}}(1,a),f_{\text{col}}(a)\in R^\times$, then we have*

$$\Delta(O,\rho;f_1,f_2;f_{\text{row}};f_{\text{col}})=\frac{(-1)^k\det f_1(a,a)^{-1}}{\det f_{\text{row}}(1,a)\det f_{\text{col}}(a)}.$$

Proof. Let D be the diagram of O depicted in Figure 5. We have

$$\begin{aligned} A(D,\rho;f_1,f_2) &= \begin{pmatrix}-1 & f_1(a,a)+f_2(a,a)\\ f_1(a,a)+f_2(a,a) & -1\end{pmatrix}=\begin{pmatrix}-1&1\\1&-1\end{pmatrix},\\ \operatorname{cor}(D,\rho;f_1,f_2) &= (-1)^2\cdot f_1(a,a)^{\frac{-1+2+1}{2}}=f_1(a,a),\\ R_{\text{row}}(D,\rho;f_{\text{row}}) &= \begin{pmatrix}f_{\text{row}}(1,a) & f_{\text{row}}(1,a)\end{pmatrix},\\ R_{\text{col}}(D,\rho;f_{\text{col}}) &= \begin{pmatrix}f_{\text{col}}(a)\\ f_{\text{col}}(a)\end{pmatrix}.\end{aligned}$$

Since $f_{\mathrm{row}}(1,a), f_{\mathrm{col}}(a) \in R^{\times}$, the matrices $R_{\mathrm{row}}(D,\rho;f_{\mathrm{row}})$ and $R_{\mathrm{col}}(D,\rho;f_{\mathrm{col}})$ are regular. Setting $\sigma = 1_{S_2}$, we have $\overline{\sigma} = 1_{S_{2k}}$. It follows that

$$\begin{aligned}
&\Delta(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}}) \\
&= \frac{\operatorname{sgn}\overline{\sigma}\operatorname{sgn}\overline{\sigma}\det \overline{A(D,\rho;f_1,f_2)}_{\overline{\sigma}(\overline{k}+k),\overline{\sigma}(\overline{k}+k)}\det \operatorname{cor}(D,\rho;f_1,f_2)^{-1}}{\det \overline{R_{\mathrm{row}}(D,\rho;f_{\mathrm{row}})}_{\overline{k},\overline{\sigma}(\overline{k})}\det \overline{R_{\mathrm{col}}(D,\rho;f_{\mathrm{col}})}_{\overline{\sigma}(\overline{k}),\overline{k}}} \\
&= \frac{\det \overline{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_{(k+1,\dots,2k),(k+1,\dots,2k)} \det f_1(a,a)^{-1}}{\det \overline{\begin{pmatrix} f_{\mathrm{row}}(1,a) & f_{\mathrm{row}}(1,a)\end{pmatrix}}_{(1,\dots,k),(1,\dots,k)} \det \overline{\begin{pmatrix} f_{\mathrm{col}}(a) \\ f_{\mathrm{col}}(a)\end{pmatrix}}_{(1,\dots,k),(1,\dots,k)}} \\
&= \frac{(-1)^k \det f_1(a,a)^{-1}}{\det f_{\mathrm{row}}(1,a)\det f_{\mathrm{col}}(a)}.
\end{aligned}$$

□

Let L be an oriented link and D a diagram of L. Let Q be a quandle and A a multiplicative abelian group. For $\rho \in \mathrm{Col}_Q(D)$, we define

$$\Psi(L,\rho;\psi) := \prod_{c\in C(D)} \psi(\rho(u_c),\rho(v_c))^{\operatorname{sgn}(c)},$$

where $\psi : Q\times Q \to A$ is a quandle 2-cocycle, which satisfies $\psi(a,a) = 1$ and $\psi(a \triangleleft b, c)\psi(a,b) = \psi(a\triangleleft c, b\triangleleft c)\psi(a,c)$ for $a,b,c \in Q$. We then define the multiset

$$\Psi(L;\psi) := \{\Psi(L,\rho;\psi) \mid \rho \in \mathrm{Col}_Q(D)\},$$

which is the multiplicative version of the quandle cocycle invariant introduced in the previous section. For a map $f : Q\times Q \to A$ satisfying

$$f(a\triangleleft b, c)f(a,b) = f(a\triangleleft c, b\triangleleft c)f(a,c),$$

we define the quandle 2-cocycle $\psi(f) : Q\times Q\to A$ by

$$\psi(f)(a,b) = f(a,a)^{-1}f(a,b),$$

where we note that $f(a\triangleleft c, a\triangleleft c) = f(a,a)$.

Proposition 6.2. *Let $R := M(k,k;F)$, where F is a field. Let (f_1,f_2) be an Alexander pair of maps $f_1,f_2 : Q\times Q \to R$. Let $f_{\mathrm{row},1},\dots,f_{\mathrm{row},m} : \mathrm{As}\, Q\times Q \to R$ be row relation maps, and let $f_{\mathrm{col},1},\dots,f_{\mathrm{col},m} : Q \to R$ be column relation maps. Let $L = K_1\cup\cdots\cup K_r$ be an oriented r-component link, and let $-L^*$ be the mirror image of L with the orientation reversed. Let $\rho : Q(L)\to Q$ be a quandle representation, and let $\rho^* : Q(-L^*) \to Q$ be the quandle representation induced from ρ with the reflection. Let D be a diagram of L. We fix $\omega_1,\dots,\omega_r \in R^{\times}$ so that $\omega_i = f_1(\rho(\alpha),\rho(\alpha))$ for some $\alpha \in \mathcal{A}(D;K_i)$. Then we have*

$$\begin{aligned}
&\Delta(-L^*,\rho^*;f_1,f_2;f_{\mathrm{row},1},\dots,f_{\mathrm{row},m};f_{\mathrm{col},1},\dots,f_{\mathrm{col},m}) \\
&= \frac{(-1)^{k(r+m)}\Delta(L,\rho;f_1,f_2;f_{\mathrm{row},1},\dots,f_{\mathrm{row},m};f_{\mathrm{col},1},\dots,f_{\mathrm{col},m})}{\det(\prod_{i=1}^{r}\omega_i^{\mathrm{lk}(K_i,L-K_i)})\Psi(L,\rho;\psi(\det\circ f_1))},
\end{aligned}$$

where $\mathrm{lk}(K,\emptyset) = 0$ for a knot K.

Proof. Let $x_1,\dots,x_n$ be the arcs of D such that $x_{n_1+\cdots+n_{i-1}+1},\dots,x_{n_1+\cdots+n_i}$ are the arcs of K_i. We set

$$\begin{aligned}
\upsilon := &\begin{pmatrix}1 & 2 & \cdots & n_1\end{pmatrix}\begin{pmatrix}n_1+1 & n_1+2 & \cdots & n_1+n_2\end{pmatrix} \\
&\cdots\begin{pmatrix}n_1+\cdots+n_{r-1}+1 & n_1+\cdots+n_{r-1}+2 & \cdots & n\end{pmatrix} \in S_n
\end{aligned}$$

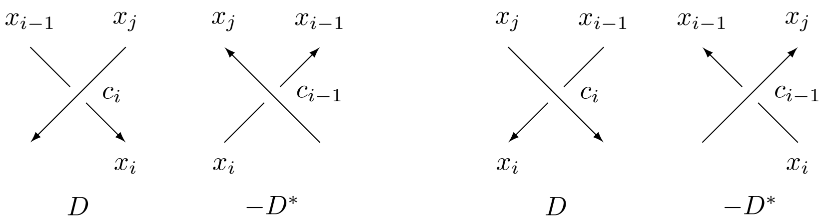

FIGURE 6

and assume that the terminal point of x_i is $c_{\upsilon(i)}$ for $i = 1, \dots, n$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the involution defined by $\varphi(x, y) = (-x, y)$. We then denote by $-D^*$ the diagram $\varphi(D)$ with the orientation reversed, which is a diagram of $-L^*$. For an arc x_i of D, we label the arc $\varphi(x_i)$ of $-D^*$ as x_i (see Figure 6). Since c_i is the crossing from which the arc x_i starts, we have $\varphi(c_i) = c_{\upsilon^{-1}(i)}$ for $i = 1, \dots, n$. From $\rho^* = \rho \circ \varphi^{-1}$, we have $\rho^*(x_i) = \rho(x_i)$. Set $d := n - m$. Because

$$\begin{aligned}
&A(-D^*, \rho^*; f_1, f_2) = A(D, \rho; f_1, f_2)_{\upsilon(\overline{n}),\overline{n}},\\
&R_{\mathrm{row}}(-D^*, \rho^*; \boldsymbol{f}_{\mathbf{row}}) = -R_{\mathrm{row}}(D, \rho; \boldsymbol{f}_{\mathbf{row}})_{\overline{m},\upsilon(\overline{n})},\\
&R_{\mathrm{col}}(-D^*, \rho^*; \boldsymbol{f}_{\mathbf{col}}) = R_{\mathrm{col}}(D, \rho; \boldsymbol{f}_{\mathbf{col}}),
\end{aligned}$$

we have

$$\begin{aligned}
A(-D^*, \rho^*; f_1, f_2)_{\sigma(\overline{d}+m),\tau(\overline{d}+m)} &= A(D, \rho; f_1, f_2)_{\sigma'(\overline{d}+m),\tau(\overline{d}+m)},\\
R_{\mathrm{row}}(-D^*, \rho^*; \boldsymbol{f}_{\mathbf{row}})_{\overline{m},\sigma(\overline{m})} &= -R_{\mathrm{row}}(D, \rho; \boldsymbol{f}_{\mathbf{row}})_{\overline{m},\sigma'(\overline{m})},\\
R_{\mathrm{col}}(-D^*, \rho^*; \boldsymbol{f}_{\mathbf{col}})_{\tau(\overline{m}),\overline{m}} &= R_{\mathrm{col}}(D, \rho; \boldsymbol{f}_{\mathbf{col}})_{\tau(\overline{m}),\overline{m}},\\
\operatorname{sgn}\sigma \operatorname{sgn}\tau &= (-1)^{r+n} \operatorname{sgn}\sigma' \operatorname{sgn}\tau,
\end{aligned}$$

where $\sigma' = \upsilon \circ \sigma$. Since

$$\begin{aligned}
&\#C_+(-D^*) = n - \#C_+(D), && \mathrm{rot}(-D^*(K_i)) = \mathrm{rot}(D(K_i)),\\
&\mathrm{wr}(-D^*(K_i)) = -\mathrm{wr}(D(K_i)), && C_-(-D^*) = C_+(D),
\end{aligned}$$

we have

$$\begin{aligned}
&\det \mathrm{cor}(-D^*, \rho^*; f_1, f_2)\\
&= (-1)^{k(n-\#C_+(D))} \prod_{i=1}^{r} \det \omega_i^{\frac{\mathrm{rot}(D(K_i)) - \mathrm{wr}(D(K_i)) + 1}{2}} \prod_{c \in C_+(D)} \det f_1(\rho(u_c), \rho(v_c)),\\
&= (-1)^{kn} \prod_{i=1}^{r} \det \omega_i^{-\mathrm{wr}(D(K_i))} \prod_{c \in C(D)} \det f_1(\rho(u_c), \rho(u_c))^{\mathrm{sgn}(c)}\\
&\quad \cdot \prod_{c \in C(D)} \bigl(\det f_1(\rho(u_c), \rho(u_c))^{-1} \det f_1(\rho(u_c), \rho(v_c))\bigr)^{\mathrm{sgn}(c)}\\
&\quad \cdot (-1)^{k\#C_+(D)} \prod_{i=1}^{r} \det \omega_i^{\frac{\mathrm{rot}(D(K_i)) + \mathrm{wr}(D(K_i)) + 1}{2}} \prod_{c \in C_-(D)} \det f_1(\rho(u_c), \rho(v_c))\\
&= (-1)^{kn} \prod_{i=1}^{r} \det \omega_i^{\mathrm{lk}(K_i, L - K_i)} \Psi(L, \rho; \psi(\det \circ f_1)) \det \mathrm{cor}(D, \rho; f_1, f_2),
\end{aligned}$$

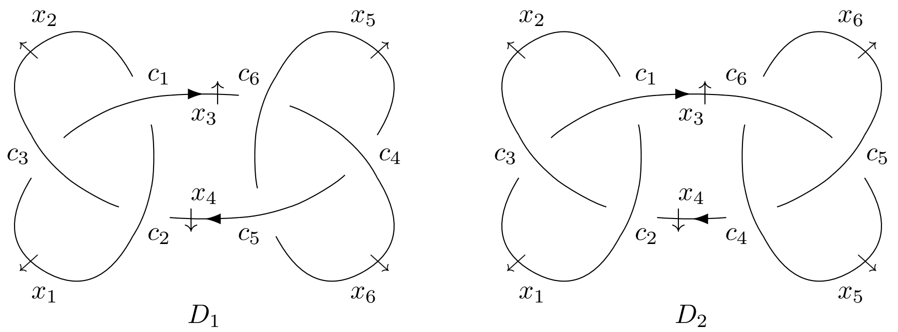

FIGURE 7. The granny knot and square knot

where the last equality follows from $\mathrm{wr}(D;K_i) = \mathrm{wr}(D(K_i)) + \mathrm{lk}(K_i, L - K_i)$. Consequently, we have

$$\begin{aligned}
&\Delta(-L^*, \rho^*; f_1, f_2; \boldsymbol{f}_{\mathbf{row}}; \boldsymbol{f}_{\mathbf{col}}) \\
&= \frac{\operatorname{sgn}\overline{\sigma}\operatorname{sgn}\overline{\tau}\det\overline{A(-D^*, \rho^*; f_1, f_2)}_{\overline{\sigma}(\overline{kd}+km),\overline{\tau}(\overline{kd}+km)}\det\operatorname{cor}(-D^*, \rho; f_1, f_2)^{-1}}{\det\overline{R_{\mathrm{row}}(-D^*, \rho^*; \boldsymbol{f}_{\mathbf{row}})}_{\overline{km},\overline{\sigma}(\overline{km})}\det\overline{R_{\mathrm{col}}(-D^*, \rho^*; \boldsymbol{f}_{\mathbf{col}})}_{\overline{\tau}(\overline{km}),\overline{km}}} \\
&= \frac{(-1)^{k(r+m)}}{\det(\prod_{i=1}^{r}\omega_i^{\mathrm{lk}(K_i, L-K_i)})\Psi(L, \rho; \psi(\det\circ f_1))} \\
&\qquad \cdot \frac{\operatorname{sgn}\overline{\sigma'}\operatorname{sgn}\overline{\tau}\det\overline{A(D, \rho; f_1, f_2)}_{\overline{\sigma'}(\overline{kd}+km),\overline{\tau}(\overline{kd}+km)}\det\operatorname{cor}(D, \rho; f_1, f_2)^{-1}}{\det\overline{R_{\mathrm{row}}(D, \rho; \boldsymbol{f}_{\mathbf{row}})}_{\overline{km},\overline{\sigma'}(\overline{km})}\det\overline{R_{\mathrm{col}}(D, \rho; \boldsymbol{f}_{\mathbf{col}})}_{\overline{\tau}(\overline{km}),\overline{km}}} \\
&= \frac{(-1)^{k(r+m)}\Delta(L, \rho; f_1, f_2; \boldsymbol{f}_{\mathbf{row}}; \boldsymbol{f}_{\mathbf{col}})}{\det(\prod_{i=1}^{r}\omega_i^{\mathrm{lk}(K_i, L-K_i)})\Psi(L, \rho; \psi(\det\circ f_1))}.
\end{aligned}$$

□

Example 6.3. Let $Q := R_3$ and $R := \mathbb{Q}[t]/(t^2+t+1)$, where we note that $t^3 = 1$ in R. Let Y be the Q-set $Q \times R^\times$ with $(y,z) \triangleleft a := (2a - y, -t^{a-y}z)$. Let (f_1, f_2) be the Alexander pair in Example 3.8, that is, $f_1(a,b) = -t^{b-a}$ and $f_2(a,b) = t^{b-a}+1$. Let f_{row} and $f_{\mathrm{col},c}$ ($c \in Q$) be the row and column relation maps in Example 3.8, that is, $f_{\mathrm{row}}((y,z),a) = -z^{-1}(t^{y-a}+1)$ and $f_{\mathrm{col},c}(a) = t^{a+c}+1$. For $c \in Q$, we define the row relation map $f_{\mathrm{row},c} : \mathrm{As}\, Q \times Q \to R$ by $f_{\mathrm{row},c}(y,a) = f_{\mathrm{row}}(\varphi_c(y), a)$, where $\varphi_c : \mathrm{As}\, Q \to Y$ is the Q-set homomorphism satisfying $\varphi_c(1) = (c, 1)$. Let K_1 and K_2 be the granny knot and square knot, respectively, that is, $K_1 = 3_1\#3_1$ and $K_2 = 3_1\#3_1^*$. Let D_1 and D_2 be respectively the oriented diagrams of K_1 and K_2 depicted in Figure 7. We note that K_1 and K_2 are invertible.

Let $\rho : Q(K_1) \to Q$ be a quandle representation defined by

$$\begin{aligned}
&\rho(x_1) = a, && \rho(x_2) = b, && \rho(x_3) = 2a+2b, \\
&\rho(x_4) = 2a+2b, && \rho(x_5) = c, && \rho(x_6) = a+b+2c,
\end{aligned}$$

where $a,b,c\in R_3$. We then have

$$A(D_1,\rho;f_1,f_2)$$
$$=\begin{pmatrix} -t^{a+2b} & -1 & t^{a+2b}+1 & 0 & 0 & 0\\ t^{a+2b}+1 & -t^{a+2b} & 0 & -1 & 0 & 0\\ -1 & t^{a+2b}+1 & -t^{a+2b} & 0 & 0 & 0\\ 0 & 0 & 0 & -t^{2a+2b+2c} & -1 & t^{2a+2b+2c}+1\\ 0 & 0 & 0 & t^{2a+2b+2c}+1 & -t^{2a+2b+2c} & -1\\ 0 & 0 & -1 & 0 & t^{2a+2b+2c}+1 & -t^{2a+2b+2c} \end{pmatrix},$$

$$R_{\mathrm{row}}(D_1,\rho;f_{\mathrm{row},0},f_{\mathrm{row},1})$$
$$=-\begin{pmatrix} t^{2b}+1 & t^{a+b}+1 & t^{2a}+1 & t^{2c}+1 & t^{2a+2b+c}+1 & t^{a+b}+1\\ t^{2b+1}+1 & t^{a+b+1}+1 & t^{2a+1}+1 & t^{2c+1}+1 & t^{2a+2b+c+1}+1 & t^{a+b+1}+1 \end{pmatrix},$$

$$R_{\mathrm{col}}(D_1,\rho;f_{\mathrm{col},0},f_{\mathrm{col},1})=\begin{pmatrix} t^{a}+1 & t^{a+1}+1\\ t^{b}+1 & t^{b+1}+1\\ t^{2a+2b}+1 & t^{2a+2b+1}+1\\ t^{2a+2b}+1 & t^{2a+2b+1}+1\\ t^{c}+1 & t^{c+1}+1\\ t^{a+b+2c}+1 & t^{a+b+2c+1}+1 \end{pmatrix},$$

$$\mathrm{cor}(D_1,\rho;f_1,f_2)=(-1)^0\cdot(-1)^{\frac{-3-6+1}{2}}\cdot(-t^{a+2b})^3(-t^{2a+2b+2c})^3=1.$$

When $a\neq b$, by setting $\sigma=1_{S_6}$, we obtain

$$\begin{aligned}
&\Delta(K_1,\rho;f_1,f_2;f_{\mathrm{row},0},f_{\mathrm{row},1};f_{\mathrm{col},0},f_{\mathrm{col},1})\\
&=\frac{\mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\sigma\det A(D_1,\rho;f_1,f_2)_{\sigma(\overline{4}+2),\sigma(\overline{4}+2)}\,\mathrm{cor}(D_1,\rho;f_1,f_2)^{-1}}{\det R_{\mathrm{row}}(D_1,\rho;f_{\mathrm{row},0},f_{\mathrm{row},1})_{\overline{2},\sigma(\overline{2})}\det R_{\mathrm{col}}(D_1,\rho;f_{\mathrm{col},0},f_{\mathrm{col},1})_{\sigma(\overline{2}),\overline{2}}}\\
&=\frac{\det\begin{pmatrix} -t^{a+2b} & 0 & 0 & 0\\ 0 & -t^{2a+2b+2c} & -1 & t^{2a+2b+2c}+1\\ 0 & t^{2a+2b+2c}+1 & -t^{2a+2b+2c} & -1\\ -1 & 0 & t^{2a+2b+2c}+1 & -t^{2a+2b+2c}\end{pmatrix}\cdot 1}{\det\begin{pmatrix} t^{2b}+1 & t^{a+b}+1\\ t^{2b+1}+1 & t^{a+b+1}+1\end{pmatrix}\det\begin{pmatrix} t^{a}+1 & t^{a+1}+1\\ t^{b}+1 & t^{b+1}+1\end{pmatrix}}\\
&=\frac{-t^{a+2b}(t^{2a+2b+2c}+t^{a+b+c}+1)}{-9t^{a+2b+1}}=\begin{cases}-(t+1)/3 & \text{if } \{a,b,c\}=\{0,1,2\},\\ 0 & \text{if } c=a\neq b \text{ or } c=b\neq a.\end{cases}
\end{aligned}$$

When $a=b\neq c$, by setting $\sigma=\begin{pmatrix}1&2&3&4&5&6\\1&5&2&3&4&6\end{pmatrix}$, we obtain

$$\begin{aligned}
&\Delta(K_1,\rho;f_1,f_2;f_{\mathrm{row},0},f_{\mathrm{row},1};f_{\mathrm{col},0},f_{\mathrm{col},1})\\
&=\frac{\mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\sigma\det A(D_1,\rho;f_1,f_2)_{\sigma(\overline{4}+2),\sigma(\overline{4}+2)}\,\mathrm{cor}(D_1,\rho;f_1,f_2)^{-1}}{\det R_{\mathrm{row}}(D_1,\rho;f_{\mathrm{row},0},f_{\mathrm{row},1})_{\overline{2},\sigma(\overline{2})}\det R_{\mathrm{col}}(D_1,\rho;f_{\mathrm{col},0},f_{\mathrm{col},1})_{\sigma(\overline{2}),\overline{2}}}\\
&=\frac{\det\begin{pmatrix} -t^{a+2b} & 0 & -1 & 0\\ t^{a+2b}+1 & -t^{a+2b} & 0 & 0\\ 0 & 0 & -t^{2a+2b+2c} & t^{2a+2b+2c}+1\\ 0 & -1 & 0 & -t^{2a+2b+2c}\end{pmatrix}\cdot 1}{\det\begin{pmatrix} t^{2b}+1 & t^{2a+2b+c}+1\\ t^{2b+1}+1 & t^{2a+2b+c+1}+1\end{pmatrix}\det\begin{pmatrix} t^{a}+1 & t^{a+1}+1\\ t^{c}+1 & t^{c+1}+1\end{pmatrix}}\\
&=\frac{3t^{2b+c}}{9t^{2b+c+1}}=-(t+1)/3.
\end{aligned}$$

Thus, we have

$$\begin{aligned}&\{\Delta(K_1,\rho;f_1,f_2;f_{\mathrm{row},0},f_{\mathrm{row},1};f_{\mathrm{col},0},f_{\mathrm{col},1})\mid\rho\text{ is nontrivial}\}\\&=\{0\ (12\text{ times}),\ -(t+1)/3\ (12\text{ times})\}.\end{aligned}$$

From Proposition 6.2, we have

$$\begin{aligned}&\{\Delta(-K_1^*,\rho;f_1,f_2;f_{\mathrm{row},0},f_{\mathrm{row},1};f_{\mathrm{col},0},f_{\mathrm{col},1})\mid\rho\text{ is nontrivial}\}\\&=\{0\ (12\text{ times}),\ (t+1)/3\ (12\text{ times})\},\end{aligned}$$

which implies $K_1\not\cong -K_1^*\cong K_1^*$. Since $K_2\cong K_2^*$, we can conclude $K_1\not\cong K_2$, which also follows from

$$\begin{aligned}&\{\Delta(K_2,\rho;f_1,f_2;f_{\mathrm{row},0},f_{\mathrm{row},1};f_{\mathrm{col},0},f_{\mathrm{col},1})\mid\rho\text{ is nontrivial}\}\\&=\{0\ (12\text{ times}),\ (t+1)/3\ (6\text{ times}),\ -(t+1)/3\ (6\text{ times})\}.\end{aligned}$$

7. The normalized (twisted) Alexander polynomial

In this section, we demonstrate how the Alexander–Conway polynomial and the normalized twisted Alexander polynomial introduced by Kitayama [15] can be obtained in our framework.

Let $L=K_1\cup\cdots\cup K_r$ be an oriented r-component link, and let D be a diagram of L. Let Q be a quandle and let $R:=M(k,k;F)$, where F is a field. Let (f_1,f_2) be an Alexander pair of maps $f_1,f_2:Q\times Q\to R$, and let $\rho:Q(L)\to Q$ be a quandle representation. We fix $\omega_1,\dots,\omega_r\in R^\times$ so that $\omega_i=f_1(\rho(\alpha),\rho(\alpha))$ for some $\alpha\in\mathcal{A}(D;K_i)$. Suppose that $(\det\omega_1)^{\frac12},\dots,(\det\omega_r)^{\frac12}\in F^\times$. We define

$$\begin{aligned}&\nabla(L,\rho;f_1,f_2;\boldsymbol{f_{\mathrm{row}}};\boldsymbol{f_{\mathrm{col}}})\\&:=\Delta(L,\rho;f_1,f_2;\boldsymbol{f_{\mathrm{row}}};\boldsymbol{f_{\mathrm{col}}})\prod_{i=1}^{r}(\det\omega_i)^{\frac{1-\mathrm{lk}(K_i,L-K_i)}{2}},\end{aligned}$$

where we note that $\det\omega_i$ does not depend on the choice of $\omega_i\in R^\times$. We then have

$$\begin{aligned}&\nabla(L,\rho;f_1,f_2;\boldsymbol{f_{\mathrm{row}}};\boldsymbol{f_{\mathrm{col}}})\\&=\frac{\operatorname{sgn}\sigma\operatorname{sgn}\tau\det\overline{A(D,\rho;f_1,f_2)}_{\sigma(\overline{kd}+km),\tau(\overline{kd}+km)}\operatorname{detcor}_\nabla(D,\rho;f_1,f_2)^{-1}}{\det\overline{R_{\mathrm{row}}(D,\rho;\boldsymbol{f_{\mathrm{row}}})}_{\overline{km},\sigma(\overline{km})}\det\overline{R_{\mathrm{col}}(D,\rho;\boldsymbol{f_{\mathrm{col}}})}_{\tau(\overline{km}),\overline{km}}},\end{aligned}$$

where

$$\begin{aligned}&\operatorname{detcor}_\nabla(D,\rho;f_1,f_2)\\&:=(-1)^{k\#C_+(D)}\prod_{i=1}^{r}(\det\omega_i)^{\frac{\mathrm{rot}(D(K_i))+\mathrm{wr}(D;K_i)}{2}}\prod_{c\in C_-(D)}\det f_1(\rho(u_c),\rho(v_c)).\end{aligned}$$

The Alexander–Conway polynomial $\nabla_L(z)$ of an oriented link L is characterized by the following:

- For the trivial knot O, we have $\nabla_O(z)=1$.
- Let D_+, D_- and D_0 be diagrams that are identical outside a disk where they are the tangles depicted in Figure 8. We call (D_+,D_-,D_0) a *skein triple*. Then the skein relation

  $$\nabla_{D_+}(z)-\nabla_{D_-}(z)=z\nabla_{D_0}(z)$$

  holds, where $\nabla_D(z)$ is the Alexander–Conway polynomial $\nabla_L(z)$ of an oriented link L represented by D.

The Alexander–Conway polynomial $\nabla_L(z)$ is a normalized Alexander polynomial.

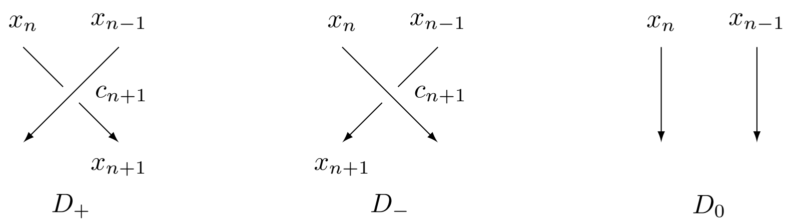

FIGURE 8

Proposition 7.1. *Let Q be a quandle and F a field. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to F$. Let $f_{\mathrm{row}} : \operatorname{As} Q \times Q \to F$ and $f_{\mathrm{col}} : Q \to F$ be row and column relation maps, respectively. Let L be an oriented link. Let $\rho : Q(L) \to Q$ be a trivial quandle representation, whose image is $\{a\}$. Set $t := f_1(a,a)^{-1}$. Suppose that $f_{\mathrm{row}}(1,a), f_{\mathrm{col}}(a), t^{1/2} \in F^\times$. Then we have*

$$\frac{\nabla_L(t^{1/2}-t^{-1/2})}{\nabla_O(t^{1/2}-t^{-1/2})} = \frac{\nabla(L,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})}{\nabla(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})},$$

where $\nabla_O(t^{1/2}-t^{-1/2}) = 1$ and

$$\nabla(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}}) = \frac{-t^{1/2}}{f_{\mathrm{row}}(1,a)f_{\mathrm{col}}(a)}.$$

Proof. Let (D_+, D_-, D_0) be a skein triple with n common crossings $c_1, \dots, c_n$ and the crossing c_{n+1} of D_+ and D_- depicted in Figure 8, and let L_+, L_- and L_0 be the oriented links represented by D_+, D_- and D_0, respectively. We use the same symbol ρ for the trivial quandle representations of $Q(L_+), Q(L_-)$ and $Q(L_0)$ to Q that send every element to a. We then have

$$\begin{aligned}
A(D_+,\rho;f_1,f_2) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1}+\boldsymbol{b}_n & \boldsymbol{a}_n & \boldsymbol{b}_{n-1} \\ \mathbf{0} & 1-t^{-1} & t^{-1} & -1 \end{pmatrix}, \\
A(D_-,\rho;f_1,f_2) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n+\boldsymbol{b}_{n-1} & \boldsymbol{b}_n \\ \mathbf{0} & -1 & 1-t^{-1} & t^{-1} \end{pmatrix}, \\
A(D_0,\rho;f_1,f_2) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1}+\boldsymbol{b}_{n-1} & \boldsymbol{a}_n+\boldsymbol{b}_n \end{pmatrix}.
\end{aligned}$$

The Laplace expansions for $\det(A(D_\pm,\rho;f_1,f_2)_{\overline{n+1},\overline{n+1}})$ along the last rows yield

$$\begin{aligned}
&\det(A(D_+,\rho;f_1,f_2)_{\overline{n+1},\overline{n+1}}) + \det(A(D_-,\rho;f_1,f_2)_{\overline{n+1},\overline{n+1}}) \\
&= (t^{-1}-1)\det(A(D_0,\rho;f_1,f_2)_{\overline{n-1}+1,\overline{n-1}+1}).
\end{aligned}$$

We have

$$\begin{aligned}
\operatorname{detcor}_\nabla(D_+,\rho;f_1,f_2) &= -t^{-1/2}\operatorname{detcor}_\nabla(D_0,\rho;f_1,f_2), \\
\operatorname{detcor}_\nabla(D_-,\rho;f_1,f_2) &= t^{-1/2}\operatorname{detcor}_\nabla(D_0,\rho;f_1,f_2),
\end{aligned}$$

since

$$\begin{aligned}
\operatorname{detcor}_\nabla(D,\rho;f_1,f_2) &= (-1)^{\#C_+(D)} t^{-\frac{\mathrm{rot}(D)+\mathrm{wr}(D)}{2}} t^{-\#C_-(D)} \\
&= (-1)^{\#C_+(D)} t^{-\frac{\mathrm{rot}(D)+\#C(D)}{2}}.
\end{aligned}$$

Since ρ is trivial, we have

$$R_{\mathrm{col}}(D_+,\rho;f_{\mathrm{col}}) = R_{\mathrm{col}}(D_-,\rho;f_{\mathrm{col}}) = \begin{pmatrix} f_{\mathrm{col}}(a) \\ \vdots \\ f_{\mathrm{col}}(a) \end{pmatrix}, \quad R_{\mathrm{col}}(D_0,\rho;f_{\mathrm{col}}) = \begin{pmatrix} f_{\mathrm{col}}(a) \\ \vdots \\ f_{\mathrm{col}}(a) \end{pmatrix}$$

and

$$\begin{aligned}
&R_{\mathrm{row}}(D_+,\rho;f_{\mathrm{row}})\\
&=\bigl(\mathrm{sgn}(c_1)f_{\mathrm{row}}(1\lhd^{i_1}a,a)\quad\cdots\quad \mathrm{sgn}(c_n)f_{\mathrm{row}}(1\lhd^{i_n}a,a)\quad f_{\mathrm{row}}(1\lhd^{i_{n+1}}a,a)\bigr),\\
&R_{\mathrm{row}}(D_-,\rho;f_{\mathrm{row}})\\
&=\bigl(\mathrm{sgn}(c_1)f_{\mathrm{row}}(1\lhd^{i_1}a,a)\quad\cdots\quad \mathrm{sgn}(c_n)f_{\mathrm{row}}(1\lhd^{i_n}a,a)\quad -f_{\mathrm{row}}(1\lhd^{i'_{n+1}}a,a)\bigr),\\
&R_{\mathrm{row}}(D_0,\rho;f_{\mathrm{row}})\\
&=\bigl(\mathrm{sgn}(c_1)f_{\mathrm{row}}(1\lhd^{i_1}a,a)\quad\cdots\quad \mathrm{sgn}(c_n)f_{\mathrm{row}}(1\lhd^{i_n}a,a)\bigr)
\end{aligned}$$

for some $i_1,\dots,i_n,i_{n+1},i'_{n+1}\in\mathbb{Z}$. The equality $f_{\mathrm{row}}(y,a)=f_{\mathrm{row}}(y\lhd a,a)f_1(a,a)$ implies $f_{\mathrm{row}}(1\lhd^{i_1}a,a)\in F^\times$. Put $d:=\mathrm{sgn}(c_1)f_{\mathrm{row}}(1\lhd^{i_1}a,a)f_{\mathrm{col}}(a)\in F^\times$. It follows that

$$\begin{aligned}
&\nabla(L_+,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})-\nabla(L_-,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})\\
&=-t^{1/2}\,\mathrm{detcor}_\nabla(D_0,\rho;f_1,f_2)^{-1}\det(A(D_+,\rho;f_1,f_2)_{\overline{n+1},\overline{n+1}})d^{-1}\\
&\quad-t^{1/2}\,\mathrm{detcor}_\nabla(D_0,\rho;f_1,f_2)^{-1}\det(A(D_-,\rho;f_1,f_2)_{\overline{n+1},\overline{n+1}})d^{-1}\\
&=(t^{1/2}-t^{-1/2})\,\mathrm{detcor}_\nabla(D_0,\rho;f_1,f_2)^{-1}\det(A(D_0,\rho;f_1,f_2)_{\overline{n-1}+1,\overline{n-1}+1})d^{-1}\\
&=(t^{1/2}-t^{-1/2})\nabla(L_0,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}}).
\end{aligned}$$

Since $\dfrac{\nabla(L,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})}{\nabla(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})}$ satisfies the conditions characterizing the Alexander–Conway polynomial $\nabla_L(t^{1/2}-t^{-1/2})$, they coincide. Since $\nabla_O(t^{1/2}-t^{-1/2})=1$, we have the equality

$$\frac{\nabla_L(t^{1/2}-t^{-1/2})}{\nabla_O(t^{1/2}-t^{-1/2})}=\frac{\nabla(L,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})}{\nabla(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})}.$$

From Proposition 6.1, we have

$$\nabla(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})=t^{-1/2}\Delta(O,\rho;f_1,f_2;f_{\mathrm{row}};f_{\mathrm{col}})=\frac{-t^{1/2}}{f_{\mathrm{row}}(1,a)f_{\mathrm{col}}(a)}.$$

□

Kitayama [15] gave a normalized twisted Alexander polynomial $\widetilde{\Delta}_{K,\rho}(t)$ for an oriented knot K and a group representation $\rho: G(K)\to GL(k;F)$, where F is a field. Fix an element $\mu\in G(K)$ represented by a meridian in $E(K)$. Let $\alpha: G(K)\to\langle t\rangle$ be the group representation that sends μ to t. We define the group representation $\rho\otimes\alpha: G(K)\to GL(k;F(t))$ by $(\rho\otimes\alpha)(x)=\alpha(x)\rho(x)$ for $x\in G(K)$. Set $Q:=\mathrm{Conj}\,GL(k;F(t))$. Let (f_1,f_2) be the Alexander pair of maps $f_1,f_2: Q\times Q\to M(k,k;F(t))$ in Example 3.6 (2) with $n=1$, that is, $f_1(a,b)=b^{-1}$ and $f_2(a,b)=b^{-1}a-b^{-1}$. Let D be a diagram of K. Let $\langle \boldsymbol{x}\,|\,\boldsymbol{s}\rangle:=\langle x_1,\dots,x_n\,|\,s_1,\dots,s_n\rangle$ be the Wirtinger presentation of $G(K)$ with respect to D, where s_i is the relation $v_i^{-1}u_iv_iw_i^{-1}$. A presentation that is obtained by removing one relation from $\langle \boldsymbol{x}\,|\,\boldsymbol{s}\rangle$ also represents $G(K)$. Set $\langle \boldsymbol{x}\,|\,\boldsymbol{s}'\rangle:=\langle x_1,\dots,x_n\,|\,s_2,\dots,s_n\rangle$. Then, the normalized twisted Alexander polynomial of (K,ρ) is determined by

$$(3)\qquad \widetilde{\Delta}_{K,\rho}(t)=\frac{\delta(\langle \boldsymbol{x}\,|\,\boldsymbol{s}'\rangle)^k}{(t^k\det\rho(\mu))^{d(\langle \boldsymbol{x}\,|\,\boldsymbol{s}'\rangle)}}\cdot\frac{\det\overline{A(D,\rho\otimes\alpha;f_1,f_2)}_{\overline{k(n-1)}+k,\overline{k(n-1)}+k}}{\det(t\rho(\mu)-1)},$$

where $\delta(\langle \boldsymbol{x}\,|\,\boldsymbol{s}'\rangle)\in\{\pm1\}$ and $d(\langle \boldsymbol{x}\,|\,\boldsymbol{s}'\rangle)\in\{n/2\,|\,n\in\mathbb{Z}\}$ are independent of k and ρ. For details, we refer the reader to [15].

Lemma 7.2. *We have*

$$\delta(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle) = (-1)^{\#C_+(D)+1}\operatorname{sgn}(c_1), \quad d(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle) = n_A(r_1) - \frac{\operatorname{rot}(D) + \#C(D)}{2}.$$

Furthermore, we have

$$\widetilde{\Delta}_{K,\rho}(t) = \frac{\left((-1)^{\#C_+(D)+1}\operatorname{sgn}(c_1)\right)^k}{(t^k \det \rho(\mu))^{n_A(r_1) - \frac{\operatorname{rot}(D)+\#C(D)}{2}}} \cdot \frac{\det \overline{A(D, \rho\otimes\alpha; f_1, f_2)}_{\overline{k(n-1)}+k, \overline{k(n-1)}+k}}{\det(t\rho(\mu) - 1)}.$$

Proof. Let ρ be the trivial group representation $\rho : G(K) \to GL(1; F)$, which is the constant map with constant value 1. Then we may identify $\rho \otimes \alpha$ with α. By Lemma 4.6 in [15], we have

$$\nabla_K(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})\widetilde{\Delta}_{K,\rho}(t).$$

We use the same symbol $\alpha : Q(K) \to Q$ for the induced quandle representation. Let Y be the Q-set $F(t)^\times$ defined by $y \triangleleft a := a^{-1}y$. Let $f_{\mathrm{row}} : Y \times Q \to F(t)$ and $f_{\mathrm{col}} : Q \to F(t)$ be, respectively, the row and column relation maps in Example 3.6 (2) with $n = 1$, that is, $f_{\mathrm{col}}(a) = a - 1$ and $f_{\mathrm{row}}(y, a) = y^{-1}$. We define the row relation map $f_{\mathrm{row},1} : \operatorname{As} Q \times Q \to F(t)$ by $f_{\mathrm{row},1}(y, a) = f_{\mathrm{row}}(\varphi(y), a)$, where $\varphi : \operatorname{As} Q \to Y$ is the Q-set homomorphism satisfying $\varphi(1) = 1$. By Proposition 7.1, we have

$$\nabla_K(t^{1/2} - t^{-1/2}) = (t^{-1/2} - t^{1/2})\nabla(K, \alpha; f_1, f_2; f_{\mathrm{row},1}; f_{\mathrm{col}}),$$

since

$$\nabla(O, \alpha; f_1, f_2; f_{\mathrm{row},1}; f_{\mathrm{col}}) = \frac{-t^{1/2}}{t - 1} = \frac{1}{t^{-1/2} - t^{1/2}}.$$

We then have

$$\widetilde{\Delta}_{K,\rho}(t) = -\nabla(K, \alpha; f_1, f_2; f_{\mathrm{row},1}; f_{\mathrm{col}}). \tag{4}$$

We have

$$\begin{aligned} R_{\mathrm{row}}(D, \alpha; f_{\mathrm{row},1}) &= \begin{pmatrix} \operatorname{sgn}(c_1)\widetilde{\alpha}(r_1)^{-1} & \cdots & \operatorname{sgn}(c_n)\widetilde{\alpha}(r_n)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{sgn}(c_1)t^{n_A(r_1)} & \cdots & \operatorname{sgn}(c_n)t^{n_A(r_n)} \end{pmatrix}, \\ R_{\mathrm{col}}(D, \alpha; f_{\mathrm{col}}) &= \begin{pmatrix} t-1 \\ \vdots \\ t-1 \end{pmatrix}. \end{aligned}$$

Setting $\sigma = \tau = 1_{S_n}$, we have

$$\begin{aligned} &\nabla(K, \rho; f_1, f_2; f_{\mathrm{row},1}; f_{\mathrm{col}}) \\ &= \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det A(D, \rho; f_1, f_2)_{\sigma(\overline{n-1}+1),\tau(\overline{n-1}+1)} \operatorname{detcor}_\nabla(D, \alpha; f_1, f_2)^{-1}}{\det R_{\mathrm{row}}(D, \alpha; f_{\mathrm{row},1})_{\overline{1},\sigma(\overline{1})} \det R_{\mathrm{col}}(D, \alpha; f_{\mathrm{col}})_{\tau(\overline{1}),\overline{1}}} \\ &= \frac{\det A(D, \alpha; f_1, f_2)_{\overline{n-1}+1,\overline{n-1}+1} \operatorname{detcor}_\nabla(D, \alpha; f_1, f_2)^{-1}}{\operatorname{sgn}(c_1)t^{n_A(r_1)}(t-1)}. \end{aligned} \tag{5}$$

From (3), we have

$$\widetilde{\Delta}_{K,\rho}(t) = \frac{\delta(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle)}{t^{d(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle)}} \frac{\det A(D, \alpha; f_1, f_2)_{\overline{n-1}+1,\overline{n-1}+1}}{t-1}. \tag{6}$$

The equalities (4)–(6) imply

$$\frac{\delta(\langle \boldsymbol{x} \mid \boldsymbol{r}' \rangle)}{t^{d(\langle \boldsymbol{x} \mid \boldsymbol{r}' \rangle)}} = -\frac{\operatorname{detcor}_\nabla(D, \alpha; f_1, f_2)^{-1}}{\operatorname{sgn}(c_1)t^{n_A(r_1)}},$$

since

$$\frac{\det A(D, \alpha; f_1, f_2)_{\overline{n-1}+1,\overline{n-1}+1}}{t-1} \neq 0,$$

which follows from $\det A(D,\alpha;f_1,f_2)_{\overline{n-1}+1,\overline{n-1}+1}|_{t=1} = \pm 1 \neq 0$. Therefore, we have

$$\frac{\delta(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle)}{t^{d(\langle \boldsymbol{x} \mid \boldsymbol{r}' \rangle)}} = \frac{-\operatorname{sgn}(c_1)(-1)^{\#C_+(D)}}{t^{n_A(r_1)}t^{-\frac{\operatorname{rot}(D)+\operatorname{wr}(D)}{2}}t^{-\#C_-(D)}} = \frac{(-1)^{\#C_+(D)+1}\operatorname{sgn}(c_1)}{t^{n_A(r_1)-\frac{\operatorname{rot}(D)+\#C(D)}{2}}},$$

which implies that

$$\delta(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle) = (-1)^{\#C_+(D)+1}\operatorname{sgn}(c_1), \quad d(\langle \boldsymbol{x} \mid \boldsymbol{s}' \rangle) = n_A(r_1) - \frac{\operatorname{rot}(D)+\#C(D)}{2}.$$

□

Definition 7.3. Set $Q := \operatorname{Conj} GL(k;F(t))$, where F is a field. Let Y be the Q-set $GL(k;F(t))$ defined by $y \triangleleft a := a^{-1}y$. Let (f_1,f_2) be the Alexander pair of maps $f_1,f_2 : Q\times Q \to M(k,k;F(t))$ in Example 3.6 (2) with $n=1$, that is, $f_1(a,b)=b^{-1}$ and $f_2(a,b) = b^{-1}a - b^{-1}$. Let $f_{\mathrm{row}} : Y \times Q \to M(k,k;F(t))$ and $f_{\mathrm{col}} : Q \to M(k,k;F(t))$ be, respectively, the row and column relation maps in Example 3.6 (2) with $n=1$, that is, $f_{\mathrm{col}}(a) = a-1$ and $f_{\mathrm{row}}(y,a) = y^{-1}$. We define the row relation map $f_{\mathrm{row},1} : \operatorname{As} Q \times Q \to M(k,k;F(t))$ by $f_{\mathrm{row},1}(y,a) = -f_{\mathrm{row}}(\varphi(y),a)$, where $\varphi : \operatorname{As} Q \to Y$ is the Q-set homomorphism satisfying $\varphi(1)=1$. Let L be an oriented link, and let $\rho : G(L) \to GL(k;F)$ be a group representation. We use the same symbol $\rho\otimes\alpha : Q(L) \to Q$ for the induced quandle representation of $\rho\otimes\alpha : G(L) \to GL(k;F(t))$. We then define

$$\nabla(L,\rho) := \nabla(L,\rho\otimes\alpha;f_1,f_2;f_{\mathrm{row},1};f_{\mathrm{col}}).$$

Proposition 7.4. *Let K be an oriented knot, and let $\rho : G(K) \to GL(k;F)$ be a group representation. Then we have*

$$\widetilde{\Delta}_{K,\rho}(t) = \nabla(K,\rho).$$

Proof. Let D be a diagram of K with n crossings. Putting

$$A := A(D,\rho\otimes\alpha;f_1,f_2), \qquad B := R_{\mathrm{row}}(D,\rho\otimes\alpha;f_{\mathrm{row},1}),$$
$$C := R_{\mathrm{col}}(D,\rho\otimes\alpha;f_{\mathrm{col}}),$$

we have

$$B = \begin{pmatrix} -\operatorname{sgn}(c_1)t^{n_A(r_1)}\widetilde{\rho}(r_1)^{-1} & \cdots & -\operatorname{sgn}(c_n)t^{n_A(r_n)}\widetilde{\rho}(r_n)^{-1}\end{pmatrix},$$
$$C = \begin{pmatrix} t\rho(x_1)-1 \\ \vdots \\ t\rho(x_n)-1 \end{pmatrix}.$$

Setting $\sigma = \tau = 1_{S_n}$, we have

$$\begin{aligned}
&\nabla(K,\rho)\\
&= \frac{\operatorname{sgn}\sigma \operatorname{sgn}\tau \det \overline{A}_{\sigma(\overline{k(n-1)}+k),\tau(\overline{k(n-1)}+k)} \operatorname{detcor}_{\nabla}(D,\rho\otimes\alpha;f_1,f_2)^{-1}}{\det \overline{B}_{\overline{k},\sigma(\overline{k})} \det \overline{C}_{\tau(\overline{k}),\overline{k}}}\\
&= \frac{\operatorname{detcor}_{\nabla}(D,\rho\otimes\alpha;f_1,f_2)^{-1}}{\det(-\operatorname{sgn}(c_1)t^{n_A(r_1)}\widetilde{\rho}(r_1)^{-1})} \cdot \frac{\det \overline{A}_{\overline{k(n-1)}+k,\overline{k(n-1)}+k}}{\det(t\rho(x_1)-1)}\\
&= \frac{((-1)^{k(\#C_+(D)+1)}(\det(t^{-1}\rho(x_1)^{-1}))^{\frac{\operatorname{rot}(D)+\#C(D)}{2}})^{-1}}{\operatorname{sgn}(c_1)^k t^{kn_A(r_1)}\det\rho(x_1)^{n_A(r_1)}} \cdot \frac{\det \overline{A}_{\overline{k(n-1)}+k,\overline{k(n-1)}+k}}{\det(t\rho(x_1)-1)}\\
&= \frac{\left((-1)^{\#C_+(D)+1}\operatorname{sgn}(c_1)\right)^k}{(t^k\det\rho(x_1))^{n_A(r_1)-\frac{\operatorname{rot}(D)+\#C(D)}{2}}} \cdot \frac{\det \overline{A}_{\overline{k(n-1)}+k,\overline{k(n-1)}+k}}{\det(t\rho(x_1)-1)}\\
&= \widetilde{\Delta}_{K,\rho}(t),
\end{aligned}$$

where the last equality follows from Lemma 7.2. □

8. Cohomologous Alexander pairs and relation maps

Let (f_1, f_2) and (g_1, g_2) be Alexander pairs of maps $f_1, f_2, g_1, g_2 : Q \times Q \to R$. Let $f_{\mathrm{row}} : \mathrm{As}\, Q \times Q \to R$ and $g_{\mathrm{row}} : \mathrm{As}\, Q \times Q \to R$ be row relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. Let $f_{\mathrm{col}} : Q \to R$ and $g_{\mathrm{col}} : Q \to R$ be column relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. Two tuples $(f_1, f_2; f_{\mathrm{row}}; f_{\mathrm{col}})$ and $(g_1, g_2; g_{\mathrm{row}}; g_{\mathrm{col}})$ are *cohomologous* if there exists a map $h : Q \to R^\times$ satisfying the following conditions:

- For any $a, b \in Q$, $h(a \triangleleft b) f_1(a, b) = g_1(a, b) h(a)$.
- For any $a, b \in Q$, $h(a \triangleleft b) f_2(a, b) = g_2(a, b) h(b)$.
- For any $a \in Q$, $h(a) f_{\mathrm{col}}(a) = g_{\mathrm{col}}(a)$.
- For any $a \in Q$ and $y \in \mathrm{As}\, Q$, $f_{\mathrm{row}}(y, a) = g_{\mathrm{row}}(y, a) h(a)$.

We then write $(f_1, f_2; f_{\mathrm{row}}; f_{\mathrm{col}}) \sim_h (g_1, g_2; g_{\mathrm{row}}; g_{\mathrm{col}})$ to specify h. For $i = 1, \ldots, m$, let $f_{\mathrm{row},i} : \mathrm{As}\, Q \times Q \to R$ and $g_{\mathrm{row},i} : \mathrm{As}\, Q \times Q \to R$ be row relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. For $i = 1, \ldots, m$, let $f_{\mathrm{col},i} : Q \to R$ and $g_{\mathrm{col},i} : Q \to R$ be column relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. When $(f_1, f_2; f_{\mathrm{row},i}; f_{\mathrm{col},i}) \sim_h (g_1, g_2; g_{\mathrm{row},i}; g_{\mathrm{col},i})$ for any $i \in \{1, \ldots, m\}$, we write $(f_1, f_2; \boldsymbol{f_{\mathrm{row}}}; \boldsymbol{f_{\mathrm{col}}}) \sim_h (g_1, g_2; \boldsymbol{g_{\mathrm{row}}}; \boldsymbol{g_{\mathrm{col}}})$.

Example 8.1. For an Alexander pair (f_1, f_2) and $a \in Q$, we define $f_1 \triangleleft a$ and $f_2 \triangleleft a$ by

$$(f_1 \triangleleft a)(x, y) = f_1(x \triangleleft a, y \triangleleft a), \qquad (f_2 \triangleleft a)(x, y) = f_2(x \triangleleft a, y \triangleleft a).$$

For a column relation map f_{col} and $a \in Q$, we define $f_{\mathrm{col}} \triangleleft a$ by

$$(f_{\mathrm{col}} \triangleleft a)(x) = f_1(x, a) f_{\mathrm{col}}(x).$$

For a row relation map f_{row} and $a \in Q$, we define $f_{\mathrm{row}} \triangleleft a$ by

$$(f_{\mathrm{row}} \triangleleft a)(y, x) = f_{\mathrm{row}}(y, x) f_1(x, a)^{-1}.$$

Putting $h(x) := f_1(x, a)$, we have

$$(f_1, f_2; f_{\mathrm{row}}; f_{\mathrm{col}}) \sim_h (f_1 \triangleleft a, f_2 \triangleleft a; f_{\mathrm{row}} \triangleleft a; f_{\mathrm{col}} \triangleleft a).$$

Example 8.2. Let (f_1, f_2), f_{row} and f_{col} be the Alexander pair and row and column relation maps in Example 3.5 (1). Let (g_1, g_2), g_{row} and g_{col} be the Alexander pair and row and column relation maps in Example 3.5 (2). We define the row relation maps $f_{\mathrm{row},1} : \mathrm{As}\, Q \times Q \to R$ and $g_{\mathrm{row},1} : \mathrm{As}\, Q \times Q \to R$ by $f_{\mathrm{row},1}(y, a) = f_{\mathrm{row}}(\varphi(y), a)$ and $g_{\mathrm{row},1}(y, a) = g_{\mathrm{row}}(\varphi(y), a)$, where $\varphi : \mathrm{As}\, Q \to Y$ is the Q-set homomorphism satisfying $\varphi(1) = 1$. We define a map $h : Q \to R$ by $h(x) = f(x) - 1$. Suppose that $h(a)$ is invertible for any $a \in Q$. Then $(f_1, f_2; f_{\mathrm{row}}; f_{\mathrm{col}}) \sim_h (g_1, g_2; g_{\mathrm{row}}; g_{\mathrm{col}})$.

Proposition 8.3. *Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let $\rho : Q(L) \to Q$ be a quandle representation. Let D be a diagram of L with n crossings. Set $R := M(k, k; F)$, where F is a field. Let (f_1, f_2) and (g_1, g_2) be Alexander pairs of maps $f_1, f_2, g_1, g_2 : Q \times Q \to R$. Let $f_{\mathrm{row},1}, \ldots, f_{\mathrm{row},m} : \mathrm{As}\, Q \times Q \to R$ and $g_{\mathrm{row},1}, \ldots, g_{\mathrm{row},m} : Q \to R$ be row relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. Let $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},m} : Q \to R$ and $g_{\mathrm{col},1}, \ldots, g_{\mathrm{col},m} : Q \to R$ be column relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. If $(f_1, f_2; \boldsymbol{f_{\mathrm{row}}}; \boldsymbol{f_{\mathrm{col}}}) \sim_h (g_1, g_2; \boldsymbol{g_{\mathrm{row}}}; \boldsymbol{g_{\mathrm{col}}})$, then we have*

$$\begin{aligned}&(\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{f_{\mathrm{row}}}), \widetilde{A}(D, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{f_{\mathrm{col}}})) \\ &\sim (\widetilde{R_{\mathrm{row}}}(D, \rho; \boldsymbol{g_{\mathrm{row}}}), \widetilde{A}(D, \rho; g_1, g_2), \widetilde{R_{\mathrm{col}}}(D, \rho; \boldsymbol{g_{\mathrm{col}}})).\end{aligned}$$

Furthermore, we have

$$\Delta(L, \rho; f_1, f_2; \boldsymbol{f_{\mathrm{row}}}; \boldsymbol{f_{\mathrm{col}}}) = \Delta(L, \rho; g_1, g_2; \boldsymbol{g_{\mathrm{row}}}; \boldsymbol{g_{\mathrm{col}}}).$$

Proof. Put

$$\begin{aligned} A(f) &:= A(D,\rho;f_1,f_2), & A(g) &:= A(D,\rho;g_1,g_2),\\ R_{\rm row}(f) &:= R_{\rm row}(D,\rho;\boldsymbol{f_{\rm row}}), & R_{\rm row}(g) &:= R_{\rm row}(D,\rho;\boldsymbol{g_{\rm row}}),\\ R_{\rm col}(f) &:= R_{\rm col}(D,\rho;\boldsymbol{f_{\rm col}}), & R_{\rm col}(g) &:= R_{\rm col}(D,\rho;\boldsymbol{g_{\rm col}}),\\ {\rm cor}(f) &:= {\rm cor}(D,\rho;f_1,f_2), & {\rm cor}(g) &:= {\rm cor}(D,\rho;g_1,g_2). \end{aligned}$$

By the proof of Theorem 9.3 in [12], we have

$$h(\rho(\boldsymbol{w}))A(f) = A(g)h(\rho(\boldsymbol{x})),$$

where $h(\rho(\boldsymbol{a}))$ is the diagonal matrix in $M(n,n;R)$ whose (i,i)-entry is $h(\rho(a_i))$. From $f_{{\rm row},i}(y,\rho(w_j)) = g_{{\rm row},i}(y,\rho(w_j))h(\rho(w_j))$, we have

$$R_{\rm row}(f) = R_{\rm row}(g)h(\rho(\boldsymbol{w})). \tag{7}$$

From $h(\rho(x_i))f_{{\rm col},j}(\rho(x_i)) = g_{{\rm col},j}(\rho(x_i))$, we have

$$h(\rho(\boldsymbol{x}))R_{\rm col}(f) = R_{\rm col}(g). \tag{8}$$

From

$$\begin{aligned} &(-1)^{\#C_+(D)}\prod_{i=1}^{r}\omega_i^{\frac{{\rm rot}(D(K_i))+{\rm wr}(D(K_i))+1}{2}}\prod_{c\in C_-(D)}h(\rho(w_c))f_1(\rho(u_c),\rho(v_c))\\ &=(-1)^{\#C_+(D)}\prod_{i=1}^{r}\omega_i^{\frac{{\rm rot}(D(K_i))+{\rm wr}(D(K_i))+1}{2}}\prod_{c\in C_-(D)}g_1(\rho(u_c),\rho(v_c))h(\rho(u_c)), \end{aligned}$$

we have

$$\begin{aligned} {\rm cor}(f)\prod_{c\in C(D)}h(\rho(w_c)) &= {\rm cor}(g)\prod_{c\in C_-(D)}h(\rho(u_c))\prod_{c\in C_+(D)}h(\rho(w_c))\\ &= {\rm cor}(g)\prod_{c\in C(D)}h(\rho(x_c)) \end{aligned}$$

in the abelianization of $R^\times$, since $x_c = w_c$ for a positive crossing c and $x_c = u_c$ for a negative crossing c. Setting $h^\Pi(\rho(\boldsymbol{w})) := \prod_{i=1}^n h(\rho(w_i))$ and $h^\Pi(\rho(\boldsymbol{x})) := \prod_{i=1}^n h(\rho(x_i))$, we have

$$\begin{aligned} &(\widetilde{R_{\rm row}}(D,\rho;\boldsymbol{f_{\rm row}}),\widetilde{A}(D,\rho;f_1,f_2),\widetilde{R_{\rm col}}(D,\rho;\boldsymbol{f_{\rm col}}))\\ &=\left(\begin{pmatrix}R_{\rm row}(g)h(\rho(\boldsymbol{w})) & \mathbf{0}\end{pmatrix},\begin{pmatrix}A(f) & \mathbf{0}\\ \mathbf{0} & {\rm cor}(f)^{-1}\end{pmatrix},\begin{pmatrix}R_{\rm col}(f)\\ \mathbf{0}\end{pmatrix}\right)\\ &\sim\left(\begin{pmatrix}R_{\rm row}(g) & \mathbf{0}\end{pmatrix},\begin{pmatrix}h(\rho(\boldsymbol{w}))A(f) & \mathbf{0}\\ \mathbf{0} & h^\Pi(\rho(\boldsymbol{w}))^{-1}{\rm cor}(f)^{-1}\end{pmatrix},\begin{pmatrix}R_{\rm col}(f)\\ \mathbf{0}\end{pmatrix}\right)\\ &\sim\left(\begin{pmatrix}R_{\rm row}(g) & \mathbf{0}\end{pmatrix},\begin{pmatrix}A(g)h(\rho(\boldsymbol{x})) & \mathbf{0}\\ \mathbf{0} & {\rm cor}(g)^{-1}h^\Pi(\rho(\boldsymbol{x}))^{-1}\end{pmatrix},\begin{pmatrix}R_{\rm col}(f)\\ \mathbf{0}\end{pmatrix}\right)\\ &\sim\left(\begin{pmatrix}R_{\rm row}(g) & \mathbf{0}\end{pmatrix},\begin{pmatrix}A(g) & \mathbf{0}\\ \mathbf{0} & {\rm cor}(g)^{-1}\end{pmatrix},\begin{pmatrix}h(\rho(\boldsymbol{x}))R_{\rm col}(f)\\ \mathbf{0}\end{pmatrix}\right)\\ &=(\widetilde{R_{\rm row}}(D,\rho;\boldsymbol{g_{\rm row}}),\widetilde{A}(D,\rho;g_1,g_2),\widetilde{R_{\rm col}}(D,\rho;\boldsymbol{g_{\rm col}})). \end{aligned}$$

□

Example 8.4. Let $G := GL(k;F)$ and $Q := {\rm Conj}\,G$, where F is a field. Let $R := M(k,k;F(t))$. Let Y be the Q-set $GL(k;F(t))$ defined by $y \triangleleft a := a^{-1}y$. Let $(f_1,f_2), f_{\rm row}$ and $f_{\rm col}$ be the Alexander pair and row and column relation maps in Example 3.6 (1), that is, $f_1(a,b) = b^{-1}$, $f_2(a,b) = 1 - b^{-1}$, $f_{\rm row}(y,a) = y^{-1}(a - 1)$ and $f_{\rm col}(a) = 1$. Let $(g_1,g_2), g_{\rm row}$ and $g_{\rm col}$ be the Alexander pair and row and column relation maps in Example 3.6 (2), that is, $g_1(a,b) = b^{-1}$, $g_2(a,b) =$

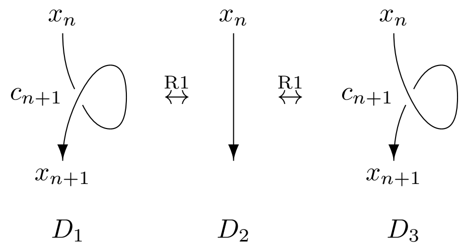

FIGURE 9

$b^{-1}a - b^{-1}$, $g_{\mathrm{row}}(y,a) = y^{-1}$ and $g_{\mathrm{col}}(a) = a - 1$. We define the row relation maps $f_{\mathrm{row},1} : \mathrm{As}\,Q \times Q \to R$ and $g_{\mathrm{row},1} : \mathrm{As}\,Q \times Q \to R$ by $f_{\mathrm{row},1}(y,a) = f_{\mathrm{row}}(\varphi(y),a)$ and $g_{\mathrm{row},1}(y,a) = g_{\mathrm{row}}(\varphi(y),a)$, where $\varphi : \mathrm{As}\,Q \to Y$ is the Q-set homomorphism satisfying $\varphi(1) = 1$. Let L be an oriented link, and let $\rho : Q(L) \to Q$ be a quandle representation. Let $\alpha : G(L) \to \langle t \rangle$ be the group representation that sends a meridian to t. We define the group representation $\rho \otimes \alpha : G(L) \to GL(k; F(t))$ by $(\rho \otimes \alpha)(x) = \alpha(x)\rho(x)$ for $x \in G(L)$. Setting $h(x) = x - 1$ in Example 8.2, we have $(f_1, f_2; f_{\mathrm{row},1}; f_{\mathrm{col}}) \sim_h (g_1, g_2; g_{\mathrm{row},1}; g_{\mathrm{col}})$. We then have

$$\Delta(L, \rho \otimes \alpha; f_1, f_2; f_{\mathrm{row},1}; f_{\mathrm{col}}) = \Delta(L, \rho \otimes \alpha; g_1, g_2; g_{\mathrm{row},1}; g_{\mathrm{col}}),$$

where we remark that the right invariant corresponds to the twisted Alexander polynomial.

9. Proof of Theorem 4.7

In this section, we give a proof of Theorem 4.7. For short, we set

$$A(D) := A(D, \rho; f_1, f_2), \qquad \mathrm{cor}(D) := \mathrm{cor}(D, \rho; f_1, f_2),$$
$$R_{\mathrm{row}}(D) := R_{\mathrm{row}}(D, \rho; \boldsymbol{f_{\mathrm{row}}}), \qquad R_{\mathrm{col}}(D) := R_{\mathrm{col}}(D, \rho; \boldsymbol{f_{\mathrm{col}}}),$$
$$\boldsymbol{f_{\mathrm{col}}}(a) := \begin{pmatrix} f_{\mathrm{col},1}(a) & \cdots & f_{\mathrm{col},m}(a) \end{pmatrix}, \qquad \boldsymbol{f_{\mathrm{row}}}(z,a) := \begin{pmatrix} f_{\mathrm{row},1}(z,a) \\ \vdots \\ f_{\mathrm{row},m}(z,a) \end{pmatrix}.$$

9.1. **Reidemeister move I.** Let D_1, D_2 and D_3 be diagrams of an oriented link L that differ by a single Reidemeister move I as shown in Figure 9. Let $c_1, \dots, c_n$ be n crossings of D_1, D_2 and D_3 that stay outside the disk in which the move is applied, and let c_{n+1} be the other crossing of D_1 and D_3 that stays within the disk. We remark again that, for each i, we denote by x_i the arc starting from a crossing c_i. Put $a := \rho(x_n) = \rho(x_{n+1})$ and $z := \widetilde{\rho}(r_{n+1})$. We then have

$$A(D_1) = \begin{pmatrix} A_{n-1} & \boldsymbol{a}'_n & \boldsymbol{a}''_n \\ \boldsymbol{0} & f_1(a,a) & f_2(a,a) - 1 \end{pmatrix}, \qquad A(D_2) = \begin{pmatrix} A_{n-1} & \boldsymbol{a}_n \end{pmatrix},$$
$$A(D_3) = \begin{pmatrix} A_{n-1} & \boldsymbol{a}'_n & \boldsymbol{a}''_n \\ \boldsymbol{0} & f_2(a,a) - 1 & f_1(a,a) \end{pmatrix},$$

where $\boldsymbol{a}'_n + \boldsymbol{a}''_n = \boldsymbol{a}_n$, since the arc x_n in D_2 is separated into the two arcs x_n and x_{n+1} in D_1 and D_3. We also have

$$R_{\mathrm{row}}(D_1) = \begin{pmatrix} B_n & \boldsymbol{f_{\mathrm{row}}}(z,a) \end{pmatrix}, \qquad R_{\mathrm{row}}(D_2) = B_n,$$
$$R_{\mathrm{row}}(D_3) = \begin{pmatrix} B_n & -\boldsymbol{f_{\mathrm{row}}}(z,a) \end{pmatrix},$$

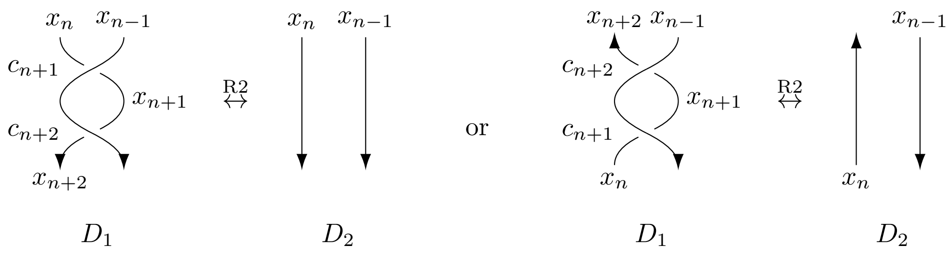

FIGURE 10

and

$$R_{\mathrm{col}}(D_1) = \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \boldsymbol{f}_{\mathbf{col}}(a) \end{pmatrix}, \quad R_{\mathrm{col}}(D_2) = \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \end{pmatrix}, \quad R_{\mathrm{col}}(D_3) = \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \boldsymbol{f}_{\mathbf{col}}(a) \end{pmatrix}.$$

In the abelianization of $R^\times$, we have

$$\mathrm{cor}(D_1) = -f_1(a,a)\,\mathrm{cor}(D_2), \qquad \mathrm{cor}(D_3) = f_1(a,a)\,\mathrm{cor}(D_2).$$

We have $B_n \boldsymbol{a}''_n - \boldsymbol{f}_{\mathbf{row}}(z,a) f_1(a,a) = \mathbf{0}$, since $R_{\mathrm{row}}(D_1)$ is a row relation matrix of $A(D_1)$. It follows that

$$\begin{aligned}
&(\widetilde{R_{\mathrm{row}}}(D_1,\rho;\boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D_1,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D_1,\rho;\boldsymbol{f}_{\mathbf{col}})) \\
&= \left(\begin{pmatrix} B_n & \boldsymbol{f}_{\mathbf{row}}(z,a) & \mathbf{0} \end{pmatrix}, \begin{pmatrix} A_{n-1} & \boldsymbol{a}'_n & \boldsymbol{a}''_n & \mathbf{0} \\ \mathbf{0} & f_1(a,a) & -f_1(a,a) & 0 \\ \mathbf{0} & 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \mathbf{0} \end{pmatrix} \right) \\
&\sim \left(\begin{pmatrix} B_n & \boldsymbol{f}_{\mathbf{row}}(z,a) & \mathbf{0} \end{pmatrix}, \begin{pmatrix} A_{n-1} & \boldsymbol{a}_n & \boldsymbol{a}''_n & \mathbf{0} \\ \mathbf{0} & 0 & -f_1(a,a) & 0 \\ \mathbf{0} & 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right) \\
&\sim \left(\begin{pmatrix} B_n & \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} A_{n-1} & \boldsymbol{a}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & -f_1(a,a) & 0 \\ \mathbf{0} & 0 & 0 & \mathrm{cor}(D_1)^{-1} \end{pmatrix}, \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right) \\
&\sim \left(\begin{pmatrix} B_n & \mathbf{0} \end{pmatrix}, \begin{pmatrix} A_{n-1} & \boldsymbol{a}_n & \mathbf{0} \\ \mathbf{0} & 0 & -f_1(a,a)\,\mathrm{cor}(D_1)^{-1} \end{pmatrix}, \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \mathbf{0} \end{pmatrix} \right) \\
&\sim \left(\begin{pmatrix} B_n & \mathbf{0} \end{pmatrix}, \begin{pmatrix} A_{n-1} & \boldsymbol{a}_n & \mathbf{0} \\ \mathbf{0} & 0 & \mathrm{cor}(D_2)^{-1} \end{pmatrix}, \begin{pmatrix} C_{n-1} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \mathbf{0} \end{pmatrix} \right) \\
&= (\widetilde{R_{\mathrm{row}}}(D_2,\rho;\boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D_2,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D_2,\rho;\boldsymbol{f}_{\mathbf{col}})),
\end{aligned}$$

where $\alpha = \mathrm{cor}(D_1)^{-1}$. In a similar manner, we have

$$\begin{aligned}
&(\widetilde{R_{\mathrm{row}}}(D_3,\rho;\boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D_3,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D_3,\rho;\boldsymbol{f}_{\mathbf{col}})) \\
&\sim (\widetilde{R_{\mathrm{row}}}(D_2,\rho;\boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D_2,\rho;f_1,f_2), \widetilde{R_{\mathrm{col}}}(D_2,\rho;\boldsymbol{f}_{\mathbf{col}})).
\end{aligned}$$

9.2. **Reidemeister move II.** Let D_1 and D_2 be diagrams of an oriented link L that differ by a single Reidemeister move II as shown in Figure 10. Let $c_1, \dots, c_n$ be n crossings of D_1 and D_2 that stay outside the disk in which the move is applied, and let c_{n+1} and c_{n+2} be the other crossings of D_1 that stay within the disk. Put $a := \rho(x_n)$, $b := \rho(x_{n-1})$ and $z := \widetilde{\rho}(r_{n+1}) = \widetilde{\rho}(r_{n+2})$. We then have

$$\begin{aligned} A(D_1) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}'_n & \boldsymbol{0} & \boldsymbol{a}''_n \\ \boldsymbol{0} & f_2(a,b) & f_1(a,b) & -1 & 0 \\ \boldsymbol{0} & f_2(a,b) & 0 & -1 & f_1(a,b) \end{pmatrix}, \\ A(D_2) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \end{pmatrix}, \end{aligned}$$

where $\boldsymbol{a}'_n + \boldsymbol{0} + \boldsymbol{a}''_n = \boldsymbol{a}_n$, since the arc x_n in D_2 is separated into the three arcs x_n, x_{n+1} and x_{n+2} in D_1. We also have

$$R_{\mathrm{row}}(D_1) = \begin{pmatrix} B_n & \boldsymbol{f_{\mathrm{row}}}(z, a \triangleleft b) & -\boldsymbol{f_{\mathrm{row}}}(z, a \triangleleft b) \end{pmatrix}, \qquad R_{\mathrm{row}}(D_2) = B_n,$$

and

$$R_{\mathrm{col}}(D_1) = \begin{pmatrix} C_{n-2} \\ \boldsymbol{f_{\mathrm{col}}}(b) \\ \boldsymbol{f_{\mathrm{col}}}(a) \\ \boldsymbol{f_{\mathrm{col}}}(a \triangleleft b) \\ \boldsymbol{f_{\mathrm{col}}}(a) \end{pmatrix}, \qquad R_{\mathrm{col}}(D_2) = \begin{pmatrix} C_{n-2} \\ \boldsymbol{f_{\mathrm{col}}}(b) \\ \boldsymbol{f_{\mathrm{col}}}(a) \end{pmatrix}.$$

In the abelianization of $R^\times$, we have

$$\mathrm{cor}(D_1) = -f_1(a,b)\,\mathrm{cor}(D_2).$$

We put

$$\begin{aligned} A_1^{\mathrm{R2}} &:= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n & \boldsymbol{0} & \boldsymbol{a}''_n \\ \boldsymbol{0} & f_2(a,b) & f_1(a,b) & -1 & 0 \\ \boldsymbol{0} & f_2(a,b) & f_1(a,b) & -1 & f_1(a,b) \end{pmatrix} \oplus (\mathrm{cor}(D_1)^{-1}), \\ A_2^{\mathrm{R2}} &:= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n & \boldsymbol{0} & \boldsymbol{a}''_n \\ \boldsymbol{0} & 0 & 0 & -1 & 0 \\ \boldsymbol{0} & 0 & 0 & -1 & f_1(a,b) \end{pmatrix} \oplus (\mathrm{cor}(D_1)^{-1}), \\ A_3^{\mathrm{R2}} &:= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n & \boldsymbol{0} & \boldsymbol{a}''_n \\ \boldsymbol{0} & 0 & 0 & -1 & 0 \\ \boldsymbol{0} & 0 & 0 & 0 & f_1(a,b) \end{pmatrix} \oplus (\mathrm{cor}(D_1)^{-1}), \\ A_4^{\mathrm{R2}} &:= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 0 & 0 & -1 & 0 \\ \boldsymbol{0} & 0 & 0 & 0 & f_1(a,b) \end{pmatrix} \oplus (\mathrm{cor}(D_1)^{-1}), \\ A_5^{\mathrm{R2}} &:= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \end{pmatrix} \oplus (-f_1(a,b)\,\mathrm{cor}(D_1)^{-1}), \end{aligned}$$

and

$$R_{\mathrm{col},1}^{\mathrm{R2}} := \begin{pmatrix} C_{n-2} \\ \boldsymbol{f_{\mathrm{col}}}(b) \\ \boldsymbol{f_{\mathrm{col}}}(a) \\ \boldsymbol{f_{\mathrm{col}}}(a \triangleleft b) \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \qquad R_{\mathrm{col},2}^{\mathrm{R2}} := \begin{pmatrix} C_{n-2} \\ \boldsymbol{f_{\mathrm{col}}}(b) \\ \boldsymbol{f_{\mathrm{col}}}(a) \\ \boldsymbol{c}_{n+1} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix},$$

where $\boldsymbol{c}_{n+1} = \boldsymbol{f_{\mathrm{col}}}(a \triangleleft b) - f_1(a,b)\boldsymbol{f_{\mathrm{col}}}(a) - f_2(a,b)\boldsymbol{f_{\mathrm{col}}}(b) = \boldsymbol{0}$. We have $B_n\boldsymbol{a}''_n - \boldsymbol{f_{\mathrm{row}}}(z, a \triangleleft b) f_1(a,b) = \boldsymbol{0}$, since $R_{\mathrm{row}}(D_1)$ is a row relation matrix of $A(D_1)$. It

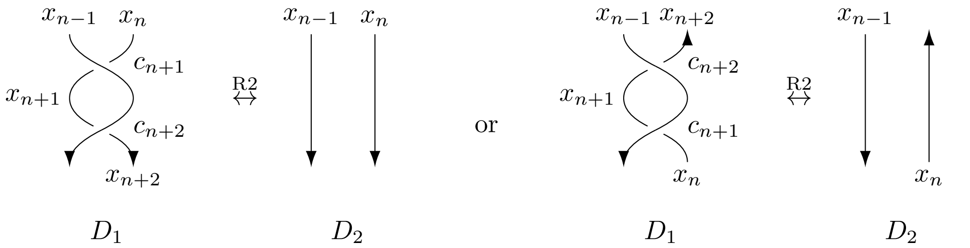

FIGURE 11

follows that

$$\begin{aligned}
&(\widetilde{R_{\mathrm{row}}}(D_1,\rho;\boldsymbol{f_{\mathbf{row}}}),\widetilde{A}(D_1,\rho;f_1,f_2),\widetilde{R_{\mathrm{col}}}(D_1,\rho;\boldsymbol{f_{\mathbf{col}}}))\\
&\sim \left(\begin{pmatrix}B_n & \boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) & -\boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) & \mathbf{0}\end{pmatrix}, A_1^{\mathrm{R2}}, R_{\mathrm{col},1}^{\mathrm{R2}}\right)\\
&\sim \left(\begin{pmatrix}B_n & \boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) & -\boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) & \mathbf{0}\end{pmatrix}, A_2^{\mathrm{R2}}, R_{\mathrm{col},2}^{\mathrm{R2}}\right)\\
&\sim \left(\begin{pmatrix}B_n & \mathbf{0} & -\boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) & \mathbf{0}\end{pmatrix}, A_3^{\mathrm{R2}}, R_{\mathrm{col},2}^{\mathrm{R2}}\right)\\
&\sim \left(\begin{pmatrix}B_n & \mathbf{0} & \boldsymbol{b}_{n+2} & \mathbf{0}\end{pmatrix}, A_4^{\mathrm{R2}}, R_{\mathrm{col},2}^{\mathrm{R2}}\right)\\
&\sim (\widetilde{R_{\mathrm{row}}}(D_2,\rho;\boldsymbol{f_{\mathbf{row}}}), A_5^{\mathrm{R2}}, \widetilde{R_{\mathrm{col}}}(D_2,\rho;\boldsymbol{f_{\mathbf{col}}}))\\
&\sim (\widetilde{R_{\mathrm{row}}}(D_2,\rho;\boldsymbol{f_{\mathbf{row}}}),\widetilde{A}(D_2,\rho;f_1,f_2),\widetilde{R_{\mathrm{col}}}(D_2,\rho;\boldsymbol{f_{\mathbf{col}}})),
\end{aligned}$$

where $\boldsymbol{b}_{n+2} = -\boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) + B_n\boldsymbol{a}_n'' f_1(a,b)^{-1} = \mathbf{0}$.

We next consider the situation depicted in Figure 11. Let D_1 and D_2 be diagrams of an oriented link L that differ by a single Reidemeister move II as shown in Figure 11. Let $c_1,\dots,c_n$ be n crossings of D_1 and D_2 that stay outside the disk in which the move is applied, and let c_{n+1} and c_{n+2} be the other crossings of D_1 that stay within the disk. Put $a := \rho(x_{n+1})$, $b := \rho(x_{n-1})$ and $z := \widetilde{\rho}(r_{n+1}) = \widetilde{\rho}(r_{n+2})$. We then have

$$\begin{aligned}
A(D_1) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n' & \mathbf{0} & \boldsymbol{a}_n'' \\ \mathbf{0} & f_2(a,b) & -1 & f_1(a,b) & 0 \\ \mathbf{0} & f_2(a,b) & 0 & f_1(a,b) & -1 \end{pmatrix},\\
A(D_2) &= \begin{pmatrix} A_{n-2} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \end{pmatrix},
\end{aligned}$$

where $\boldsymbol{a}_n' + \mathbf{0} + \boldsymbol{a}_n'' = \boldsymbol{a}_n$, since the arc x_n in D_2 is separated into the three arcs x_n, x_{n+1} and x_{n+2} in D_1. We also have

$$R_{\mathrm{row}}(D_1) = \begin{pmatrix} B_n & -\boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b) & \boldsymbol{f_{\mathbf{row}}}(z,a\triangleleft b)\end{pmatrix}, \qquad R_{\mathrm{row}}(D_2) = B_n,$$

and

$$R_{\mathrm{col}}(D_1) = \begin{pmatrix} C_{n-2} \\ \boldsymbol{f_{\mathbf{col}}}(b) \\ \boldsymbol{f_{\mathbf{col}}}(a\triangleleft b) \\ \boldsymbol{f_{\mathbf{col}}}(a) \\ \boldsymbol{f_{\mathbf{col}}}(a\triangleleft b)\end{pmatrix}, \qquad R_{\mathrm{col}}(D_2) = \begin{pmatrix} C_{n-2} \\ \boldsymbol{f_{\mathbf{col}}}(b) \\ \boldsymbol{f_{\mathbf{col}}}(a\triangleleft b)\end{pmatrix}.$$

In the abelianization of $R^\times$, we have

$$\mathrm{cor}(D_1) = -f_1(a,b)\,\mathrm{cor}(D_2).$$

In a similar manner as the previous situation, we have

$$\begin{aligned}
&(\widetilde{R_{\mathrm{row}}}(D_1,\rho;\boldsymbol{f_{\mathbf{row}}}),\widetilde{A}(D_1,\rho;f_1,f_2),\widetilde{R_{\mathrm{col}}}(D_1,\rho;\boldsymbol{f_{\mathbf{col}}}))\\
&\sim (\widetilde{R_{\mathrm{row}}}(D_2,\rho;\boldsymbol{f_{\mathbf{row}}}),\widetilde{A}(D_2,\rho;f_1,f_2),\widetilde{R_{\mathrm{col}}}(D_2,\rho;\boldsymbol{f_{\mathbf{col}}})).
\end{aligned}$$

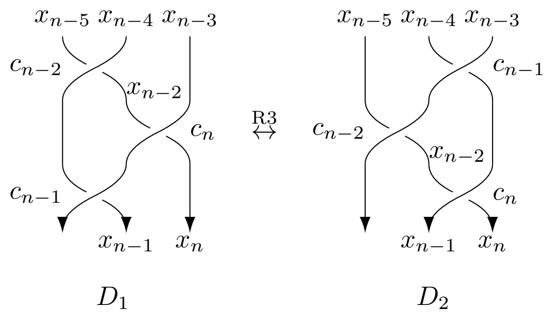

FIGURE 12

9.3. **Reidemeister move III.** Let D_1 and D_2 be diagrams of an oriented link L that differ by a single Reidemeister move III as shown in Figure 12. Let $c_1, \dots, c_{n-3}$ be $n-3$ crossings of D_1 and D_2 that stay outside the disk in which the move is applied, and let c_{n-2}, c_{n-1} and c_n be the other crossings of D_1 and D_2 as shown in Figure 12. Put $a := \rho(x_{n-5})$, $b := \rho(x_{n-4})$, $c := \rho(x_{n-3})$ and $z := (\widetilde{\rho}(r_n) \triangleleft^{-1} c) \triangleleft^{-1} b$. We then have

$$A(D_1) = \begin{pmatrix} A_{n-6} & \boldsymbol{a}_{n-5} & \boldsymbol{a}_{n-4} & \boldsymbol{a}_{n-3} & \mathbf{0} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \\ \mathbf{0} & f_1(a,b) & f_2(a,b) & 0 & -1 & 0 & 0 \\ \mathbf{0} & 0 & f_1(b,c) & f_2(b,c) & 0 & -1 & 0 \\ \mathbf{0} & 0 & 0 & f_2(a \triangleleft b, c) & f_1(a \triangleleft b, c) & 0 & -1 \end{pmatrix},$$

$$A(D_2) = \begin{pmatrix} A_{n-6} & \boldsymbol{a}_{n-5} & \boldsymbol{a}_{n-4} & \boldsymbol{a}_{n-3} & \mathbf{0} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \\ \mathbf{0} & f_1(a,c) & 0 & f_2(a,c) & -1 & 0 & 0 \\ \mathbf{0} & 0 & f_1(b,c) & f_2(b,c) & 0 & -1 & 0 \\ \mathbf{0} & 0 & 0 & 0 & f_1(a \triangleleft c, b \triangleleft c) & f_2(a \triangleleft c, b \triangleleft c) & -1 \end{pmatrix}$$

and

$$R_{\mathrm{row}}(D_1) = \begin{pmatrix} B_{n-3} & \boldsymbol{f}_{\mathbf{row}}(z \triangleleft b, a \triangleleft b) & \boldsymbol{f}_{\mathbf{row}}(z \triangleleft c, b \triangleleft c) & \boldsymbol{f}_{\mathbf{row}}((z \triangleleft b) \triangleleft c, (a \triangleleft b) \triangleleft c) \end{pmatrix},$$

$$R_{\mathrm{row}}(D_2) = \begin{pmatrix} B_{n-3} & \boldsymbol{f}_{\mathbf{row}}(z \triangleleft c, a \triangleleft c) & \boldsymbol{f}_{\mathbf{row}}((z \triangleleft a) \triangleleft c, b \triangleleft c) & \boldsymbol{f}_{\mathbf{row}}((z \triangleleft b) \triangleleft c, (a \triangleleft b) \triangleleft c) \end{pmatrix}.$$

We also have

$$R_{\mathrm{col}}(D_1) = \begin{pmatrix} C_{n-6} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \boldsymbol{f}_{\mathbf{col}}(b) \\ \boldsymbol{f}_{\mathbf{col}}(c) \\ \boldsymbol{f}_{\mathbf{col}}(a \triangleleft b) \\ \boldsymbol{f}_{\mathbf{col}}(b \triangleleft c) \\ \boldsymbol{f}_{\mathbf{col}}((a \triangleleft b) \triangleleft c) \end{pmatrix}, \qquad R_{\mathrm{col}}(D_2) = \begin{pmatrix} C_{n-6} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \boldsymbol{f}_{\mathbf{col}}(b) \\ \boldsymbol{f}_{\mathbf{col}}(c) \\ \boldsymbol{f}_{\mathbf{col}}(a \triangleleft c) \\ \boldsymbol{f}_{\mathbf{col}}(b \triangleleft c) \\ \boldsymbol{f}_{\mathbf{col}}((a \triangleleft b) \triangleleft c) \end{pmatrix},$$

and

$$\mathrm{cor}(D_1, \rho; f_1, f_2) = \mathrm{cor}(D_2, \rho; f_1, f_2).$$

We put

$$A_1^{\mathrm{R3}} := \begin{pmatrix} A_{n-6} & \boldsymbol{a}_{n-5} & \boldsymbol{a}_{n-4} & \boldsymbol{a}_{n-3} & \mathbf{0} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \\ \mathbf{0} & f_1(a,b) & f_2(a,b) & 0 & -1 & 0 & 0 \\ \mathbf{0} & 0 & f_1(b,c) & f_2(b,c) & 0 & -1 & 0 \\ \mathbf{0} & a_{n,n-5} & a_{n,n-4} & a_{n,n-3} & 0 & 0 & -1 \end{pmatrix} \oplus (\mathrm{cor}(D_1)^{-1}),$$

$$A_2^{\mathrm{R3}} := \begin{pmatrix} A_{n-6} & \boldsymbol{a}_{n-5} & \boldsymbol{a}_{n-4} & \boldsymbol{a}_{n-3} & \mathbf{0} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \\ \mathbf{0} & 0 & 0 & 0 & -1 & 0 & 0 \\ \mathbf{0} & 0 & f_1(b,c) & f_2(b,c) & 0 & -1 & 0 \\ \mathbf{0} & a_{n,n-5} & a_{n,n-4} & a_{n,n-3} & 0 & 0 & -1 \end{pmatrix} \oplus (\mathrm{cor}(D_1)^{-1}),$$

$$A_3^{\mathrm{R3}} := \begin{pmatrix} A_{n-6} & \boldsymbol{a}_{n-5} & \boldsymbol{a}_{n-4} & \boldsymbol{a}_{n-3} & \mathbf{0} & \boldsymbol{a}_{n-1} & \boldsymbol{a}_n \\ \mathbf{0} & f_1(a,c) & 0 & f_2(a,c) & -1 & 0 & 0 \\ \mathbf{0} & 0 & f_1(b,c) & f_2(b,c) & 0 & -1 & 0 \\ \mathbf{0} & a_{n,n-5} & a_{n,n-4} & a_{n,n-3} & 0 & 0 & -1 \end{pmatrix} \oplus (\mathrm{cor}(D_2)^{-1}),$$

where

$$\begin{aligned} a_{n,n-5} &= f_1(a \triangleleft b, c) f_1(a,b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a,c), \\ a_{n,n-4} &= f_1(a \triangleleft b, c) f_2(a,b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b,c), \\ a_{n,n-3} &= f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a,c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b,c). \end{aligned}$$

We put

$$R_{\mathrm{row}}^{\mathrm{R3}} := \begin{pmatrix} B_{n-3} & \boldsymbol{b}_{n-2} & \boldsymbol{f}_{\mathbf{row}}(z \triangleleft c, b \triangleleft c) & \boldsymbol{f}_{\mathbf{row}}((z \triangleleft b) \triangleleft c, (a \triangleleft b) \triangleleft c) & \mathbf{0} \end{pmatrix},$$

where $\boldsymbol{b}_{n-2} = \boldsymbol{f}_{\mathbf{row}}(z \triangleleft b, a \triangleleft b) - \boldsymbol{f}_{\mathbf{row}}((z \triangleleft b) \triangleleft c, (a \triangleleft b) \triangleleft c) f_1(a \triangleleft b, c) = \mathbf{0}$. We also put

$$R_{\mathrm{col}}^{\mathrm{R3}} := \begin{pmatrix} C_{n-6} \\ \boldsymbol{f}_{\mathbf{col}}(a) \\ \boldsymbol{f}_{\mathbf{col}}(b) \\ \boldsymbol{f}_{\mathbf{col}}(c) \\ \boldsymbol{c}_{n-2} \\ \boldsymbol{f}_{\mathbf{col}}(b \triangleleft c) \\ \boldsymbol{f}_{\mathbf{col}}((a \triangleleft b) \triangleleft c) \\ \mathbf{0} \end{pmatrix},$$

where $\boldsymbol{c}_{n-2} = \boldsymbol{f}_{\mathbf{col}}(a \triangleleft b) - f_1(a,b)\boldsymbol{f}_{\mathbf{col}}(a) - f_2(a,b)\boldsymbol{f}_{\mathbf{col}}(b) = \mathbf{0}$. It follows that

$$\begin{aligned} &(\widetilde{R_{\mathrm{row}}}(D_1, \rho; \boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D_1, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D_1, \rho; \boldsymbol{f}_{\mathbf{col}})) \\ &\sim \left(R_{\mathrm{row}}^{\mathrm{R3}}, A_1^{\mathrm{R3}}, \widetilde{R_{\mathrm{col}}}(D_1, \rho; \boldsymbol{f}_{\mathbf{col}}))\right) \\ &\sim (R_{\mathrm{row}}^{\mathrm{R3}}, A_2^{\mathrm{R3}}, R_{\mathrm{col}}^{\mathrm{R3}}) \\ &\sim \left(R_{\mathrm{row}}^{\mathrm{R3}}, A_3^{\mathrm{R3}}, \widetilde{R_{\mathrm{col}}}(D_2, \rho; \boldsymbol{f}_{\mathbf{col}}))\right) \\ &\sim (\widetilde{R_{\mathrm{row}}}(D_2, \rho; \boldsymbol{f}_{\mathbf{row}}), \widetilde{A}(D_2, \rho; f_1, f_2), \widetilde{R_{\mathrm{col}}}(D_2, \rho; \boldsymbol{f}_{\mathbf{col}})), \end{aligned}$$

where the third equivalence follows from

$$\boldsymbol{c}_{n-2} = \mathbf{0} = \boldsymbol{f}_{\mathbf{col}}(a \triangleleft c) - f_1(a,c)\boldsymbol{f}_{\mathbf{col}}(a) - f_2(a,c)\boldsymbol{f}_{\mathbf{col}}(c)$$

and the last equivalence follows from

$$\begin{aligned} &\boldsymbol{b}_{n-2} = \mathbf{0} = \boldsymbol{f}_{\mathbf{row}}(z \triangleleft c, a \triangleleft c) - \boldsymbol{f}_{\mathbf{row}}((z \triangleleft b) \triangleleft c, (a \triangleleft b) \triangleleft c) f_1(a \triangleleft c, b \triangleleft c), \\ &\boldsymbol{f}_{\mathbf{row}}(z \triangleleft c, b \triangleleft c) \\ &= \boldsymbol{f}_{\mathbf{row}}((z \triangleleft a) \triangleleft c, b \triangleleft c) - \boldsymbol{f}_{\mathbf{row}}((z \triangleleft b) \triangleleft c, (a \triangleleft b) \triangleleft c) f_2(a \triangleleft c, b \triangleleft c). \end{aligned}$$

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