DERIVATIVES WITH MCQ ALEXANDER PAIRS FOR HANDLEBODY-KNOTS

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Abstract. In this paper, we introduce the f-derivative for multiple conjugation quandles with an MCQ Alexander pair f, where a multiple conjugation quandle is an algebra whose axioms are motivated from handlebody-knot theory, and an MCQ Alexander pair is a pair of maps related to a linear extension of a multiple conjugation quandle. Using this, we define f-twisted Alexander matrices for handlebody-knots equipped with their multiple conjugation quandle representations. We show that this matrix produces strictly stronger invariants than the (twisted) Alexander matrix for handlebody-knots.

1. Introduction

The (twisted) Alexander matrix of a knot group is obtained by using the free derivative introduced by Fox [6]. Alexander [1] defined a classical knot invariant, called the Alexander polynomial, which is a generator of the elementary ideal of the Alexander matrix of the knot group. Lin [18] and Wada [22] generalized it to the twisted Alexander polynomial of a knot equipped with a group representation. Topological properties of knots, such as the genus, fiberedness, and so on, appear in the Alexander polynomials, and the properties are extended to the twisted Alexander polynomials [5, 8, 9, 10, etc.]. More generally, by using the free derivative, we can obtain invariants of groups equipped with their group representations.

A quandle [17, 19] is an algebra whose axioms are motivated from knot theory. Oshiro and the first author [16] introduced the f-derivative for quandles with a pair of maps f called an Alexander pair. An Alexander pair is a dynamical cocycle [2] corresponding to a linear extension of a quandle. They defined invariants of quandles equipped with their quandle representations by using an f-derivative. The (twisted) Alexander polynomial for knots can be obtained in their framework.

A multiple conjugation quandle (MCQ) [12] is an algebra whose axioms correspond to the Reidemeister moves for handlebody-knots, where a handlebody-knot is a handlebody embedded in the 3-sphere S^3, which is a generalization of a knot to higher genera. A handlebody-knot can be also regarded as a quotient structure of a spatial graph. The second author [20] introduced a pair of maps called an MCQ Alexander pair, which is an MCQ version of an Alexander pair. The first author [13] introduced the fundamental MCQ of a handlebody-knot, whose representations correspond to MCQ colorings for the handlebody-knot. However, it is not easy to distinguish two MCQs in general. Then counting the number of MCQ representations of the fundamental MCQ of a handlebody-knot to an MCQ

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gives an elementary combinatorial invariant, called the MCQ coloring number. In this paper, we introduce the f-derivative for MCQs with an MCQ Alexander pair f and define the f-twisted Alexander matrix for handlebody-knots equipped with their MCQ representations. Furthermore, we see that this matrix produces strictly stronger invariants than the (twisted) Alexander matrix for handlebody-knots and spatial graphs [15].

This paper is organized as follows. In Section 2, we recall the notions of a quandle and a multiple conjugation quandle (MCQ). In Section 3, we review the notion of an MCQ Alexander pair and give some examples of MCQ Alexander pairs. In Section 4, we recall the notions of an MCQ presentation and the fundamental MCQ of a handlebody-knot. We see that two handlebody-knots equipped with their MCQ representations are equivalent if and only if their fundamental MCQs with the MCQ representations are related by a finite sequence of some transformations. In Section 5, we introduce the f-derivative for MCQs with an MCQ Alexander pair f. In Section 6, we introduce f-twisted Alexander matrices and handlebody-knot invariants derived from the matrices with an MCQ Alexander pair f. In Section 7, we show that the (twisted) Alexander matrix for handlebody-knots is recoverable from an f-twisted Alexander matrix for some MCQ Alexander pair f. In Section 8, calculating our invariants, we distinguish two handlebody-knots whose complements have isomorphic fundamental groups. We emphasize that they can not be distinguished by invariants derived from the (twisted) Alexander matrices for handlebody-knots.

2. Multiple conjugation quandles

A *quandle* [17, 19] is a non-empty set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

(Q1) For any $a \in Q$, $a \triangleleft a = a$.
(Q2) For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
(Q3) For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote the iterated map $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. For quandles $(Q_1, \triangleleft_1)$ and $(Q_2, \triangleleft_2)$, a *quandle homomorphism* $f : Q_1 \to Q_2$ is defined to be a map from Q_1 to Q_2 satisfying $f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)$ for any $a, b \in Q_1$.

Let G be a group and n an integer. We define a binary operation $\triangleleft$ on G by $a \triangleleft b = b^{-n}ab^n$. Then, $(G, \triangleleft)$ is a quandle. We call it the *n-fold conjugation quandle* of G and denote it by $\mathrm{Conj}_n G$. The 1-fold conjugation quandle of G is called the *conjugation quandle* of G and denoted by $\mathrm{Conj} G$. We define another binary operation $\triangleleft$ on G by $a \triangleleft b = ba^{-1}b$. Then, $(G, \triangleleft)$ is a quandle. We call it the *core quandle* of G and denote it by $\mathrm{Core} G$. For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n. We define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = 2b - a$. Then, $(\mathbb{Z}_n, \triangleleft)$ is a quandle. We call it the *dihedral quandle* of order n and denote it by R_n. Let Q be an $R[t^{\pm 1}]$-module for a commutative ring R. We define a binary operation $\triangleleft$ on Q by $a \triangleleft b = ta + (1 - t)b$. Then Q is a quandle, called an *Alexander quandle*.

We define the *type* of a quandle Q by

$$\mathrm{type}\, Q = \min\{n \in \mathbb{Z}_{>0} \mid x \triangleleft^n y = x \text{ for any } x, y \in Q\},$$

where we set $\min \emptyset := \infty$ for the empty set $\emptyset$, and $\mathbb{Z}_{>0}$ denotes the set of positive integers. We note that $(Q, \triangleleft^i)$ is also a quandle for any $i \in \mathbb{Z}$, and any finite quandle

is of finite type. For a quandle Q, an *extension* of Q is a quandle $\widetilde{Q}$ which has a surjective homomorphism $f : \widetilde{Q} \to Q$ such that the cardinalities of the inverse images $f^{-1}(a)$ and $f^{-1}(b)$ coincide for any $a, b \in Q$.

Let $(Q, \triangleleft)$ be a quandle and R a ring. The pair of maps $f_1, f_2 : Q \times Q \to R$ is *Alexander pair* [16] if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a, a) + f_2(a, a) = 1$.
- For any $a, b \in Q$, $f_1(a, b)$ is invertible.
- For any $a, b, c \in Q$,

$$\begin{aligned} &f_1(a \triangleleft b, c) f_1(a, b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a, c), \\ &f_1(a \triangleleft b, c) f_2(a, b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b, c), \\ &f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c). \end{aligned}$$

An Alexander pair is a dynamical cocycle [2] corresponding to a linear extension of a quandle. Many examples of Alexander pairs are given in [16].

Definition 2.1 ([12])**.** A *multiple conjugation quandle (MCQ)* X is a disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a binary operation $\triangleleft : X \times X \to X$ satisfying the following axioms:

- For any $a, b \in G_\lambda$, $a \triangleleft b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x \triangleleft e_\lambda = x$ and $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$, where $a \triangleleft x, b \triangleleft x \in G_\mu$ for some $\mu \in \Lambda$.

In this paper, we often omit parentheses. When doing so, we apply binary operations from left on expressions, except for group operations, which we always apply first. For example, we write $a \triangleleft_1 b \triangleleft_2 cd \triangleleft_3 (e \triangleleft_4 f \triangleleft_5 g)$ for $((a \triangleleft_1 b) \triangleleft_2 (cd)) \triangleleft_3 ((e \triangleleft_4 f) \triangleleft_5 g)$ simply, where each $\triangleleft_i$ is a binary operation, and c and d are elements of the same group. For an MCQ $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$, we denote by G_a the group G_λ containing $a \in X$, and we denote by e_a the identity of G_a.

We remark that an MCQ itself is a quandle. For two MCQs $X_1 = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ and $X_2 = \bigsqcup_{\mu \in M} G_\mu$, an *MCQ homomorphism* $f : X_1 \to X_2$ is defined to be a map from X_1 to X_2 satisfying $f(x \triangleleft y) = f(x) \triangleleft f(y)$ for any $x, y \in X_1$ and $f(ab) = f(a)f(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. An MCQ homomorphism $\rho : X_1 \to X_2$ is also called an *MCQ representation* of X_1 to X_2. We denote by $\mathrm{Hom}(X_1, X_2)$ the set of MCQ homomorphisms from X_1 to X_2. We call a bijective MCQ homomorphism an *MCQ isomorphism*. When there exists an MCQ isomorphism from X_1 to X_2, we call that X_1 and X_2 are *isomorphic*, denoted by $X_1 \cong X_2$. Let $\rho_1 : X_1 \to Y$ and $\rho_2 : X_2 \to Y$ be MCQ representations. We say (X_1, ρ_1) and (X_2, ρ_2) are *isomorphic*, denoted by $(X_1, \rho_1) \cong (X_2, \rho_2)$, if there exists an MCQ isomorphism $f : X_1 \to X_2$ such that $\rho_1 = \rho_2 \circ f$. For an MCQ $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$, an *extension* of X is an MCQ $\widetilde{X}$ which has a surjective MCQ homomorphism $f : \widetilde{X} \to X$ such that the cardinalities of the inverse images $f^{-1}(x)$ and $f^{-1}(y)$ coincide for any $x, y \in X$.

We recall the definition of a G-family of quandles. A G-family of quandles is an algebraic system which yields an MCQ.

Definition 2.2 ([14])**.** Let G be a group with identity element e. A *G-family of quandles* is a non-empty set X with a family of binary operations $\triangleleft^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms:

- For any $x \in X$ and $g \in G$, $x \triangleleft^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x \triangleleft^e y = x$ and $x \triangleleft^{gh} y = (x \triangleleft^g y) \triangleleft^h y$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x \triangleleft^g y) \triangleleft^h z = (x \triangleleft^h z) \triangleleft^{h^{-1}gh} (y \triangleleft^h z)$.

Let R be a ring and G a group with identity element e. Let X be a right $R[G]$-module, where $R[G]$ is the group ring of G over R. Then $(X, \{\triangleleft^g\}_{g\in G})$ is a G-family of quandles, called a *G-family of Alexander quandles*, with $x \triangleleft^g y = xg + y(e-g)$ [14]. Let $(Q, \triangleleft)$ be a quandle and put $k := \operatorname{type} Q$. Then $(Q, \{\triangleleft^i\}_{i\in\mathbb{Z}_k})$ is a $\mathbb{Z}_k$-family of quandles, where we put $\mathbb{Z}_\infty := \mathbb{Z}$.

Let $(X, \{\triangleleft^g\}_{g\in G})$ be a G-family of quandles. Then $X \times G = \bigsqcup_{x\in X}(\{x\} \times G)$ is an MCQ with

$$(x, g) \triangleleft (y, h) := (x \triangleleft^h y, h^{-1}gh), \qquad (x, g)(x, h) := (x, gh)$$

for any $x, y \in X$ and $g, h \in G$ [12]. We call it the *associated MCQ* of $(X, \{\triangleleft^g\}_{g\in G})$. The associated MCQ $X \times G$ is an extension of G, where we regard G as an MCQ with the conjugation operation.

3. MCQ Alexander pairs

In this section, we recall the notion of MCQ Alexander pairs and give some examples of them. Throughout this paper, we assume that every ring has the multiplicative identity $1 \neq 0$. For a ring R, we denote by $R^\times$ the group of units of R.

Definition 3.1 ([20])**.** Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be an MCQ and R a ring. The pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to R$ is an *MCQ Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a, b \in G_\lambda$,
$$f_1(a, b) + f_2(a, b) = f_1(a, a^{-1}b). \tag{1-i}$$
- For any $a, b \in G_\lambda$ and $x \in X$,
$$f_1(a, x) = f_1(b, x), \tag{2-i}$$
$$f_2(ab, x) = f_2(a, x) + f_1(b \triangleleft x, a^{-1} \triangleleft x) f_2(b, x). \tag{2-ii}$$
- For any $x \in X$ and $a, b \in G_\lambda$,
$$f_1(x, e_\lambda) = 1, \tag{3-i}$$
$$f_1(x, ab) = f_1(x \triangleleft a, b) f_1(x, a), \tag{3-ii}$$
$$f_2(x, ab) = f_1(x \triangleleft a, b) f_2(x, a). \tag{3-iii}$$
- For any $x, y, z \in X$,
$$f_1(x \triangleleft y, z) f_1(x, y) = f_1(x \triangleleft z, y \triangleleft z) f_1(x, z), \tag{4-i}$$
$$f_1(x \triangleleft y, z) f_2(x, y) = f_2(x \triangleleft z, y \triangleleft z) f_1(y, z), \tag{4-ii}$$
$$f_2(x \triangleleft y, z) = f_1(x \triangleleft z, y \triangleleft z) f_2(x, z) + f_2(x \triangleleft z, y \triangleleft z) f_2(y, z). \tag{4-iii}$$

As same as an Alexander pair for a quandle, an MCQ Alexander pair corresponds to a linear extension of an MCQ [20]. We note that an MCQ Alexander pair is an Alexander pair. We call $(1,0)$ the *trivial MCQ Alexander pair*, where 0 and 1 respectively denote the zero map and the constant map that sends all elements of the domain to the multiplicative identity 1 of the ring.

We give some examples of MCQ Alexander pairs. In Section 7, we see that the MCQ Alexander pairs in Examples 3.2 and 3.3 are related to the Alexander matrix and the twisted Alexander matrix for handlebody-knots, respectively.

Let R be a ring. We denote by $M(m,n;R)$ the set of $m \times n$ matrices over R and by $GL(n;R)$ the set of $n \times n$ invertible matrices over R. Let G_0 be the abelian group

$$\left\langle t_1, \dots, t_r \,\middle|\, t_1^{k_1}, \dots, t_r^{k_r}, [t_i, t_j] \ (1 \le i < j \le r) \right\rangle,$$

where $k_1, \dots, k_r \in \mathbb{Z}_{\ge 0}$, and $[t_i, t_j]$ denotes the commutator of t_i and t_j. We remark that the group ring $R[G_0]$ can be identified with the quotient ring of the Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1)$.

Example 3.2. We set maps $f_1, f_2 : G_0 \times G_0 \to R[G_0]$ by

$$f_1(a,b) = b^{-1}, \ f_2(a,b) = b^{-1}a - b^{-1}.$$

Then the pair (f_1, f_2) is an MCQ Alexander pair.

Example 3.3. We set maps $f_1, f_2 : GL(k, R[G_0]) \times GL(k, R[G_0]) \to M(k,k; R[G_0])$ by

$$f_1(a,b) = b^{-1}, \ f_2(a,b) = b^{-1}a - b^{-1}.$$

Then the pair (f_1, f_2) is an MCQ Alexander pair.

Example 3.4. Let X be an MCQ, R a ring and $f : X \to R^\times$ an MCQ homomorphism. We set maps $f_1, f_2 : X \times X \to R$ by

$$f_1(x,y) = f(y)^{-1}, \ f_2(x,y) = f(y)^{-1}(f(x) - 1).$$

Then the pair (f_1, f_2) is an MCQ Alexander pair.

By using the following proposition, we can construct MCQ Alexander pairs from Alexander pairs.

Proposition 3.5. *Let $(Q, \triangleleft)$ be a quandle and assume $k := \operatorname{type} Q < \infty$. Let R be a ring and let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$ satisfying*

$$\prod_{i=1}^{k} f_1(x \triangleleft^{k-i} y, y) = 1 \qquad \text{and} \qquad \sum_{i=1}^{k} f_1(x,x)^i = 0$$

for any $x, y \in Q$. Let $X := Q \times \mathbb{Z}_k$ be the associated MCQ of a $\mathbb{Z}_k$-family of quandles $(Q, \{\triangleleft^i\}_{i \in \mathbb{Z}_k})$. We define maps $\widetilde{f_1}, \widetilde{f_2} : X \times X \to R$ by

$$\widetilde{f_1}((x,a),(y,b)) = \prod_{i=1}^{\overline{b}} f_1(x \triangleleft^{b-i} y, y),$$

$$\widetilde{f_2}((x,a),(y,b)) = \left(\prod_{i=1}^{\overline{b-1}} f_1(x \triangleleft^{b-i} y, y) \right) \sum_{j=1}^{\overline{a}} f_1(x \triangleleft y, x \triangleleft y)^{j-a} f_2(x,y),$$

where for any $l \in \mathbb{Z}$, we denote by $\bar{l}$ the integer satisfying $1 \leq \bar{l} \leq k$ and $l \equiv \bar{l}$ mod k. Then the pair $(\widetilde{f_1}, \widetilde{f_2})$ is an MCQ Alexander pair.

Proof. We remark that

$$\prod_{i=1}^{k} f_1(x \lhd^{l-i} y, y) = 1, \text{ especially } f_1(x,x)^k = 1,$$
$$\sum_{i=1}^{k} f_1(x,x)^{l+i} = 0$$

for any $x, y \in Q$ and $l \in \mathbb{Z}$, and that $\overline{l_1} + \overline{l_2} \equiv \overline{l_1 + l_2}$ mod k for any $l_1, l_2 \in \mathbb{Z}$. Since (f_1, f_2) is an Alexander pair, we have the following equalities:

$$f_1(x \lhd^c y, x \lhd^c y)^{-a} \left(\prod_{i=1}^{\overline{c-1}} f_1(x \lhd^{c-i} y, y) \right) = \left(\prod_{i=1}^{\overline{c-1}} f_1(x \lhd^{c-i} y, y) \right) f_1(x \lhd y, x \lhd y)^{-a}, \tag{1}$$

$$\left(\prod_{i=1}^{\overline{c}} f_1((x \lhd^b y) \lhd^{c-i} z, z) \right) \left(\prod_{i=1}^{\overline{b}} f_1(x \lhd^{b-i} y, y) \right)$$
$$= \left(\prod_{i=1}^{\overline{b}} f_1((x \lhd^{b-i} y) \lhd^c z, y \lhd^c z) \right) \left(\prod_{i=1}^{\overline{c}} f_1(x \lhd^{c-i} z, z) \right), \tag{2}$$

$$f_2(x \lhd^c z, y \lhd^c z) \left(\prod_{i=1}^{\overline{c}} f_1(y \lhd^{c-i} z, z) \right) = \left(\prod_{i=1}^{\overline{c}} f_1((x \lhd y) \lhd^{c-i} z, z) \right) f_2(x,y), \tag{3}$$

$$\left(\sum_{j=1}^{\overline{a}} f_1((x \lhd y) \lhd^c z, (x \lhd y) \lhd^c z)^{j-a} \right) \left(\prod_{i=1}^{\overline{c}} f_1((x \lhd y) \lhd^{c-i} z, z) \right)$$
$$= \left(\prod_{i=1}^{\overline{c}} f_1((x \lhd y) \lhd^{c-i} z, z) \right) \left(\sum_{j=1}^{\overline{a}} f_1(x \lhd y, x \lhd y)^{j-a} \right), \tag{4}$$

$$f_2(x \lhd z, y \lhd z) \left(\sum_{j=1}^{\overline{b}} f_1(y \lhd z, y \lhd z)^{j-b} \right)$$
$$= \sum_{j=1}^{\overline{b}} \left(\prod_{i=0}^{\overline{j-b-1}} f_1((x \lhd^{k-i} y) \lhd z, y \lhd z) \right) f_2((x \lhd^{b-j} y) \lhd z, y \lhd z) \tag{5}$$

for any $a, b, c \in \mathbb{Z}$ and $x, y, z \in Q$. By the above equalities (1)–(5), the axioms (2-ii), (4-i), (4-ii) and (4-iii) in Definition 3.1 hold. It is easy to see that the axioms (1-i), (2-i), (3-i), (3-ii) and (3-iii) in Definition 3.1 hold. Therefore $(\widetilde{f_1}, \widetilde{f_2})$ is an MCQ Alexander pair. □

Example 3.6. Let Q be a quandle and assume $k := \operatorname{type} Q < \infty$. Let R be a ring and let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R[t^{\pm 1}]/(1 + t + \cdots + t^{k-1})$ defined by $f_1(a,b) = t$ and $f_2(a,b) = 1 - t$. Let $X := Q \times \mathbb{Z}_k$ be the associated MCQ of a $\mathbb{Z}_k$-family of quandles $(Q, \{\lhd^i\}_{i \in \mathbb{Z}_k})$. Then, by Proposition 3.5, we have

the MCQ Alexander pair $(\widetilde{f}_1, \widetilde{f}_2)$ of maps $\widetilde{f}_1, \widetilde{f}_2 : X \times X \to R[t^{\pm 1}]/(1+t+\cdots+t^{k-1})$ defined by

$$\widetilde{f}_1((x,a),(y,b)) = t^b,\ \widetilde{f}_2((x,a),(y,b)) = t^b(t^{-a}-1),$$

which coincides with the case when we set $f : X \to R[t^{\pm 1}]/(1+t+\cdots+t^{k-1})$ by $f(a,x) = t^a$ in Example 3.4.

Example 3.7. Let $Q := \operatorname{Core} G$ be the core quandle of a group G. Let R be a ring and let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R[G]$ defined by $f_1(a,b) = -ba^{-1}$ and $f_2(a,b) = 1 + ba^{-1}$. Since $\operatorname{type} Q = 2$, we have the associated MCQ $X := Q \times \mathbb{Z}_2$ of a $\mathbb{Z}_2$-family of quandles $(Q, \{\triangleleft^i\}_{i\in\mathbb{Z}_2})$. Then, by Proposition 3.5, we have the MCQ Alexander pair $(\widetilde{f}_1, \widetilde{f}_2)$ of maps $\widetilde{f}_1, \widetilde{f}_2 : X \times X \to R[G]$ defined by

$$\widetilde{f}_1((x,a),(y,b)) = \begin{cases} 1 & \text{if } b = 0, \\ -yx^{-1} & \text{otherwise,} \end{cases}$$

$$\widetilde{f}_2((x,a),(y,b)) = \begin{cases} 0 & \text{if } a = 0, \\ -1 - xy^{-1} & \text{if } a = 1 \text{ and } b = 0, \\ 1 + yx^{-1} & \text{if } a = 1 \text{ and } b = 1. \end{cases}$$

Example 3.8. Let Q be the conjugacy class of the symmetric group of order 4 consisting of cyclic permutations of length 4. We define a binary operation $\triangleleft$ on Q by $a \triangleleft b = b^{-1}ab$. Then $(Q, \triangleleft)$ is a quandle. We write the elements of Q by $g_1 := (1234), g_2 := (1423), g_3 := (1324), g_4 := (1432), g_5 := (1324), g_6 := (1243)$ and define the map $f_1 : Q \times Q \to \mathbb{Z}_4[t^{\pm 1}]/(t^4-1)$ by

$$f_1(x,y) = \begin{cases} t & \text{if } (x,y) \in \left\{ \begin{array}{c} (g_1,g_3),(g_1,g_4),(g_2,g_1),(g_2,g_3),(g_2,g_5),(g_3,g_1), \\ (g_3,g_5),(g_3,g_6),(g_4,g_1),(g_5,g_1),(g_5,g_2),(g_5,g_6), \\ (g_6,g_1),(g_6,g_2),(g_6,g_3),(g_6,g_5) \end{array} \right\}, \\ t^2 & \text{if } (x,y) \in \{(g_1,g_5),(g_5,g_3)\}, \\ t^3 & \text{if } (x,y) \in \{(g_1,g_6),(g_3,g_2)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then the pair $(f_1, 0)$ is an Alexander pair, where 0 denotes the zero map. Since $\operatorname{type} Q = 4$, we have the associated MCQ $X := Q \times \mathbb{Z}_4$ of the $\mathbb{Z}_4$-family of quandles $(Q, \{\triangleleft^i\}_{i\in\mathbb{Z}_4})$. Then, by Proposition 3.5, we have the MCQ Alexander pair $(\widetilde{f}_1, 0)$, where $\widetilde{f}_1$ is the map $\widetilde{f}_1 : X \times X \to \mathbb{Z}_4[t^{\pm 1}]/(t^4-1)$ defined by $\widetilde{f}_1((x,a),(y,b)) = \prod_{i=1}^{\overline{b}} f_1(x \triangleleft^{b-i} y, y)$.

Remark 3.9. In general, a quandle 2-cocycle $\theta : Q \times Q \to A$ induces an Alexander pair $(f_\theta, 0)$, where Q and A denote a quandle and an abelian group, respectively, and $f_\theta : Q \times Q \to R[A]$ is the map defined by $f_\theta(a,b) = \theta(a,b)$ for a ring R. The Alexander pair $(f_1, 0)$ in Example 3.8 is obtained through this process from a quandle 2-cocycle (refer to [4, Appendix]). Taniguchi [21] showed that, for classical links, the 0-th elementary ideal of the f-twisted Alexander matrix [16] with an Alexander pair $f = (f_\theta, 0)$ can be realized from a quandle cocycle invariant [3] with a quandle 2-cocycle θ.

Proposition 3.10. *Let $\rho : X \to Y$ be an MCQ representation. If (f_1, f_2) is an MCQ Alexander pair of maps $f_1, f_2 : Y \times Y \to R$, then $(f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$ is also an MCQ Alexander pair.*

Proof. This proposition can be verified by direct calculation. □

4. MCQ presentations and the fundamental MCQ of a handlebody-link

In this section, we review the notions of MCQ presentations and the fundamental MCQ of a handlebody-link. For details see [13].

The free MCQ $F_{\mathsf{MCQ}}(S_\Lambda)$ over a given set of pairwise disjoint sets $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ is a free object in the category of MCQs. It consists of all MCQ words that can be built from elements of $\bigcup_{\lambda\in\Lambda} S_\lambda$ as described below. For $k \geq 0$, we set

$$\begin{aligned}
&\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; 0) := \bigcup_{\lambda\in\Lambda} S_\lambda,\\
&\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k+1)\\
&:= \left\{ a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n} \;\middle|\; \begin{array}{l} n \geq 1,\ \varepsilon_1, \ldots, \varepsilon_n \in \{0, 1, -1\},\ a_1, \ldots, a_n \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k),\\ a_1, \ldots, a_n \text{ represent multiplicable elements of } F_{\mathsf{MCQ}}(S_\Lambda) \end{array} \right\}\\
&\quad \cup \{x \lhd y \,|\, x, y \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k)\},
\end{aligned}$$

where $a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n}$ and $x \lhd y$ are symbols, and we put parentheses in appropriate places. For example,

$$a \lhd a, a \lhd b, b \lhd a, b \lhd b, a, a^0, a^{-1}, b, b^0, b^{-1}, aa, aa^{-1}, a^{-1}aa, bb^{-1}, bb^0b^{-1}$$

are elements of $\mathcal{W}_{\mathsf{MCQ}}(\{\{a\}, \{b\}\}; 1)$, and

$$(a \lhd b) \lhd b^{-1}, aa \lhd (a \lhd b), (a \lhd b) \lhd aa, b^{-1} \lhd (a \lhd b), (a \lhd b)^0, (b^{-1})^{-1}, (a \lhd b)(a \lhd b)$$

are elements of $\mathcal{W}_{\mathsf{MCQ}}(\{\{a\}, \{b\}\}; 2)$. We define $\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda) := \bigcup_{k=0}^{\infty} \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k)$ and call its element *MCQ word in* S_Λ. Two MCQ words in S_Λ are *multiplicable* if they represent multiplicable elements of $F_{\mathsf{MCQ}}(S_\Lambda)$.

Lemma 4.1 ([13])**.** *Let $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. For $w_1, w_2 \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$, w_1 and w_2 represent the same element in $F_{\mathsf{MCQ}}(S_\Lambda)$ if and only if w_1 and w_2 are related by a finite sequence of the following local replacements on MCQ words:*

$$\begin{aligned}
&ab^\varepsilon b^{-\varepsilon}c \leftrightarrow ab^0c \leftrightarrow ac,\\
&a \lhd b \leftrightarrow b^{-1}ab,\\
&x \lhd a^0 \leftrightarrow x,\ \ x \lhd ab \leftrightarrow (x \lhd a) \lhd b,\\
&(x \lhd y) \lhd z \leftrightarrow (x \lhd z) \lhd (y \lhd z),\\
&ab \lhd x \leftrightarrow (a \lhd x)(b \lhd x),
\end{aligned}$$

where $\varepsilon \in \{0, 1, -1\}$ and $a, b, c, x, y, z \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ such that a, b, c are multiplicable.

Every MCQ has a *presentation* $\langle S_\Lambda \,|\, R\rangle$, where $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ is a set of pairwise disjoint sets, and $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$. It is also denoted $\langle S_\lambda\ (\lambda \in \Lambda) \,|\, R\rangle$. We call S_Λ the *generating set* of $\langle S_\Lambda \,|\, R\rangle$ and an element of R a *relator* of $\langle S_\Lambda \,|\, R\rangle$. A relator (a, b) is also written as $a = b$. For $x \in \bigcup_{\lambda\in\Lambda} S_\lambda$, we use the same symbol x for the element of $\langle S_\Lambda \,|\, R\rangle$ represented by x. A presentation $\langle S_\Lambda \,|\, R\rangle$ is

called a *finite presentation* if $\bigcup_{\lambda\in\Lambda} S_\lambda$ and R are finite. For a finitely presented MCQ, we often write

$$\begin{aligned}&\langle x_{1,1},\dots,x_{1,n_1};\dots;x_{l,1},\dots,x_{l,n_l}\,|\,r_1,\dots,r_m\rangle\\&:=\langle\{x_{1,1},\dots,x_{1,n_1}\},\dots,\{x_{l,1},\dots,x_{l,n_l}\}\,|\,\{r_1,\dots,r_m\}\rangle.\end{aligned}$$

We define the MCQ isomorphisms

$$\begin{aligned}&f_{\text{T1-1}}:\langle S_\Lambda\,|\,R\rangle\to\langle S_\Lambda\,|\,R,(x,x)\rangle\ (x\in F_{\mathsf{MCQ}}(S_\Lambda)),\\&f_{\text{T1-2}}:\langle S_\Lambda\,|\,R,(a,b)\rangle\to\langle S_\Lambda\,|\,R,(a,b),(b,a)\rangle,\\&f_{\text{T1-3}}:\langle S_\Lambda\,|\,R,(a,b),(b,c)\rangle\to\langle S_\Lambda\,|\,R,(a,b),(b,c),(a,c)\rangle,\\&f_{\text{T1-4}}:\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2)\rangle\to\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2),(a_1\triangleleft b_1,a_2\triangleleft b_2)\rangle,\\&f_{\text{T1-5}}:\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2)\rangle\to\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2),(a_1b_1^{-1},a_2b_2^{-1})\rangle\\&(a_i\text{ and }b_i\text{ are multiplicable}),\\&f_{\text{T2}}:\langle S_\Lambda\,|\,R\rangle\to\langle S_\Lambda,\{y\}\,|\,R,(y,w)\rangle\ (y\notin F_{\mathsf{MCQ}}(S_\Lambda),w\in F_{\mathsf{MCQ}}(S_\Lambda)),\\&f_{\text{T3-1}}:\langle S_\Lambda,S_\mu,S_\nu\,|\,R,(a^0,b^0)\rangle\to\langle S_\Lambda,S_\mu\cup S_\nu\,|\,R,(a^0,b^0)\rangle\ (a\in S_\mu,b\in S_\nu)\end{aligned}$$

by

$$\begin{aligned}&f_{\text{T1-}i}(x)=x\ (i=1,\dots,5),\\&f_{\text{T2}}(y)=y,\\&f_{\text{T3-1}}(z)=z\end{aligned}$$

for $x,y\in\bigcup_{\lambda\in\Lambda}S_\lambda$ and $z\in\bigcup_{\lambda\in\Lambda}S_\lambda\cup S_\mu\cup S_\nu$, respectively, where in the transformations, $\langle S_\Lambda,S_1,\dots,S_n\,|\,R,r_1,\dots,r_m\rangle$ stands for $\langle S_\Lambda\cup\{S_1,\dots S_n\}\,|\,R\cup\{r_1,\dots,r_m\}\rangle$. We define the following transformations on presentations with MCQ representations:

(T1-1) $(\langle S_\Lambda\,|\,R\rangle,\rho)\leftrightarrow(\langle S_\Lambda\,|\,R,(x,x)\rangle,\rho\circ f_{\text{T1-1}}^{-1})\ (x\in F_{\mathsf{MCQ}}(S_\Lambda))$,
(T1-2) $(\langle S_\Lambda\,|\,R,(a,b)\rangle,\rho)\leftrightarrow(\langle S_\Lambda\,|\,R,(a,b),(b,a)\rangle,\rho\circ f_{\text{T1-2}}^{-1})$,
(T1-3) $(\langle S_\Lambda\,|\,R,(a,b),(b,c)\rangle,\rho)\leftrightarrow(\langle S_\Lambda\,|\,R,(a,b),(b,c),(a,c)\rangle,\rho\circ f_{\text{T1-3}}^{-1})$,
(T1-4) $(\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2)\rangle,\rho)\leftrightarrow(\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2),(a_1\triangleleft b_1,a_2\triangleleft b_2)\rangle,\rho\circ f_{\text{T1-4}}^{-1})$,
(T1-5) $(\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2)\rangle,\rho)\leftrightarrow(\langle S_\Lambda\,|\,R,(a_1,a_2),(b_1,b_2),(a_1b_1^{-1},a_2b_2^{-1})\rangle,\rho\circ f_{\text{T1-5}}^{-1})$ (a_i and b_i are multiplicable),
(T2) $(\langle S_\Lambda\,|\,R\rangle,\rho)\leftrightarrow(\langle S_\Lambda,\{y\}\,|\,R,(y,w)\rangle,\rho\circ f_{\text{T2}}^{-1})\ (y\notin F_{\mathsf{MCQ}}(S_\Lambda),w\in F_{\mathsf{MCQ}}(S_\Lambda))$,
(T3-1) $(\langle S_\Lambda,S_\mu,S_\nu\,|\,R,(a^0,b^0)\rangle,\rho)\leftrightarrow(\langle S_\Lambda,S_\mu\cup S_\nu\,|\,R,(a^0,b^0)\rangle,\rho\circ f_{\text{T3-1}}^{-1})\ (a\in S_\mu,b\in S_\nu)$.

In this paper, we often use the same symbol ρ for $\rho\circ f_{\text{T1-}i}^{-1}$, $\rho\circ f_{\text{T2}}^{-1}$ and $\rho\circ f_{\text{T3-1}}^{-1}$. For finitely presented MCQs $\langle S_\Lambda\,|\,R\rangle$ and $\langle S'_{\Lambda'}\,|\,R'\rangle$, and MCQ representations $\rho:\langle S_\Lambda\,|\,R\rangle\to X$ and $\rho':\langle S'_{\Lambda'}\,|\,R'\rangle\to X$, we write $(\langle S_\Lambda\,|\,R\rangle,\rho)\sim_T(\langle S'_{\Lambda'}\,|\,R'\rangle,\rho')$ if they are transformed into each other by a finite sequence of the transformations (T1-1)–(T1-5), (T2) and (T3-1). Clearly, if $(\langle S_\Lambda\,|\,R\rangle,\rho)\sim_T(\langle S'_{\Lambda'}\,|\,R'\rangle,\rho')$, then $(\langle S_\Lambda\,|\,R\rangle,\rho)\cong(\langle S'_{\Lambda'}\,|\,R'\rangle,\rho')$.

In the following, we recall the fundamental MCQ of a handlebody-link and its Wirtinger presentation. A *handlebody-link* [11] is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent

graph embedded in S^3. In this paper, a trivalent graph may contain circle components. Two handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of R1–R6 moves depicted in Figure 1 [11].

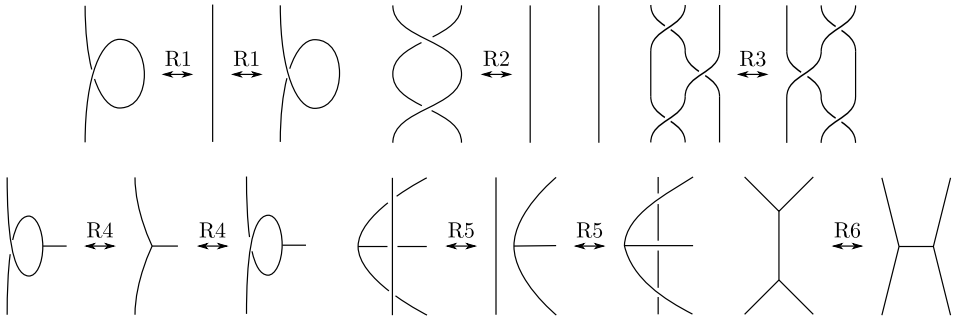

FIGURE 1. The Reidemeister moves for handlebody-links.

Let D be a diagram of a handlebody-link. A *Y-orientation* of D is a collection of orientations of all edges of D without sources and sinks with respect to the orientation as shown in Figure 2, where an edge of D is a piece of a curve each of whose endpoints is a vertex. In this paper, a circle component of D is also regarded as an edge of D. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on a diagram. A vertex of a Y-oriented diagram can be allocated a sign; the vertex is said to have a sign $+1$ or -1. The standard convention of the signs is shown in Figure 2. It is known that every diagram has a Y-orientation.

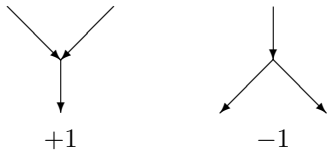

FIGURE 2. Y-orientations.

Let H be a handlebody-link represented by a Y-oriented diagram D. We denote by $C(D)$, $V(D)$ and $\mathcal{A}(D)$ the sets of crossings, vertices and arcs of D, respectively. For each $c \in C(D)$, we denote by v_c the over-arc of c, and we denote by u_c and w_c the under-arcs of c such that the normal orientation of v_c points from u_c to w_c as illustrated in the left of Figure 3. For each $\tau \in V(D)$, if τ has a sign $+1$ (resp. -1), we denote by w_τ the arc whose initial (resp. terminal) vertex is τ, and we denote by u_τ and v_τ the arcs incident to τ such that the normal orientation of w_τ points from u_τ to v_τ as illustrated in the center and right of Figure 3. We denote by $\mathcal{A}^{\sqcup}(D)$ the quotient set of $\mathcal{A}(D)$ by the equivalence relation generated by $\bigcup_{\tau \in V(D)} \{u_\tau, v_\tau, w_\tau\}^2$, that is, two arcs $x, x' \in \mathcal{A}(D)$ are equivalent if there exist arcs $x_1, x_2, \ldots, x_n \in \mathcal{A}(D)$ such that $x = x_1, x' = x_n$, and that x_i and x_{i+1} have a common vertex of D for each i. For example, for the Y-oriented diagram D of a handlebody-knot depicted in Figure 4, we have $\mathcal{A}^{\sqcup}(D) =$

$\{\{x_1, x_2, x_3\}, \{x_4, \ldots, x_{10}\}, \{x_{11}\}, \ldots, \{x_{14}\}\}$. Then we define

$$\mathrm{MCQ}(D) := \langle \mathcal{A}^{\sqcup}(D) \,|\, r_c, r_\tau \ (c \in C(D), \tau \in V(D)) \rangle \,, \tag{6}$$

where r_c and r_τ denote the relators $(u_c \triangleleft v_c, w_c)$ and $(u_\tau v_\tau, w_\tau)$, respectively. The isomorphism class of $\mathrm{MCQ}(D)$ does not depend on the choice of a diagram D of H and its Y-orientation [13]. We then define $\mathrm{MCQ}(H) := \mathrm{MCQ}(D)$ and call it the *fundamental MCQ* of H. The presentation (6) is called the *Wirtinger presentation of* $\mathrm{MCQ}(H)$ *with respect to* D. We obtain the presentation $G(D)$ of the fundamental group $G(H) := \pi_1(S^3 - H)$ by replacing $\mathcal{A}^{\sqcup}(D)$ by $\mathcal{A}(D)$, r_c by $v_c^{-1} u_c v_c w_c^{-1}$ and r_τ by $u_\tau v_\tau w_\tau^{-1}$ in (6), which is called the *Wirtinger presentation of* $G(H)$ *with respect to* D.

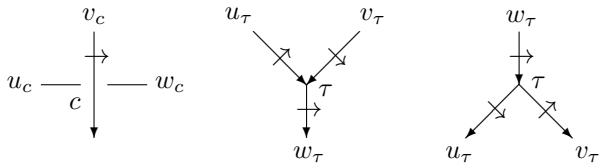

FIGURE 3. Notations of arcs.

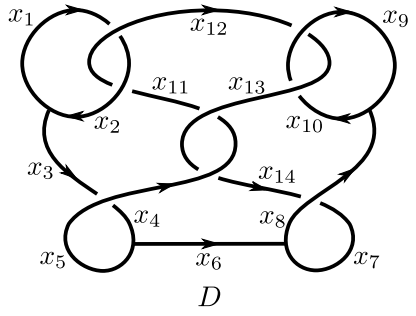

FIGURE 4. A Y-oriented diagram of a handlebody-knot.

Let D be a Y-oriented diagram of a handlebody-link H and let X be an MCQ. An X*-coloring* of D is a map $C : \mathcal{A}(D) \to X$ satisfying the conditions

$$C(u_c) \triangleleft C(v_c) = C(w_c) \qquad \text{and} \qquad C(u_\tau) C(v_\tau) = C(w_\tau)$$

for each crossing $c \in C(D)$ and vertex $\tau \in V(D)$. We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. An X-coloring of D can be regarded as an MCQ representation of $\mathrm{MCQ}(D)$ to X. We can then identify $\mathrm{Col}_X(D)$ with $\mathrm{Hom}(\mathrm{MCQ}(D), X)$. Hence its cardinality gives an invariant for the handlebody-link, called the *MCQ coloring number*.

Let D be a Y-oriented diagram of a handlebody-link H and D' a Y-oriented diagram of H obtained by changing the Y-orientation of D. We then obtain the MCQ isomorphism $f_{(D,D')} : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D')$ sending x into $x^{\varepsilon(x)}$ for any $x \in \mathcal{A}(D)$, where $\varepsilon(x) = 1$ if the Y-orientations of D and D' coincide on x; otherwise $\varepsilon(x) = -1$. Moreover, let D'' a Y-oriented diagram of H obtained by applying one of Reidemeister moves preserving the Y-orientation to D once. We then obtain a unique MCQ isomorphism $f_{(D,D'')} : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D'')$ sending x into x for

any $x \in \mathcal{A}(D \cap D'')$, where $\mathcal{A}(D \cap D'')$ denotes the set of arcs in the outside of the disk where the move is applied. Let H and H' be handlebody-links represented by Y-oriented diagrams D and D', respectively. Let $\rho : \mathrm{MCQ}(D) \to X$ and $\rho' : \mathrm{MCQ}(D') \to X$ be MCQ representations. Then (H, ρ) and (H', ρ') are *equivalent*, denoted by $(H, \rho) \cong (H', \rho')$, if there exists a sequence $D = D_1 \leftrightarrow \cdots \leftrightarrow D_n = D'$ of Reidemeister moves and Y-orientation changes such that $\rho' = \rho \circ f^{-1}_{(D_1, D_2)} \circ \cdots \circ f^{-1}_{(D_{n-1}, D_n)}$. If $H \cong H'$, then for any MCQ representation $\rho : \mathrm{MCQ}(D) \to X$, there exists a unique MCQ representation $\rho' : \mathrm{MCQ}(D') \to X$ such that $(H, \rho) \cong (H', \rho')$. Then we have the following lemma.

Lemma 4.2. *Let H and H' be handlebody-links represented by Y-oriented diagrams D and D', respectively. Let $\rho : \mathrm{MCQ}(D) \to X$ and $\rho' : \mathrm{MCQ}(D') \to X$ be MCQ representations. If $(H, \rho) \cong (H', \rho')$, then it follows $(\mathrm{MCQ}(D), \rho) \sim_T (\mathrm{MCQ}(D'), \rho')$.*

Proof. Assume $(H, \rho) \cong (H', \rho')$. By the definition, there exists a sequence $D = D_1 \leftrightarrow \cdots \leftrightarrow D_n = D'$ of Reidemeister moves and Y-orientation changes such that $\rho' = \rho \circ f^{-1}_{(D_1, D_2)} \circ \cdots \circ f^{-1}_{(D_{n-1}, D_n)}$. By [13], each MCQ isomorphism $f_{(D_i, D_{i+1})}$ can be realized as a composition of $f^{\pm 1}_{\text{T1-1}}, \ldots, f^{\pm 1}_{\text{T1-5}}, f^{\pm 1}_{\text{T2}}$ and $f^{\pm 1}_{\text{T3-1}}$. Then we have $(\mathrm{MCQ}(D), \rho) \sim_T (\mathrm{MCQ}(D'), \rho')$. □

5. Derivatives with MCQ Alexander pairs

Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a finite set of pairwise disjoint finite sets and $x_1, \ldots, x_n$ the elements of $\bigcup_{\lambda \in \Lambda} S_\lambda$. Let $X = \langle S_\Lambda \mid \{r_1, \ldots, r_m\} \rangle$ be a finitely presented MCQ. Let $F_{\mathsf{MCQ}}(S_\Lambda)$ be the free MCQ on S_Λ and $\mathrm{pr} : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ be the canonical projection. We often omit "pr" to represent $\mathrm{pr}(x)$ as x. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. We denote by G_μ each direct summand of $F_{\mathsf{MCQ}}(S_\Lambda)$, where μ is an element of an index set $\overline{\Lambda}$, that is, $F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{\mu \in \overline{\Lambda}} G_\mu$.

Definition 5.1. For $j \in \{1, \ldots, n\}$, the *f-derivative with respect to x_j* is a map $\frac{\partial_f}{\partial x_j} : F_{\mathsf{MCQ}}(S_\Lambda) \to R$ satisfying

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(x \triangleleft y) = f_1(x, y)\frac{\partial_f}{\partial x_j}(x) + f_2(x, y)\frac{\partial_f}{\partial x_j}(y),\\
&\frac{\partial_f}{\partial x_j}(ab) = \frac{\partial_f}{\partial x_j}(a) + f_1(a, a^{-1})\frac{\partial_f}{\partial x_j}(b),\\
&\frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}
\end{aligned}$$

for any $x, y \in F_{\mathsf{MCQ}}(S_\Lambda)$, $a, b \in G_\mu$ and $i \in \{1, \ldots, n\}$, where δ_{ij} denotes the Kronecker delta.

Theorem 5.2. *For $j \in \{1, \ldots, n\}$, the f-derivative $\frac{\partial_f}{\partial x_j} : F_{\mathsf{MCQ}}(S_\Lambda) \to R$ is well-defined.*

Proof. We temporarily regard the f-derivative $\frac{\partial_f}{\partial x_j}$ as the map $\frac{\partial_f}{\partial x_j} : \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda) \to R$ defined by using the following equalities inductively:

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(x \triangleleft y) = f_1(x,y)\frac{\partial_f}{\partial x_j}(x) + f_2(x,y)\frac{\partial_f}{\partial x_j}(y),\\
&\frac{\partial_f}{\partial x_j}(ab) = \frac{\partial_f}{\partial x_j}(a) + f_1(a,a^{-1})\frac{\partial_f}{\partial x_j}(b),\\
&\frac{\partial_f}{\partial x_j}(a^{-1}) = -f_1(a,a)\frac{\partial_f}{\partial x_j}(a),\\
&\frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij},\\
&\frac{\partial_f}{\partial x_j}(a^0) = 0
\end{aligned}$$

for $x, y, a, b \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$, where a and b are multiplicable. This map is well-defined since for any $a, b, c \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ which are multiplicable, we have

$$\begin{aligned}
\frac{\partial_f}{\partial x_j}((ab)c) &= \frac{\partial_f}{\partial x_j}(ab) + f_1(ab, b^{-1}a^{-1})\frac{\partial_f}{\partial x_j}(c)\\
&= \frac{\partial_f}{\partial x_j}(a) + f_1(a,a^{-1})\frac{\partial_f}{\partial x_j}(b) + f_1(ab, b^{-1}a^{-1})\frac{\partial_f}{\partial x_j}(c)\\
&= \frac{\partial_f}{\partial x_j}(a) + f_1(a,a^{-1})\frac{\partial_f}{\partial x_j}(b) + f_1(a,a^{-1})f_1(b,b^{-1})\frac{\partial_f}{\partial x_j}(c)\\
&= \frac{\partial_f}{\partial x_j}(a) + f_1(a,a^{-1})\frac{\partial_f}{\partial x_j}(bc)\\
&= \frac{\partial_f}{\partial x_j}(a(bc)).
\end{aligned}$$

For any $x_1, x_2, y_1, y_2, a_1, a_2, b_1, b_2 \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ satisfying that a_i, b_i are multiplicable for $i = 1, 2$, if

$$\frac{\partial_f}{\partial x_j}(x_1) = \frac{\partial_f}{\partial x_j}(x_2), \quad \frac{\partial_f}{\partial x_j}(y_1) = \frac{\partial_f}{\partial x_j}(y_2), \quad \frac{\partial_f}{\partial x_j}(a_1) = \frac{\partial_f}{\partial x_j}(a_2), \quad \frac{\partial_f}{\partial x_j}(b_1) = \frac{\partial_f}{\partial x_j}(b_2)$$

and $x_1 = x_2$, $y_1 = y_2$, $a_1 = a_2$, $b_1 = b_2$ in $F_{\mathsf{MCQ}}(S_\Lambda)$, then we have

$$\frac{\partial_f}{\partial x_j}(x_1 \triangleleft y_1) = \frac{\partial_f}{\partial x_j}(x_2 \triangleleft y_2), \qquad \frac{\partial_f}{\partial x_j}(a_1 b_1) = \frac{\partial_f}{\partial x_j}(a_2 b_2).$$

Hence, by Lemma 4.1, the well-definedness of the f-derivative $\frac{\partial_f}{\partial x_j} : F_{\mathsf{MCQ}}(S_\Lambda) \to R$ follows from the following equalities:

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(b^\varepsilon b^{-\varepsilon}) = \frac{\partial_f}{\partial x_j}(b^0) = 0,\\
&\frac{\partial_f}{\partial x_j}(a \triangleleft b) = \frac{\partial_f}{\partial x_j}(b^{-1}ab),\\
&\frac{\partial_f}{\partial x_j}(x \triangleleft a^0) = \frac{\partial_f}{\partial x_j}(x),\\
&\frac{\partial_f}{\partial x_j}(x \triangleleft (ab)) = \frac{\partial_f}{\partial x_j}((x \triangleleft a) \triangleleft b),\\
&\frac{\partial_f}{\partial x_j}((x \triangleleft y) \triangleleft z) = \frac{\partial_f}{\partial x_j}((x \triangleleft z) \triangleleft (y \triangleleft z)),\\
&\frac{\partial_f}{\partial x_j}((ab) \triangleleft x) = \frac{\partial_f}{\partial x_j}((a \triangleleft x)(b \triangleleft x)),
\end{aligned}$$

where $\varepsilon \in \{0, 1, -1\}$ and $a, b, c, x, y, z \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ such that a, b, c are multiplicable. We can see that these equalities hold by direct calculation, where the fourth and sixth equalities follow from [20, Lemma 2.6]. This completes the proof. □

Proposition 5.3. *For any $\mu \in \overline{\Lambda}$, $a \in G_\mu$ and $j \in \{1, \dots, n\}$, we have*

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(e_\mu) = 0,\\
&\frac{\partial_f}{\partial x_j}(a^{-1}) = -f_1(a, a)\frac{\partial_f}{\partial x_j}(a).
\end{aligned}$$

Proof. The first equality follows from

$$\frac{\partial_f}{\partial x_j}(e_\mu e_\mu) = \frac{\partial_f}{\partial x_j}(e_\mu) + f_1(e_\mu, e_\mu)\frac{\partial_f}{\partial x_j}(e_\mu) = \frac{\partial_f}{\partial x_j}(e_\mu) + \frac{\partial_f}{\partial x_j}(e_\mu).$$

The second equality follows from

$$\frac{\partial_f}{\partial x_j}(e_\mu) = \frac{\partial_f}{\partial x_j}(a^{-1}a) = \frac{\partial_f}{\partial x_j}(a^{-1}) + f_1(a^{-1}, a)\frac{\partial_f}{\partial x_j}(a).$$

□

6. f-twisted Alexander matrices for handlebody-links

Let R be a ring. We denote by $M(m, n; R)$ the set of $m \times n$ matrices over R and by $GL(n; R)$ the set of $n \times n$ invertible matrices over R. We say that two matrices A_1 and A_2 over R are *equivalent*, denoted by $A_1 \sim A_2$, if they are related by a finite sequence of the following transformations:

- $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n) \leftrightarrow (\boldsymbol{a}_1, \dots, \boldsymbol{a}_i + \boldsymbol{a}_j r, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n)$ $(r \in R)$,

- $\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \leftrightarrow \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + r\boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}$ $(r \in R)$, $\quad$ • $A \leftrightarrow \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix}$, $\quad$ • $A \leftrightarrow \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$.

By the definitions, we have the following equivalences (for details see [16]):

- $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n) \sim (\boldsymbol{a}_1, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_n)$,
- $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_n) \sim (\boldsymbol{a}_1, \dots, \boldsymbol{a}_i u, \dots, \boldsymbol{a}_n)$ $(u \in R^\times)$,
- $\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}$, $\quad$ • $\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ u\boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}$ $(u \in R^\times)$.

Let R be a commutative ring, and let $A \in M(m, n; R)$. A *k-minor* of A is the determinant of a $k \times k$ submatrix of A. For any $d \in \mathbb{Z}_{\geq 0}$, the *$d$-th elementary ideal* $E_d(A)$ of A is the ideal of R generated by all $(n-d)$-minors of A if $n - m \leq d < n$, and

$$E_d(A) := \begin{cases} 0 & \text{if } d < n - m, \\ R & \text{if } n \leq d. \end{cases}$$

Suppose that R is a GCD domain. Then the *d-th Alexander invariant* $\Delta_d(A)$ of A is the greatest common divisor of all $(n-d)$-minors of A if $n - m \leq d < n$, and

$$\Delta_d(A) := \begin{cases} 0 & \text{if } d < n - m, \\ 1 & \text{if } n \leq d. \end{cases}$$

We remark that $\Delta_d(A)$ coincides with the greatest common divisor of generators of $E_d(A)$ and is determined up to unit multiple. If $A \sim B$, then $E_d(A) = E_d(B)$ and $\Delta_d(A) \doteq \Delta_d(B)$, where "$\doteq$" means "is equal to, up to multiplication by a unit". See [7] for more details.

Remark 6.1. Let R be a commutative ring. We can regard a matrix in $M(m, n; M(k, k; R))$ as a matrix in $M(km, kn; R)$. We call such matrices *flat matrices*. We note that equivalent matrices are equivalent as flat matrices.

For an MCQ representation $\rho : X \to Y$ and an MCQ Alexander pair $f = (f_1, f_2)$ of maps $f_1, f_2 : Y \times Y \to R$, we set $f \circ (\rho \times \rho) := (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$, which is an MCQ Alexander pair by Proposition 3.10. For a relator $r = (r_1, r_2)$, we define

$$\frac{\partial_f}{\partial x_j}(r) := \frac{\partial_f}{\partial x_j}(r_1) - \frac{\partial_f}{\partial x_j}(r_2).$$

Definition 6.2. Let $X = \langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m \rangle$ be a finitely presented MCQ and $\rho : X \to Y$ an MCQ representation. Let $f = (f_1, f_2)$

be an MCQ Alexander pair of maps $f_1, f_2 : Y \times Y \to R$. The *$f$-twisted Alexander matrix* of (X, ρ) (with respect to the presentation $\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle$) is

$$A(X, \rho; f_1, f_2) = \begin{pmatrix} \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_m) \end{pmatrix}.$$

The following proposition shows that the equivalence class of an f-twisted Alexander matrix is invariant under the transformations (T1-1)–(T1-5), (T2) and (T3-1) with MCQ representations.

Proposition 6.3. *Let $X = \langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle$ and $X' = \langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle$ be finitely presented MCQs, and let $\rho : X \to Y$ and $\rho' : X' \to Y$ be MCQ representations. Let (f_1, f_2) be an MCQ Alexander pair of maps $f_1, f_2 : Y \times Y \to R$. If $(X, \rho) \sim_T (X', \rho')$, then we have*

$$A(X, \rho; f_1, f_2) \sim A(X', \rho'; f_1, f_2).$$

Especially, we have

$$E_d(A(X, \rho; f_1, f_2)) = E_d(A(X', \rho'; f_1, f_2))$$

if R is a commutative ring, and we have

$$\Delta_d(A(X, \rho; f_1, f_2)) \doteq \Delta_d(A(X', \rho'; f_1, f_2))$$

if R is a GCD domain.

Proof. Put $f := (f_1, f_2)$ and $f_i^\rho := f_i \circ (\rho \times \rho)$ for each $i = 1, 2$, that is, $f \circ (\rho \times \rho) = (f_1^\rho, f_2^\rho)$. We show that the equivalence class of an f-twisted Alexander matrix $A(X, \rho, f_1, f_2)$ is invariant under the transformations (T1-1)–(T1-5), (T2) and (T3-1). We remark again that the same symbol ρ is used for $\rho \circ f_{\text{T1-}i}^{-1}$, $\rho \circ f_{\text{T2}}^{-1}$ and $\rho \circ f_{\text{T3-1}}^{-1}$ on the transformations. We set $A := A(\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle, \rho; f_1, f_2)$ and $A' := A(\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle, \rho'; f_1, f_2)$. We denote by $\boldsymbol{a}_i$ the i-th row vector of A, and we denote by $a_{i,j}$ (resp. $a'_{i,j}$) the (i, j) entry of A (resp. A').

At first, we check (T1-1)–(T1-5). For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \,|\, r_1, \ldots, r_m, x = x \rangle \ (x \in F_{\mathsf{MCQ}}(\boldsymbol{x})),$$

we have

$$A = \begin{pmatrix} \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_m) \end{pmatrix} \sim \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix} = A'.$$

For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \,|\, r_1, \ldots, r_m, a = b \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \,|\, r_1, \ldots, r_m, a = b, b = a \rangle,$$

we have

$$A = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \mathbf{0} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ -\boldsymbol{a}_{m+1} \end{pmatrix} = A'.$$

For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a = b, b = c \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a = b, b = c, a = c \rangle,$$

we have

$$A = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \boldsymbol{0} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \boldsymbol{a}_{m+1} + \boldsymbol{a}_{m+2} \end{pmatrix} = A'.$$

For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a_1 = a_2, b_1 = b_2 \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a_1 = a_2, b_1 = b_2, a_1 \triangleleft b_1 = a_2 \triangleleft b_2 \rangle,$$

we have

$$A = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \boldsymbol{0} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ f_1^\rho(a_1, b_1)\boldsymbol{a}_{m+1} + f_2^\rho(a_1, b_1)\boldsymbol{a}_{m+2} \end{pmatrix} = A'.$$

For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a_1 = a_2, b_1 = b_2 \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a_1 = a_2, b_1 = b_2, a_1 b_1^{-1} = a_2 b_2^{-1} \rangle,$$

where a_i and b_i are multiplicable, we have

$$A = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \boldsymbol{0} \end{pmatrix} \sim \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \boldsymbol{a}_{m+1} - f_1^\rho(a_1, a_1^{-1}) f_1^\rho(b_1, b_1)\boldsymbol{a}_{m+2} \end{pmatrix} = A'.$$

Next, we check (T2). For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n; y \,|\, r_1, \dots, r_m, y = w \rangle \ (y \notin F_{\mathsf{MCQ}}(\boldsymbol{x}), w \in F_{\mathsf{MCQ}}(\boldsymbol{x})),$$

we have

$$A = \begin{pmatrix} \frac{\partial f \circ (\rho \times \rho)}{\partial x_1}(r_1) & \cdots & \frac{\partial f \circ (\rho \times \rho)}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial f \circ (\rho \times \rho)}{\partial x_1}(r_m) & \cdots & \frac{\partial f \circ (\rho \times \rho)}{\partial x_n}(r_m) \end{pmatrix} \sim \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} \sim \begin{pmatrix} A & \boldsymbol{0} \\ -\frac{\partial w}{\partial \boldsymbol{x}} & 1 \end{pmatrix} = A',$$

where $-\frac{\partial w}{\partial \boldsymbol{x}}$ is the row vector $\left(-\frac{\partial f \circ (\rho \times \rho)}{\partial x_1}(w), \dots, -\frac{\partial f \circ (\rho \times \rho)}{\partial x_n}(w) \right)$.

Finally, we check (T3-1). For the presentations

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_{l'}; x_{l'+1}, \dots, x_n \,|\, r_1, \dots, r_m, a^0 = b^0 \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m, a^0 = b^0 \rangle,$$

where $a \in \{x_l, \dots, x_{l'}\}$ and $b \in \{x_{l'+1}, \dots, x_n\}$, we have $A = A'$ easily.

This completes the proof. □

Let H be a handlebody-link represented by a Y-oriented diagram D. Let $\rho : \mathrm{MCQ}(D) \to X$ be an MCQ representation, which can be regarded as an X-coloring of D. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. Then we define the *$f$-twisted Alexander matrix of (H, ρ) (with respect to D)* by

$$A(H, \rho; f_1, f_2) = A(\mathrm{MCQ}(D), \rho; f_1, f_2).$$

We also define

$$E_d(H, \rho; f_1, f_2) := E_d(A(\mathrm{MCQ}(D), \rho; f_1, f_2)),$$
$$\Delta_d(H, \rho; f_1, f_2) := \Delta_d(A(\mathrm{MCQ}(D), \rho; f_1, f_2))$$

if R is a commutative ring or a GCD domain, respectively.

By Lemma 4.2 and Proposition 6.3, we have the following theorem.

Theorem 6.4. *Let H and H' be handlebody-links represented by Y-oriented diagrams D and D', respectively. Let $\rho : \mathrm{MCQ}(D) \to X$ and $\rho' : \mathrm{MCQ}(D') \to X$ be MCQ representations. Let (f_1, f_2) be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. If $(H, \rho) \cong (H', \rho')$, then we have*

$$A(H, \rho; f_1, f_2) \sim A(H', \rho'; f_1, f_2).$$

Especially, we have

$$E_d(H, \rho; f_1, f_2) = E_d(H', \rho'; f_1, f_2)$$

if R is a commutative ring, and we have

$$\Delta_d(H, \rho; f_1, f_2) \doteq \Delta_d(H', \rho'; f_1, f_2)$$

if R is a GCD domain.

We calculate our invariants of the genus g trivial handlebody-knot for any MCQ Alexander pair f.

Proposition 6.5. *Let O_g be the trivial handlebody-knot of genus g. Let D_g be the Y-oriented diagram of O_g illustrated in Figure 5. For any MCQ representation $\rho : \mathrm{MCQ}(D_g) \to X$ and MCQ Alexander pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to R$, we have*

$$A(O_g, \rho; f_1, f_2) \sim \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \in M(1, g; R).$$

Especially, we have

$$E_d(O_g, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < g, \\ R & \text{if } g \leq d \end{cases}$$

if R is a commutative ring, and we have

$$\Delta_d(O_g, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < g, \\ 1 & \text{if } g \leq d \end{cases}$$

if R is a GCD domain.

Proof. Assume that $g \geq 2$. The Wirtinger presentation $\mathrm{MCQ}(D_g)$ is given by

$$\left\langle x_1, \dots, x_{3g-3} \,\middle|\, \begin{array}{l} x_1x_2 = x_1,\ x_{3g-3}x_{3g-4} = x_{3g-3}, \\ x_{3i}x_{3i+1} = x_{3i-1},\ x_{3i}x_{3i+1} = x_{3i+2}\ (1 \leq i \leq g-2) \end{array} \right\rangle.$$

We put $a_i := f_1 \circ (\rho \times \rho)(x_i, x_i^{-1}) \in R^\times$. We then have

$$\begin{aligned} &A(O_g, \rho; f_1, f_2) \\ &= \begin{pmatrix} 0 & a_1 & & & & & & & & & & \\ 0 & -1 & 1 & a_3 & & & & & & & & \\ & & 1 & a_3 & -1 & & & \cdots & & & 0 & \\ & & & & -1 & 1 & a_6 & & & & & \\ & & & & & 1 & a_6 & -1 & & & & \\ & & & & \vdots & & & \ddots & & & \vdots & \\ & & & & & & & & -1 & 1 & a_{3g-6} & \\ & & & & 0 & & & \cdots & & 1 & a_{3g-6} & -1 & 0 \\ & & & & & & & & & & & a_{3g-3} & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \in M(1, g; R). \end{aligned}$$

It is easy to see that $A(O_1, \rho; f_1, f_2) \sim \begin{pmatrix} 0 \end{pmatrix} \in M(1, 1; R)$. □

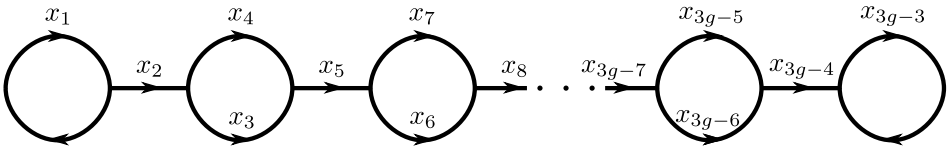

FIGURE 5. A Y-oriented diagram D_g of the genus g trivial handlebody-knot O_g.

Remark 6.6. In the same way as proof of Proposition 6.5, we have the following statement. Let $O_{g_1,\dots,g_m}$ be the m-component trivial handlebody-link the genera of whose components are $g_1, \dots, g_m$, respectively. Let $D_{g_1,\dots,g_m}$ be a Y-oriented diagram of $O_{g_1,\dots,g_m}$. Put $g := g_1 + \cdots + g_m$. For any MCQ representation $\rho : \mathrm{MCQ}(D_{g_1,\dots,g_m}) \to X$ and MCQ Alexander pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to M(n, n; R)$, we have

$$A(O_{g_1,\dots,g_m}, \rho; f_1, f_2) \sim \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \in M(1, ng; R),$$

where we regard $A(O_{g_1,\dots,g_m}, \rho; f_1, f_2)$ as a flat matrix. Especially, we have

$$E_d(O_{g_1,\dots,g_m}, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < ng, \\ R & \text{if } ng \leq d \end{cases}$$

if R is a commutative ring, and we have

$$\Delta_d(O_{g_1,\dots,g_m}, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < ng, \\ 1 & \text{if } ng \leq d \end{cases}$$

if R is a GCD domain.

7. (Twisted) Alexander matrices for handlebody-links

In this section, we recall the notion of the (twisted) Alexander matrix for handlebody-links [15] and see that it can be realized as an f-twisted Alexander matrix for some MCQ Alexander pair f.

Let H be a handlebody-link represented by a Y-oriented diagram D. We then have the Wirtinger presentations

$$\begin{aligned}\mathrm{MCQ}(D) &= \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \,|\, r_1, \dots, r_m \rangle,\\ G(D) &= \langle x_1, \dots, x_k, \dots, x_l, \dots, x_n \,|\, r_1^{\mathsf{Grp}}, \dots, r_m^{\mathsf{Grp}} \rangle,\end{aligned}$$

where

$$r_i = \begin{cases} (u_i \triangleleft v_i, w_i) & \text{if } 1 \le i \le m',\\ (u_i v_i, w_i) & \text{if } m'+1 \le i \le m,\end{cases}$$

$$r_i^{\mathsf{Grp}} = \begin{cases} v_i^{-1} u_i v_i w_i^{-1} & \text{if } 1 \le i \le m',\\ u_i v_i w_i^{-1} & \text{if } m'+1 \le i \le m.\end{cases}$$

See Section 4. Let R be a commutative ring. Set $G := GL(k;R)$. For a group representation $\rho : G(D) \to G$, we use the same symbol ρ for the induced MCQ representation of $\mathrm{MCQ}(D)$ to G sending x_i into $\rho(x_i)$, where we regard G as an MCQ with the conjugation operation. Put $S := \{x_1, \dots, x_n\}$, and let $F_{\mathsf{Grp}}(S)$ be the free group on S and G_0 the abelian group

$$\left\langle t_1, \dots, t_r \,\middle|\, t_1^{k_1}, \dots, t_r^{k_r}, [t_i, t_j]\ (1 \le i < j \le r) \right\rangle,$$

where $k_1, \dots, k_r \in \mathbb{Z}_{\ge 0}$. The group ring $R[G_0]$ can be identified with the quotient ring of the Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1)$. The Fox derivative [6] with respect to x_j is the R-homomorphism

$$\frac{\partial_{\mathsf{Grp}}}{\partial x_j} : R[F_{\mathsf{Grp}}(S)] \to R[F_{\mathsf{Grp}}(S)]$$

satisfying

$$\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(pq) = \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(p) + p\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(q) \qquad \text{and} \qquad \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(x_i) = \delta_{ij}.$$

Let $\mathrm{pr} : F_{\mathsf{Grp}}(S) \to G(D)$ be the canonical projection and $\alpha : G(D) \to G_0$ an surjective homomorphism. We denote the linear extensions of pr, α and ρ by the same symbols $\mathrm{pr} : R[F_{\mathsf{Grp}}(S)] \to R[G(D)]$, $\alpha : R[G(D)] \to R[G_0]$ and $\rho : R[G(D)] \to M(k,k;R)$, respectively. Then the R-homomorphism

$$\rho \otimes \alpha : R[G(D)] \to M(k,k;R[G_0])$$

is defined by

$$(\rho \otimes \alpha)\left(\sum r_i g_i\right) = \sum r_i \rho(g_i)\alpha(g_i)\ (r_i \in R, g_i \in G(D)).$$

The *Alexander matrix of* $G(D)$ *associated with* α is the $m \times n$ matrix

$$\left((\alpha \circ \mathrm{pr})\left(\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(r_i^{\mathsf{Grp}})\right)\right)$$

over $R[G_0]$. The *twisted Alexander matrix of $G(D)$ associated with α and ρ* is the matrix

$$\left(((\rho \otimes \alpha) \circ \mathrm{pr}) \left(\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(r_i^{\mathsf{Grp}}) \right) \right),$$

which we regard as a $km \times kn$ matrix over $R[G_0]$. These matrices produce invariants for handlebody-links by evaluating them with ideals and greatest common divisors as we see in [15].

For the induced MCQ representations $\alpha : \mathrm{MCQ}(D) \to G_0$ and $\rho : \mathrm{MCQ}(D) \to G$, we define the MCQ representation $\rho \cdot \alpha : \mathrm{MCQ}(D) \to GL(k; R[G_0])$ by $(\rho \cdot \alpha)(x) = \rho(x)\alpha(x)$. Under the following proposition, the (twisted) Alexander matrices for handlebody-links can be obtained in our framework.

Proposition 7.1. *(1) Let $f = (f_1, f_2)$ be the MCQ Alexander pair in Example 3.2, that is, $f_1, f_2 : G_0 \times G_0 \to R[G_0]$ defined by $f_1(a, b) = b^{-1}$ and $f_2(a, b) = b^{-1}a - b^{-1}$. Let $\alpha : G(D) \to G_0$ be a surjective homomorphism. Then the f-twisted Alexander matrix $A(H, \alpha; f_1, f_2)$ coincides with the Alexander matrix of $G(D)$ associated with α.*

(2) Set $G := GL(k; R)$. Let $f = (f_1, f_2)$ be the MCQ Alexander pair in Example 3.3, that is, $f_1, f_2 : GL(k, R[G_0]) \times GL(k, R[G_0]) \to M(k, k; R[G_0])$ defined by $f_1(a, b) = b^{-1}$ and $f_2(a, b) = b^{-1}a - b^{-1}$. Let $\alpha : G(D) \to G_0$ be a surjective homomorphism and $\rho : G(D) \to G$ a group representation. Then the f-twisted Alexander matrix $A(H, \rho \cdot \alpha; f_1, f_2)$ coincides with the twisted Alexander matrix of $G(D)$ associated with α and ρ.

Proof. We prove (2). Put $f_i^{\rho\cdot\alpha} := f_i \circ (\rho \cdot \alpha \times \rho \cdot \alpha)$ for each $i = 1, 2$, that is, $f \circ (\rho \cdot \alpha \times \rho \cdot \alpha) = (f_1^{\rho\cdot\alpha}, f_2^{\rho\cdot\alpha})$. We remark again that we often omit "pr" to represent $\mathrm{pr}(x)$ as x. We note that $\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(u_i)$, $\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i)$ and $\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(w_i)$ are equal to 0 or 1 for each i and j. For $1 \le i \le m'$, we have

$$\begin{aligned}
&((\rho \otimes \alpha) \circ \mathrm{pr}) \left(\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i^{-1}u_i v_i w_i^{-1}) \right) \\
&= ((\rho \otimes \alpha) \circ \mathrm{pr}) \left(-v_i^{-1}\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i) + v_i^{-1}\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(u_i) + v_i^{-1}u_i\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i) - v_i^{-1}u_i v_i w_i^{-1}\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(w_i) \right) \\
&= ((\rho \otimes \alpha) \circ \mathrm{pr})(v_i^{-1})\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(u_i) + ((\rho \otimes \alpha) \circ \mathrm{pr})(v_i^{-1}u_i - v_i^{-1})\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i) - \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(w_i) \\
&= f_1^{\rho\cdot\alpha}(u_i, v_i)\frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(u_i) + f_2^{\rho\cdot\alpha}(u_i, v_i)\frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(v_i) - \frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(w_i) \\
&= \frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(u_i \triangleleft v_i = w_i).
\end{aligned}$$

For $m'+1 \leq i \leq m$, we have

$$\begin{aligned}
&((\rho\otimes\alpha)\circ \mathrm{pr})\left(\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(u_i v_i w_i^{-1})\right)\\
&= ((\rho\otimes\alpha)\circ \mathrm{pr})\left(\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(u_i) + u_i\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i) - u_i v_i w_i^{-1}\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(w_i)\right)\\
&= \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(u_i) + ((\rho\otimes\alpha)\circ \mathrm{pr})(u_i)\frac{\partial_{\mathsf{Grp}}}{\partial x_j}(v_i) - \frac{\partial_{\mathsf{Grp}}}{\partial x_j}(w_i)\\
&= \frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(u_i) + f_1^{\rho\cdot\alpha}(u_i,u_i^{-1})\frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(v_i) - \frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(w_i)\\
&= \frac{\partial_{f\circ(\rho\cdot\alpha\times\rho\cdot\alpha)}}{\partial x_j}(u_i v_i = w_i).
\end{aligned}$$

Therefore the f-twisted Alexander matrix $A(H,\rho\cdot\alpha;f_1,f_2)$ coincides with the twisted Alexander matrix of $G(D)$ associated with α and ρ.

In the same way, we can prove (1). □

8. f-twisted Alexander matrices and handlebody-knot complements

In this section, calculating our invariants, we distinguish two handlebody-knots whose complements have isomorphic fundamental groups, which can not be distinguished by invariants derived from the (twisted) Alexander matrices. That is, f-twisted Alexander matrices produce strictly stronger invariants than (twisted) Alexander matrices for handlebody-knots.

Example 8.1. Let H_1 and H_2 be the handlebody-knots represented by the Y-oriented diagrams D_1 and D_2 depicted in Figure 6, respectively. We note that H_1 and H_2 have complements whose fundamental groups are isomorphic. The Wirtinger presentations $\mathrm{MCQ}(D_1)$ and $\mathrm{MCQ}(D_2)$ are given by

$$\left\langle \begin{array}{c} x_1,x_{10},x_{11};x_7,x_8,x_{12};\\ x_2;x_3;x_4;x_5;x_6;x_9 \end{array} \middle| \begin{array}{c} x_1\triangleleft x_6=x_2,\ x_2\triangleleft x_7=x_3,\ x_3\triangleleft x_5=x_4,\ x_4\triangleleft x_3=x_5,\\ x_5\triangleleft x_4=x_6,\ x_6\triangleleft x_2=x_7,\ x_8\triangleleft x_{11}=x_9,\ x_9\triangleleft x_{12}=x_{10},\\ x_{11}\triangleleft x_9=x_{12},\ x_1x_{11}=x_{10},\ x_{12}x_7=x_8 \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{c} x_1,x_{10},x_{11};x_7,x_8,x_{12};\\ x_2;x_3;x_4;x_5;x_6;x_9 \end{array} \middle| \begin{array}{c} x_1\triangleleft x_6=x_2,\ x_2\triangleleft x_7=x_3,\ x_4\triangleleft x_5=x_3,\ x_5\triangleleft x_6=x_4,\\ x_6\triangleleft x_4=x_5,\ x_6\triangleleft x_2=x_7,\ x_8\triangleleft x_{11}=x_9,\ x_9\triangleleft x_{12}=x_{10},\\ x_{11}\triangleleft x_9=x_{12},\ x_1x_{11}=x_{10},\ x_{12}x_7=x_8 \end{array} \right\rangle,$$

respectively. Let X and $(\widetilde{f}_1,0)$ be the MCQ and the MCQ Alexander pair of maps $\widetilde{f}_1,0 : X\times X\to \mathbb{Z}_4[t^{\pm1}]/(t^4-1)$ in Example 3.8, respectively. Putting $a_i^\rho := \widetilde{f}_1\circ(\rho\times\rho)(u_{c_i},v_{c_i})$ for each MCQ representation $\rho:\mathrm{MCQ}(D_1)\to X$ and

each crossing c_i of D_1, we have

$$A\left(H_1,\rho;\widetilde{f}_1,0\right)=\begin{pmatrix} a_1^\rho & -1 & & & & & & & & \\ & a_2^\rho & -1 & & & & & & & \\ & & a_3^\rho & -1 & & & & & & \\ & & & a_4^\rho & -1 & & & & & \\ & & & & a_5^\rho & -1 & & & & \\ & & & & & a_6^\rho & -1 & & & \\ & & 0 & & & & & a_7^\rho & -1 & \\ & & & & & & & & a_8^\rho & -1 \\ 1 & & & & & & & -1 & 1 & \\ & & & & & & 1 & -1 & & 1 \end{pmatrix}$$
$$\sim \left(a_1^\rho a_2^\rho a_3^\rho a_4^\rho a_5^\rho a_6^\rho a_7^\rho a_8^\rho \quad a_7^\rho a_8^\rho a_9^\rho - 1\right) \in M(1,2;\mathbb{Z}_4[t^{\pm1}]/(t^4-1)).$$

On the other hand, putting $b_i^\rho := \widetilde{f}_1 \circ (\rho\times\rho)(u_{c_i}, v_{c_i})$ for each MCQ representation $\rho : \mathrm{MCQ}(D_2) \to X$ and each crossing c_i of D_2, we have

$$A\left(H_2,\rho;\widetilde{f}_1,0\right)=\begin{pmatrix} b_1^\rho & -1 & & & & & & & & \\ & b_2^\rho & -1 & & & & & & & \\ & & -1 & b_3^\rho & & & & & & \\ & & & -1 & b_4^\rho & & & & & \\ & & & & -1 & b_5^\rho & & & & \\ & & & & & b_6^\rho & -1 & & & \\ & & 0 & & & & & b_7^\rho & -1 & \\ & & & & & & & & b_8^\rho & -1 \\ 1 & & & & & & & -1 & 1 & \\ & & & & & & 1 & -1 & & 1 \end{pmatrix}$$
$$\sim \left(b_1^\rho b_2^\rho b_6^\rho - b_3^\rho b_4^\rho b_5^\rho b_9^\rho \quad b_7^\rho b_8^\rho b_9^\rho - 1\right) \in M(1,2;\mathbb{Z}_4[t^{\pm1}]/(t^4-1)).$$

Evaluating these matrices for each MCQ representation ρ, we obtain the multisets

$$\left\{\Delta_1\left(H_1,\rho;\widetilde{f}_1,0\right)\,\middle|\,\rho\in\mathrm{Hom}(\mathrm{MCQ}(D_1),X)\right\}=\left\{\begin{array}{c} 0\text{ (96 times)}, t+3\text{ (384 times)},\\ t^2+3\text{ (768 times)}\end{array}\right\}$$

and

$$\left\{\Delta_1\left(H_2,\rho;\widetilde{f}_1,0\right)\,\middle|\,\rho\in\mathrm{Hom}(\mathrm{MCQ}(D_2),X)\right\}=\{0\text{ (864 times)}, t+3\text{ (384 times)}\}.$$

Therefore H_1 and H_2 are not equivalent by Theorem 6.4.

Remark 8.2. We emphasize again that, in Example 8.1, H_1 and H_2 have complements whose fundamental groups are isomorphic. This implies that f-twisted Alexander matrices yield strictly stronger invariants than (twisted) Alexander matrices for handlebody-knots. In particular, it also indicates that the fundamental MCQ distinguishes handlebody-knots whose complements have isomorphic fundamental groups. Then it is natural to consider the following problem.

Problem 8.3. Does the fundamental MCQ distinguish handlebody-knots whose complements are homeomorphic? In particular, do our invariants distinguish them?

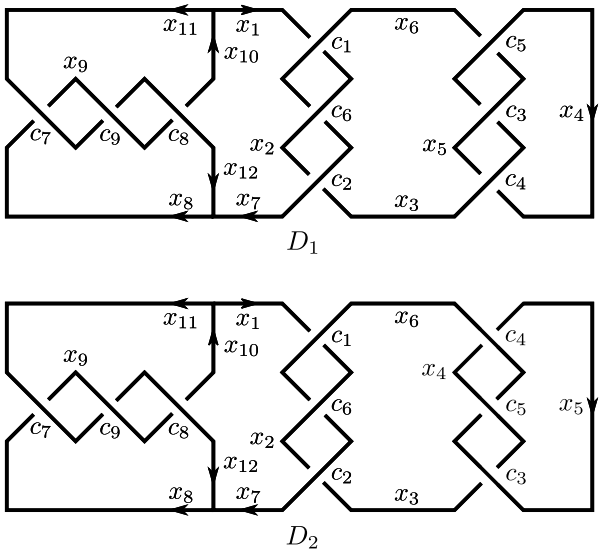

FIGURE 6. Y-oriented diagrams of two handlebody-knots whose complements have isomorphic fundamental groups.

We can easily see that the fundamental MCQ distinguishes handlebody-"links" (not handlebody-"knots") whose complements are homeomorphic. Let H_1 and H_2 be the 2-component handlebody-links represented by the Y-oriented diagrams D_1 and D_2 depicted in Figure 7, respectively. We note that H_1 and H_2 have homeomorphic complements. Set

$$M := (a_{ij}) = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 & 1 \\ 3 & 2 & 1 & 6 & 2 & 4 \\ 2 & 1 & 3 & 3 & 6 & 5 \\ 5 & 6 & 4 & 4 & 1 & 2 \\ 4 & 5 & 6 & 1 & 5 & 3 \\ 6 & 4 & 5 & 2 & 3 & 6 \end{pmatrix}$$

and $Q_M := \{1, 2, 3, 4, 5, 6\}$. We define a binary operation $\triangleleft$ on Q_M by $i \triangleleft j = a_{ij}$. Then $(Q_M, \triangleleft)$ is a quandle, called the quandle $QS6_1$. Since $\operatorname{type} Q_M = 2$, we have the associated MCQ $X := Q_M \times \mathbb{Z}_2$ of the $\mathbb{Z}_2$-family of quandles $(Q_M, \{\triangleleft^i\}_{i \in \mathbb{Z}_2})$. Then we obtain

$$\#\mathrm{Hom}(\mathrm{MCQ}(D_1), X) = 192 \qquad \text{and} \qquad \#\mathrm{Hom}(\mathrm{MCQ}(D_2), X) = 168,$$

where $\#S$ denotes the cardinality of the set S. Hence we have $\mathrm{MCQ}(D_1) \ncong \mathrm{MCQ}(D_2)$, which implies that H_1 and H_2 are not equivalent.

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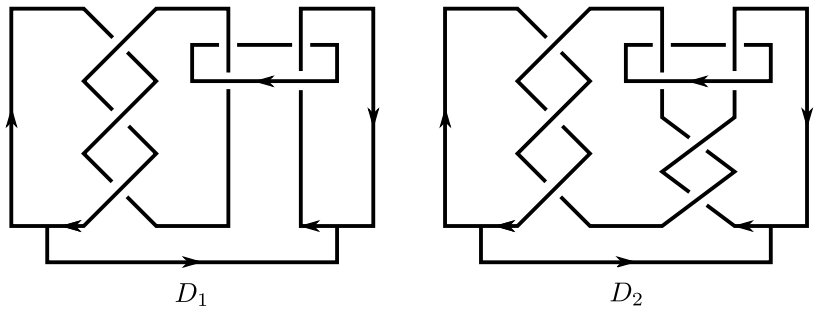

FIGURE 7. Y-oriented diagrams of two handlebody-links whose complements are homeomorphic.

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