Shade quandle presentations for oriented links

Atsushi Ishii, Kengo Kawamura, Kanako Oshiro and Yuta Taniguchi

Abstract

The purpose of this paper is to introduce an enriched presentation, called a shade quandle presentation, containing informations that can be used to normalize Alexander type invariants. We also introduce transformations on shade quandle presentations and show that two shade quandle presentations of an oriented link are related by the transformations. We see that the transformations are finer than the strong Tietze transformations to normalize Alexander type invariants.

1 Introduction

The Alexander polynomial [1] is a well used classical link invariant, and there are many different ways to compute the polynomial invariant. One of the most famous ones is the Fox's method, which is called Fox calculus. The Alexander polynomial by Fox [5] was defined for a finitely presentable group (for example a knot group), and it is an invariant of not only classical links but also groups. The Alexander polynomial was generalized by Lin [12] and Wada [15] who introduced a twisted Alexander polynomial. In particular, the version defined by Wada was given for a finitely presentable group with a group representation, and was obtained by using a similar method as Fox calculus.

We note that the (twisted) Alexander polynomials obtained via Fox calculus are determined up to multiplication by units. Hence, studies for the normalizations, i.e., studies for removing this multiplicative ambiguity, were sometimes seen as important. It is well known that the Conway polynomial [3] is regarded as a normalization of the Alexander polynomial. A normalization of the twisted Alexander polynomial of a knot was introduced by Kitayama [11].

A quandle is an algebraic structure introduced by Joyce [10] and Matveev [13], which satisfies three axioms corresponding to the Reidemeister moves on link diagrams, where we note that a quandle is a generalization of a group. In [7], the first and third authors defined a quandle version of Fox calculus. In [8], a quandle twisted Alexander invariant of a finite presentable quandle with a quandle representation was defined by using the quandle version of Fox calculus, where we note that this invariant can be regarded as a generalization of a twisted Alexander polynomial. The quandle twisted Alexander invariant for oriented links is normalized in [9] via diagrams.

In this paper, we focus on the Wada's study given in [15]. He introduced the strong Tietze transformations which relate two group presentations obtained from the same link group, where the ordinary Tietze transformations relate two group presentations of the same group. Thanks to the strong Tietze transformations, the multiplicative ambiguity of the twisted Alexander polynomials is decreased somewhat. Besides, the strong Tietze transformations also made a contribution for Kitayama's normalization. However, the multiplicative ambiguity is not completely removed even if we adopt the strong Tietze transformations.

The purpose of this paper is to introduce an enriched presentation containing informations that can be used to normalize Alexander type invariants. Concretely, we introduce a shade quandle presentation and equivalence transformations on the presentations. A shade quandle presentation is an extended quandle presentation, and the equivalence transformations can be regarded as the shade quandle version of Tietze transformations. We also introduce how to obtain a shade quandle presentation from a diagram of an oriented link and we show that the equivalence classes of shade quandle presentations give an oriented link invariant (Theorem 3.2). Using a quandle version of Fox calculus for a shade quandle presentation, we obtain a triple of matrices and show that such triples give an invariant of shade quandle presentations (Proposition 7.3). Note that it was shown in [9] that from such triples of matrices, we can obtain Alexander type invariants such as the Alexander polynomial [1], the Conway polynomial [3], the twisted Alexander polynomial [12, 15], quandle twisted Alexander invariants [7, 8, 9], quandle 2-cocycle invariants [2], and so on, and they are uniquely determined without the multiplicative ambiguity.

This paper is organized as follows. In Section 2, we recall the definitions of a quandle and a quandle coloring. In Section 3, shade quandle presentations and transformations on the presentations are introduced. It is also explained how to obtain a shade quandle presentation from a diagram of an oriented link, and one of our main results, Theorem 3.2, is mentioned. Section 4 is devoted to prove Theorem 3.2. Section 5 presents that the equivalence relation on shade quandle presentations is a finer relation than the strong Tietze transformations. In Section 6, we give shade quandle presentations for closed braids with a braid group action and also give an explicit formula of shade quandle presentations for torus links. In Section 7, we recall the definitions of an Alexander pair and relation maps and introduce three matrices obtained from a shade quandle presentation. It is shown that the equivalent shade quandle presentations induce the equivalent triples of matrices.

2 Quandles

In this section, we recall the definitions of a quandle and a quandle coloring, which is regarded as a quandle homomorphism from the fundamental quandle to a quandle.

A *quandle* [10, 13] is a non-empty set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

- For any $a \in Q$, $a \triangleleft a = a$.
- For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
- For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. Let $(Q_1, \triangleleft_1)$ and $(Q_2, \triangleleft_2)$ be quandles. A *quandle homomorphism* from Q_1 to Q_2 is defined to be a map $f : Q_1 \to Q_2$ satisfying $f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)$ for any $a, b \in Q_1$. We denote

Figure 1: Crossings and regions

by $\mathrm{Aut}(Q)$ the set of all quandle automorphisms of a quandle Q. Then $\mathrm{Aut}(Q)$ forms a group with the composition of maps and acts on Q by $\varphi \cdot x = \varphi(x)$ for $\varphi \in \mathrm{Aut}(Q)$ and $x \in Q$. The *inner automorphism group* $\mathrm{Inn}(Q)$ of Q is a subgroup of $\mathrm{Aut}(Q)$ generated by $\{\triangleleft a \mid a \in Q\}$. We denote by $\mathrm{orb}(a)$ the orbit of $a \in Q$ under the action of $\mathrm{Inn}(Q)$ on Q. That is, $\mathrm{orb}(a) = \{\varphi(a) \mid \varphi \in \mathrm{Inn}(Q)\}$. We set $\mathrm{Orb}(Q) := \{\mathrm{orb}(x) \mid x \in Q\}$.

For a quandle $(Q, \triangleleft)$, a *Q-set* is a non-empty set Y equipped with a map $\triangleleft : Y \times Q \to Y$ satisfying the following axioms:

- For any $a \in Q$, the map $\triangleleft a : Y \to Y$ defined by $\triangleleft a(y) = y \triangleleft a$ is bijective.
- For any $y \in Y$ and $a, b \in Q$, we have $(y \triangleleft a) \triangleleft b = (y \triangleleft b) \triangleleft (a \triangleleft b)$.

Here, we note that we use the same symbol $\triangleleft$ as the binary operation of Q for the map of a Q-set. We denote $(\triangleleft a)^n : Y \to Y$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$. The *associated group* $\mathrm{As}(Q)$ of a quandle Q is a group defined by the presentation:

$$\langle x \ (x \in Q) \mid x \triangleleft y = y^{-1}xy \ (x, y \in Q) \rangle.$$

Then $\mathrm{As}(Q)$ is a Q-set with $y \triangleleft a = ya$.

Let L be an oriented link, and let D be a diagram of L in $\mathbb{R}^2$. We denote by $C(D)$ the set of crossings of D. We denote by $\mathcal{R}(D)$ the set of complementary regions of D. We denote by $\mathcal{A}(D)$ the set of arcs of D. We denote by $\mathcal{SA}(D)$ the set of curves obtained by removing all crossings from the arcs of D. We call an element of $\mathcal{SA}(D)$ a *semi-arc* of D. The normal orientation is obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. For a crossing $c \in C(D)$, we denote by v_c the over-arc of c and denote by u_c, w_c the under-arcs of c such that the normal orientation of v_c points from u_c to w_c (see the left picture of Figure 1). We denote by $r(\alpha)$ and $r'(\alpha)$ the regions facing a semi-arc α such that the normal orientation of α points from $r(\alpha)$ to $r'(\alpha)$ (see the right picture of Figure 1). Let Q be a quandle, and let Y be a Q-set. A *Q-coloring* of D is a map $C : \mathcal{A}(D) \to Q$ satisfying the condition

$$C(u_c) \triangleleft C(v_c) = C(w_c)$$

for each crossing $c \in C(D)$. We denote by $\mathrm{Col}_Q(D)$ the set of Q-colorings of D. A *Q_Y-coloring* C_Y of D is an extension of a Q-coloring C of D that assigns an element of Y to each region of D satisfying the condition

$$C_Y(r(\alpha)) \triangleleft C(\bar{\alpha}) = C_Y(r'(\alpha))$$

for each semi-arc $\alpha \in \mathcal{SA}(D)$, where $\bar{\alpha}$ is the arc from which the semi-arc α originates. We note that the colors of the regions are determined by those of the arcs and one region.

Let D be a diagram of an oriented link L. We denote by $F_{\mathsf{Qnd}}(S)$ the free quandle on a set S. For $R \subset F_{\mathsf{Qnd}}(S) \times F_{\mathsf{Qnd}}(S)$, we denote by $N_{\mathsf{Qnd}}(R)$ the minimal quandle congruence relation including R. We then have a quandle $\langle S \,|\, R \rangle = F_{\mathsf{Qnd}}(S)/N_{\mathsf{Qnd}}(R)$. We often write $a = b$ for $(a, b) \in R$. We set $-(a, b) := (b, a)$ for $(a, b) \in R$. We define $r_c \in F_{\mathsf{Qnd}}(\mathcal{A}(D)) \times F_{\mathsf{Qnd}}(\mathcal{A}(D))$ to be the relation $w_c = u_c \triangleleft v_c$ for a positive crossing c and the relation $u_c = w_c \triangleleft^{-1} v_c$ for a negative crossing c. We then have a presentation of the fundamental quandle $Q(L)$ with respect to the diagram of L:

$$Q(L) \cong \langle \mathcal{A}(D) \,|\, \{r_c \,|\, c \in C(D)\} \rangle.$$

A *quandle representation* ρ of $Q(L)$ to Q is a quandle homomorphism $\rho : Q(L) \to Q$. From the presentation of $Q(L)$, a Q-coloring C of D can be regarded as a quandle representation of $Q(L)$ to Q. We then often use ρ instead of C. For a quandle representation $\rho : Q(L) \to Q$, we denote by $\widetilde{\rho}$ the $Q_{\mathrm{As}(Q)}$-coloring of D that is the extension of the Q-coloring ρ satisfying $\widetilde{\rho}(r_{\mathrm{out}}) = 1$, where r_{out} is the outermost region of the link diagram D. For further details, we refer the reader to [4, 10].

3 Shade quandle presentations

In this section, we introduce the notion of a shade quandle presentation and transformations on the presentations and explain how to obtain a shade quandle presentation from a diagram of an oriented link.

We call the form

$$\langle x_1, \ldots, x_n; \mu \,|\, r_1, \ldots, r_m; y_1, \ldots, y_m \rangle$$

a *shade quandle presentation*, where $S = \{x_1, \ldots, x_n\}$ is a set, $R = \{r_1, \ldots, r_m\} \subset F_{\mathsf{Qnd}}(S) \times F_{\mathsf{Qnd}}(S)$, $y_1, \ldots, y_m \in \mathbb{Z}[\mathrm{As}(\langle S \,|\, R \rangle)]$ and $\mu : \mathrm{Orb}(\langle S \,|\, R \rangle) \to \mathbb{Z}$ is a map. Putting $\mu_i := \mu(\mathrm{orb}(x_i))$, we also write it as

$$\langle x_1, \ldots, x_n; \mu_1, \ldots, \mu_n \,|\, r_1, \ldots, r_m; y_1, \ldots, y_m \rangle.$$

For a map $\mu : \mathrm{Orb}(\langle S \,|\, R \rangle) \to \mathbb{Z}$, we define the map $\mu|_{a=p} : \mathrm{Orb}(\langle S \,|\, R \rangle) \to \mathbb{Z}$ by

$$\mu|_{a=p}(O) = \begin{cases} p & \text{if } O = \mathrm{orb}(a), \\ \mu(O) & \text{otherwise.} \end{cases}$$

We then define transformations on shade quandle presentations:

(S1) $\langle x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n; \mu \,|\, r_1, \ldots, r_k, \ldots, r_l, \ldots, r_m; y_1, \ldots, y_k, \ldots, y_l, \ldots, y_m \rangle$
$\leftrightarrow \langle x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n; \mu \,|\, r_1, \ldots, r_l, \ldots, r_k, \ldots, r_m; y_1, \ldots, y_l, \ldots, y_k, \ldots, y_m \rangle$,

(S2) $\langle \boldsymbol{x}; \mu \,|\, r_1, \ldots, r_i, \ldots, r_j, \ldots, r_m; y_1, \ldots, y_i, \ldots, y_j, \ldots, y_m \rangle$
$\leftrightarrow \langle \boldsymbol{x}; \mu \,|\, r_1, \ldots, r_j, \ldots, -r_i, \ldots, r_m; y_1, \ldots, y_j, \ldots, -y_i, \ldots, y_m \rangle$,

(S3) $\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a_1 \triangleleft^{\varepsilon} b, a_1 = a_2; \boldsymbol{y}, z, w \rangle$
$\leftrightarrow \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a_2 \triangleleft^{\varepsilon} b, a_1 = a_2; \boldsymbol{y}, z, w - z \triangleleft^{-\varepsilon} b \rangle \quad (\varepsilon = -1, 0, 1)$,

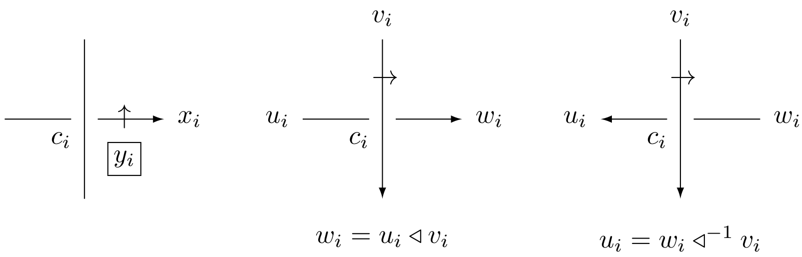

Figure 2: Relators on crossings

(S4) $\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \lhd b_1, b_1 = b_2; \boldsymbol{y}, z, w \rangle$
$\leftrightarrow \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \lhd b_2, b_1 = b_2; \boldsymbol{y}, z, w + z \lhd^{-1} b_2 - (z \lhd c) \lhd^{-1} b_2 \rangle$,

(S5) $\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \lhd^{-1} b_1, b_1 = b_2; \boldsymbol{y}, z, w \rangle$
$\leftrightarrow \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \lhd^{-1} b_2, b_1 = b_2; \boldsymbol{y}, z, w - z + z \lhd c \rangle$,

(S6) $\langle \boldsymbol{x}; \mu|_{a=p} \,|\, \boldsymbol{r}, a = b; \boldsymbol{y}, z \rangle \leftrightarrow \langle \boldsymbol{x}; \mu|_{a=p+1} \,|\, \boldsymbol{r}, a = b \lhd a; \boldsymbol{y}, z \lhd a \rangle$,

(S7) $\langle x_1, \ldots, x_n; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle \leftrightarrow \langle x_1, \ldots, x_n, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_{n+1} = w; \boldsymbol{y}, 0 \rangle$
$(x_{n+1} \notin F_{\mathsf{Qnd}}(\{x_1, \ldots, x_n\}),\ w \in F_{\mathsf{Qnd}}(\{x_1, \ldots, x_n\}))$,

where a bold symbol indicates a sequence of the symbols. Two shade quandle presentations are said to be *equivalent* ($\sim$) if they are related by a finite sequence of these transformations.

Hereafter, we assume that link diagrams satisfy the condition that every component has at least one undercrossing. Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link, and let D be a diagram of L. Let $D(K_i)$ be the diagram of K_i that is obtained by removing the other components from D. Let $c_1, \ldots, c_n$ be the crossings of D. We denote by x_i the arc starting from a crossing c_i for $i = 1, \ldots, n$. We set $u_i := u_{c_i}$, $v_i := v_{c_i}$, $w_i := w_{c_i}$ and $r_i := r_{c_i}$. See Figure 2. Let $K_{[i]}$ be the component of L such that x_i is an arc of $K_{[i]}$. We define $\mu_i \in \mathbb{Z}$ by

$$\mu_i = \frac{\mathrm{rot}(D(K_{[i]})) + \mathrm{wr}(D(K_{[i]})) + 1}{2}, \tag{3.1}$$

where $\mathrm{rot}(\,\cdot\,)$ stands for the rotation number and $\mathrm{wr}(\,\cdot\,)$ stands for the writhe. For an arc $\alpha \in \mathcal{A}(D)$, we use the same symbol α to represent the semi-arc that shares an initial point with the arc α. We set

$$y_i := \widetilde{\mathrm{id}_{Q(L)}}(r(x_i)) \in \mathrm{As}(Q(L))$$

for the $Q(L)$-coloring $\mathrm{id}_{Q(L)} : Q(L) \to Q(L)$. We then define

$$\widetilde{Q}(D) := \langle x_1, \ldots, x_n; \mu_1, \ldots, \mu_n \,|\, r_1, \ldots, r_n; y_1, \ldots, y_n \rangle.$$

Example 3.1. Let D be the diagram of a two-component link as illustrated in Figure 3. Then, we have

$$\widetilde{Q}(D) = \langle x_1, x_2, x_3, x_4, x_5; 3, 3, 3, 3, 0 \,|\, x_1 = x_4 \lhd x_2, x_2 = x_1 \lhd x_3, x_3 = x_2 \lhd x_1, \\ x_4 = x_3 \lhd^{-1} x_5, x_5 = x_5 \lhd^{-1} x_4; x_2, x_3, x_1, x_5^{-1}, x_5^{-1} \rangle.$$

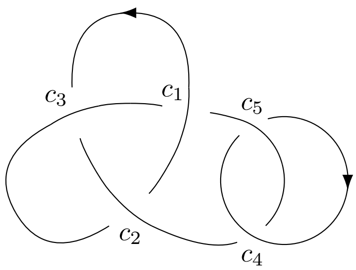

Figure 3: An oriented link diagram

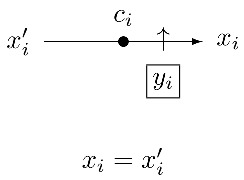

Figure 4: Relators on bivalent vertices

Theorem 3.2. *Let D_1, D_2 be diagrams of an oriented link L. Then we have $\widetilde{Q}(D_1) \sim \widetilde{Q}(D_2)$.*

We prove this theorem in the next section.

4 Proof of Theorem 3.2

In this section, we prove Theorem 3.2. First of all, we extend the definition of $\widetilde{Q}(D)$ to oriented link diagrams D with bivalent vertices.

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link. Let D be a diagram obtained by adding finite bivalent vertices on a diagram of L. An *arc* of D is a piece of a curve such that the end points of the piece are undercrossings or bivalent vertices. Let $c_1, \ldots, c_n$ be the crossings and bivalent vertices of D. We denote by x_i the arc whose initial point is c_i and denote by x_i' the arc whose terminal point is c_i for $i = 1, \ldots, n$. We define $r_i \in F_{\mathsf{Qnd}}(\mathcal{A}(D)) \times F_{\mathsf{Qnd}}(\mathcal{A}(D))$ to be the relation $w_i = u_i \triangleleft v_i$ if c_i is a positive crossing, the relation $u_i = w_i \triangleleft^{-1} v_i$ if c_i is a negative crossing, and the relation $x_i = x_i'$ if c_i is a bivalent vertex. See Figure 4. We also define $y_1, \ldots, y_n$ and μ in the same way as in the previous section. We then define

$$\widetilde{Q}(D) := \langle x_1, \ldots, x_n; \mu \,|\, r_1, \ldots, r_n; y_1, \ldots, y_n \rangle.$$

It is easy to see that two oriented link diagrams with bivalent vertices represent the same link if and only if they are related by a finite sequence of the moves depicted in Figures 5–13. Then, it is sufficient to show the invariance of $\widetilde{Q}(D)$ for these moves. Each of Figures 5–13 indicates diagrams D_1, D_2 (and D_3) that are identical outside a disk where they are the tangles depicted in the figure. Let $c_1, \ldots, c_{n-1}$ be crossings and bivalent vertices of D_1, D_2 (and

D_3) that stay outside the disk, and let $c_n, c_{n+1}, \ldots$ be the other crossings and bivalent vertices of D_1, D_2 (and D_3) that stay within the disk.

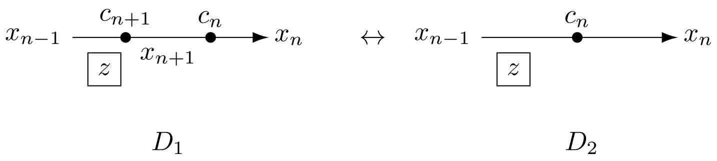

Figure 5:

For Figure 5, we have

$$\begin{aligned}\widetilde{Q}(D_1) &= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, z\rangle \\ &\sim \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, 0\rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}; \boldsymbol{y}, z\rangle = \widetilde{Q}(D_2).\end{aligned}$$

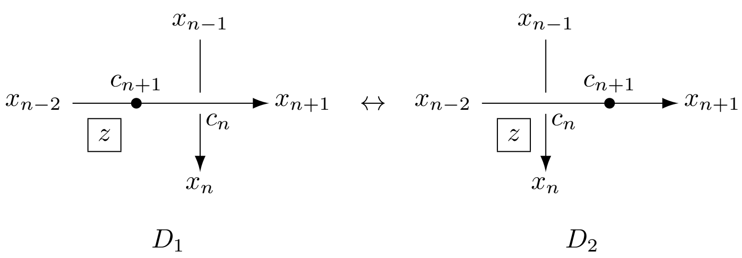

Figure 6:

For Figure 6, we have

$$\begin{aligned}\widetilde{Q}(D_1) &= \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft^{-1} x_{n+1}, x_{n+1} = x_{n-2}; \boldsymbol{y}, z, z\rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft^{-1} x_{n-2}, x_{n+1} = x_{n-2}; \boldsymbol{y}, z, z \triangleleft x_n\rangle = \widetilde{Q}(D_2).\end{aligned}$$

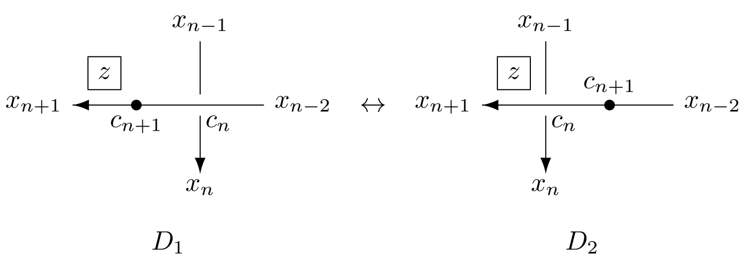

Figure 7:

For Figure 7, we have

$$\begin{aligned}\widetilde{Q}(D_2) &= \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft x_{n+1}, x_{n+1} = x_{n-2}; \boldsymbol{y}, z \triangleleft x_{n+1}, z \triangleleft x_{n-1}\rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft x_{n-2}, x_{n+1} = x_{n-2}; \boldsymbol{y}, z \triangleleft x_{n+1}, z\rangle = \widetilde{Q}(D_1).\end{aligned}$$

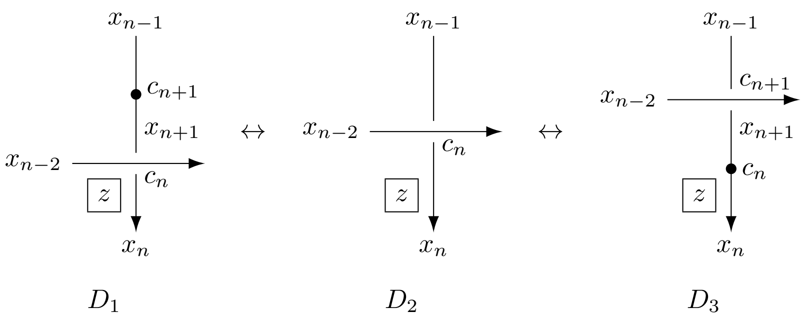

Figure 8:

For Figure 8, we have

$$\begin{aligned}\widetilde{Q}(D_1) &= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1} \triangleleft^{-1} x_{n-2}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, z \triangleleft x_{n-2} \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft^{-1} x_{n-2}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, 0 \rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft^{-1} x_{n-2}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2)\end{aligned}$$

and

$$\begin{aligned}\widetilde{Q}(D_3) &= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1} \triangleleft^{-1} x_{n-2}; \boldsymbol{y}, z, z \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft^{-1} x_{n-2}, x_{n+1} = x_{n-1} \triangleleft^{-1} x_{n-2}; \boldsymbol{y}, z, 0 \rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft^{-1} x_{n-2}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2).\end{aligned}$$

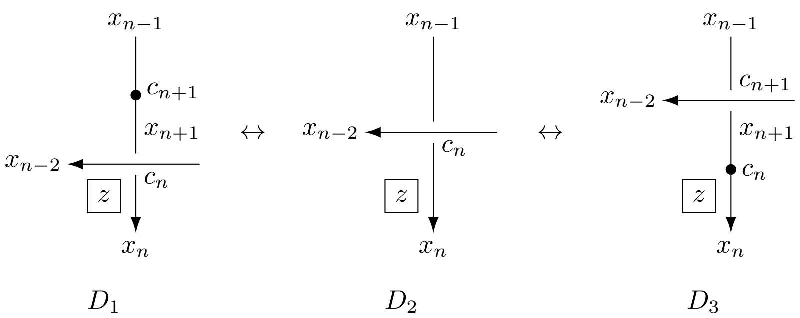

Figure 9:

For Figure 9, we have

$$\begin{aligned}\widetilde{Q}(D_1) &= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1} \triangleleft x_{n-2}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, z \triangleleft^{-1} x_{n-2} \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft x_{n-2}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, 0 \rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft x_{n-2}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2)\end{aligned}$$

and

$$\begin{aligned}\widetilde{Q}(D_3) &= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1} \triangleleft x_{n-2}; \boldsymbol{y}, z, z \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft x_{n-2}, x_{n+1} = x_{n-1} \triangleleft x_{n-2}; \boldsymbol{y}, z, 0 \rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1} \triangleleft x_{n-2}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2).\end{aligned}$$

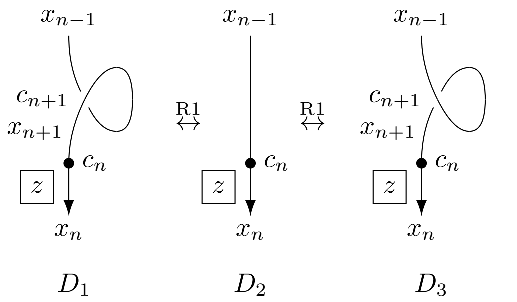

Figure 10:

For Figure 10, we have

$$
\begin{aligned}
\widetilde{Q}(D_1) &= \langle \boldsymbol{x}, x_{n+1}; \mu|_{x_n=p+1} \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1} \triangleleft x_{n+1}; \boldsymbol{y}, z, z \triangleleft x_n \rangle \\
&\sim \langle \boldsymbol{x}, x_{n+1}; \mu|_{x_n=p} \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, z \rangle \\
&\sim \langle \boldsymbol{x}, x_{n+1}; \mu|_{x_n=p} \,|\, \boldsymbol{r}, x_n = x_{n-1}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, 0 \rangle \\
&\sim \langle \boldsymbol{x}; \mu|_{x_n=p} \,|\, \boldsymbol{r}, x_n = x_{n-1}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{Q}(D_3) &= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1} \triangleleft^{-1} x_{n-1}; \boldsymbol{y}, z, z \rangle \\
&= \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+1}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, z \rangle \\
&\sim \langle \boldsymbol{x}, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}, x_{n+1} = x_{n-1}; \boldsymbol{y}, z, 0 \rangle \\
&\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2).
\end{aligned}
$$

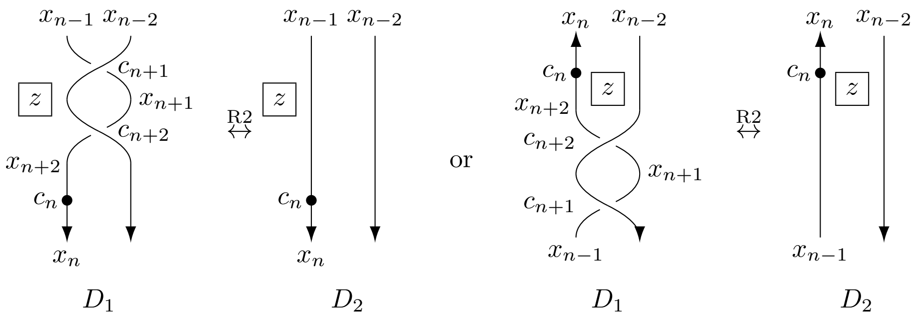

Figure 11:

For Figure 11, we have

$$
\begin{aligned}
\widetilde{Q}(D_1) &= \langle \boldsymbol{x}, x_{n+1}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+2}, x_{n+1} = x_{n-1} \triangleleft x_{n-2}, \\
&\qquad\qquad x_{n+2} = x_{n+1} \triangleleft^{-1} x_{n-2}; \boldsymbol{y}, z, z \triangleleft x_{n-2}, z \rangle \\
&\sim \langle \boldsymbol{x}, x_{n+1}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+2}, x_{n+1} = x_{n-1} \triangleleft x_{n-2}, x_{n+2} = x_{n-1}; \\
&\qquad\qquad \boldsymbol{y}, z, 0, z \rangle \\
&\sim \langle \boldsymbol{x}, x_{n+1}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}, x_{n+1} = x_{n-1} \triangleleft x_{n-2}, x_{n+2} = x_{n-1}; \\
&\qquad\qquad \boldsymbol{y}, z, 0, 0 \rangle \\
&\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}; \boldsymbol{y}, z \rangle = \widetilde{Q}(D_2).
\end{aligned}
$$

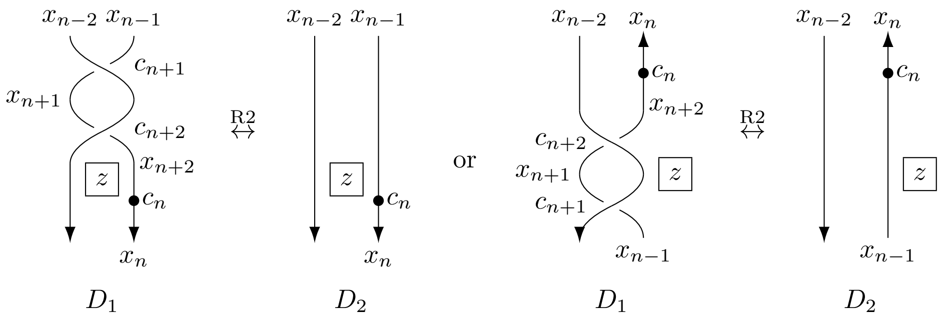

Figure 12:

For Figure 12, we have

$$\begin{aligned}
\widetilde{Q}(D_1) = \langle \boldsymbol{x}, x_{n+1}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+2}, x_{n+1} = x_{n-1} \lhd^{-1} x_{n-2},& \\
x_{n+2} = x_{n+1} \lhd x_{n-2}; \boldsymbol{y}, z, z \lhd^{-1} x_{n-2}, z\rangle& \\
\sim \langle \boldsymbol{x}, x_{n+1}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n+2}, x_{n+1} = x_{n-1} \lhd^{-1} x_{n-2},& \\
x_{n+2} = x_{n-1}; \boldsymbol{y}, z, 0, z\rangle& \\
\sim \langle \boldsymbol{x}, x_{n+1}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}, x_{n+1} = x_{n-1} \lhd^{-1} x_{n-2},& \\
x_{n+2} = x_{n-1}; \boldsymbol{y}, z, 0, 0\rangle& \\
\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-1}; \boldsymbol{y}, z\rangle = \widetilde{Q}(D_2).&
\end{aligned}$$

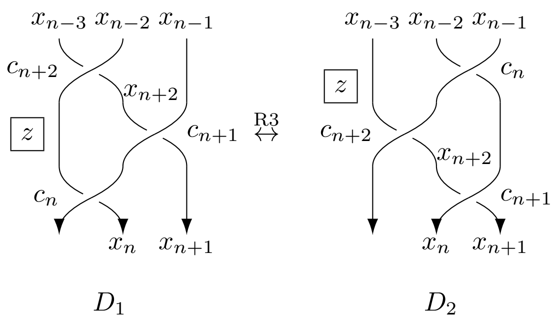

Figure 13:

For Figure 13, we have

$$\begin{aligned}\widetilde{Q}(D_1) &= \langle \boldsymbol{x}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-2} \triangleleft x_{n-1}, x_{n+1} = x_{n+2} \triangleleft x_{n-1}, \\ &\qquad x_{n+2} = x_{n-3} \triangleleft x_{n-2}; \boldsymbol{y}, z \triangleleft x_{n-1}, (z \triangleleft x_{n-2}) \triangleleft x_{n-1}, z \triangleleft x_{n-2} \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-2} \triangleleft x_{n-1}, x_{n+1} = (x_{n-3} \triangleleft x_{n-2}) \triangleleft x_{n-1}, \\ &\qquad x_{n+2} = x_{n-3} \triangleleft x_{n-2}; \boldsymbol{y}, z \triangleleft x_{n-1}, (z \triangleleft x_{n-2}) \triangleleft x_{n-1}, 0 \rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-2} \triangleleft x_{n-1}, x_{n+1} = (x_{n-3} \triangleleft x_{n-2}) \triangleleft x_{n-1}; \\ &\qquad \boldsymbol{y}, z \triangleleft x_{n-1}, (z \triangleleft x_{n-2}) \triangleleft x_{n-1} \rangle \\ &\sim \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-2} \triangleleft x_{n-1}, x_{n+1} = (x_{n-3} \triangleleft x_{n-1}) \triangleleft x_n; \\ &\qquad \boldsymbol{y}, (z \triangleleft x_{n-3}) \triangleleft x_{n-1}, (z \triangleleft x_{n-1}) \triangleleft x_n \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-2} \triangleleft x_{n-1}, x_{n+1} = (x_{n-3} \triangleleft x_{n-1}) \triangleleft x_n, \\ &\qquad x_{n+2} = x_{n-3} \triangleleft x_{n-1}; \boldsymbol{y}, (z \triangleleft x_{n-3}) \triangleleft x_{n-1}, (z \triangleleft x_{n-1}) \triangleleft x_n, 0 \rangle \\ &\sim \langle \boldsymbol{x}, x_{n+2}; \mu \,|\, \boldsymbol{r}, x_n = x_{n-2} \triangleleft x_{n-1}, x_{n+1} = x_{n+2} \triangleleft x_n, \\ &\qquad x_{n+2} = x_{n-3} \triangleleft x_{n-1}; \boldsymbol{y}, (z \triangleleft x_{n-3}) \triangleleft x_{n-1}, (z \triangleleft x_{n-1}) \triangleleft x_n, z \triangleleft x_{n-1} \rangle \\ &= \widetilde{Q}(D_2).\end{aligned}$$

5 Strong Tietze transformations

In this section, we see that the equivalence relation on shade quandle presentations derived from the transformations (S1)–(S7) is a finer relation than the equivalence relation on group presentations derived from strong Tietze transformations.

We denote by $F_{\mathsf{Grp}}(S)$ the free group on a set S. For $R \subset F_{\mathsf{Grp}}(S)$, we denote by $N_{\mathsf{Grp}}(R)$ the normal subgroup of $F_{\mathsf{Grp}}(S)$ generated by R. We then have a group $\langle S \,|\, R \rangle = F_{\mathsf{Grp}}(S)/N_{\mathsf{Grp}}(R)$. The Tietze transformation theorem [14] states that two finite presentations of a group are related by a finite sequence of the following transformations:

(T0) $\langle x_1, \dots, x_i, \dots, x_j, \dots, x_n \,|\, \boldsymbol{r} \rangle \leftrightarrow \langle x_1, \dots, x_j, \dots, x_i, \dots, x_n \,|\, \boldsymbol{r} \rangle$,
$\langle \boldsymbol{x} \,|\, r_1, \dots, r_i, \dots, r_j, \dots, r_m \rangle \leftrightarrow \langle \boldsymbol{x} \,|\, r_1, \dots, r_j, \dots, r_i, \dots, r_m \rangle$,

(T1) $\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle \leftrightarrow \langle \boldsymbol{x} \,|\, \boldsymbol{r}, r \rangle \quad (r \in N_{\mathsf{Grp}}(R))$,

(T2) $\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle \leftrightarrow \langle \boldsymbol{x}, x_{n+1} \,|\, \boldsymbol{r}, x_{n+1} w^{-1} \rangle \quad (x_{n+1} \notin F_{\mathsf{Grp}}(\boldsymbol{x}),\ w \in F_{\mathsf{Grp}}(\boldsymbol{x}))$,

where a bold symbol indicates a sequence of the symbols. Wada [15] showed that two Wirtinger presentations of a link are related by a finite sequence of the following transformations:

(ST0) $\langle x_1, \dots, x_i, \dots, x_j, \dots, x_n \,|\, \boldsymbol{r} \rangle \leftrightarrow \langle x_1, \dots, x_j, \dots, x_i, \dots, x_n \,|\, \boldsymbol{r} \rangle$,
$\langle \boldsymbol{x} \,|\, r_1, \dots, r_i, \dots, r_j, \dots, r_m \rangle \leftrightarrow \langle \boldsymbol{x} \,|\, r_1, \dots, r_j, \dots, r_i, \dots, r_m \rangle$,

(ST1) $\langle \boldsymbol{x} \,|\, r_1, \dots, r_i, \dots, r_m \rangle \leftrightarrow \langle \boldsymbol{x} \,|\, r_1, \dots, r_i^{-1}, \dots, r_m \rangle$,

(ST2) $\langle \boldsymbol{x} \,|\, r_1, \dots, r_i, \dots, r_m \rangle \leftrightarrow \langle \boldsymbol{x} \,|\, r_1, \dots, w r_i w^{-1}, \dots, r_m \rangle \quad (w \in F_{\mathsf{Grp}}(\boldsymbol{x}))$,

(ST3) $\langle \boldsymbol{x} \,|\, r_1, \dots, r_i, \dots, r_m \rangle \leftrightarrow \langle \boldsymbol{x} \,|\, r_1, \dots, r_i r_k, \dots, r_m \rangle \quad (k \neq i)$,

(ST4) $\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle \leftrightarrow \langle \boldsymbol{x}, x_{n+1} \,|\, \boldsymbol{r}, x_{n+1} w^{-1} \rangle \quad (x_{n+1} \notin F_{\mathsf{Grp}}(\boldsymbol{x}),\ w \in F_{\mathsf{Grp}}(\boldsymbol{x}))$, We write $\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle \sim_{\mathrm{ST}} \langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle$ if they are related by a finite sequence of the transformations (ST0)–(ST4).

For a relation $r = (a, b) \in F_{\mathsf{Qnd}}(\boldsymbol{x}) \times F_{\mathsf{Qnd}}(\boldsymbol{x})$, we set

$$\overline{r} := ab^{-1} \in \mathrm{As}(F_{\mathsf{Qnd}}(\boldsymbol{x})) = F_{\mathsf{Grp}}(\boldsymbol{x}).$$

For example, we have $\overline{(a \triangleleft b, c)} = b^{-1}abc^{-1}$. For a sequence of relations $\boldsymbol{r}$, we denote by $\overline{\boldsymbol{r}}$ the sequence $\overline{r_1}, \ldots, \overline{r_m}$.

Proposition 5.1. *Let $\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle$ and $\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle$ be shade quandle presentations. If $\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle \sim \langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle$, then $\langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}} \rangle \sim_{\mathrm{ST}} \langle \boldsymbol{x}' \,|\, \overline{\boldsymbol{r}'} \rangle$.*

Proof. It is sufficient to show the invariance of $\langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}} \rangle$ under the transformations (S1)–(S7). The invariance under the transformation (S1) follows from (ST0). The invariance under the transformation (S2) follows from (ST1), since we have $\overline{-r_i} = \overline{r_i}^{-1}$. For the transformation (S3)

$$\begin{aligned} &\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a_1 \triangleleft^{\varepsilon} b, a_1 = a_2; \boldsymbol{y}, z, w \rangle \\ &\leftrightarrow \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a_2 \triangleleft^{\varepsilon} b, a_1 = a_2; \boldsymbol{y}, z, w - z \triangleleft^{-\varepsilon} b \rangle, \end{aligned}$$

we have

$$\begin{aligned} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, cb^{-\varepsilon}a_1^{-1}b^{\varepsilon}, a_1a_2^{-1} \rangle &\overset{\text{(ST2)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, b^{\varepsilon}cb^{-\varepsilon}a_1^{-1}, a_1a_2^{-1} \rangle \\ &\overset{\text{(ST3)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, b^{\varepsilon}cb^{-\varepsilon}a_2^{-1}, a_1a_2^{-1} \rangle \\ &\overset{\text{(ST2)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, cb^{-\varepsilon}a_2^{-1}b^{\varepsilon}, a_1a_2^{-1} \rangle. \end{aligned}$$

For the transformation (S4)

$$\begin{aligned} &\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \triangleleft b_1, b_1 = b_2; \boldsymbol{y}, z, w \rangle \\ &\leftrightarrow \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \triangleleft b_2, b_1 = b_2; \boldsymbol{y}, z, w + z \triangleleft^{-1} b_2 - (z \triangleleft c) \triangleleft^{-1} b_2 \rangle, \end{aligned}$$

we have

$$\begin{aligned} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, cb_1^{-1}a^{-1}b_1, b_1b_2^{-1} \rangle &\overset{\text{(ST2)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, a^{-1}b_1cb_1^{-1}, b_1b_2^{-1} \rangle \\ &\overset{\text{(ST3)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, a^{-1}b_1cb_2^{-1}, b_1b_2^{-1} \rangle \\ &\overset{\text{(ST2)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, b_1cb_2^{-1}a^{-1}, b_1b_2^{-1} \rangle \\ &\overset{\text{(ST1)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, ab_2c^{-1}b_1^{-1}, b_1b_2^{-1} \rangle \\ &\overset{\text{(ST3)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, ab_2c^{-1}b_2^{-1}, b_1b_2^{-1} \rangle \\ &\overset{\text{(ST1)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, b_2cb_2^{-1}a^{-1}, b_1b_2^{-1} \rangle \\ &\overset{\text{(ST2)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, cb_2^{-1}a^{-1}b_2, b_1b_2^{-1} \rangle. \end{aligned}$$

In a similar manner, the invariance under the transformation (S5) follows from (ST1)–(ST3). For the transformation (S6)

$$\langle \boldsymbol{x}; \mu|_{a=p} \,|\, \boldsymbol{r}, a = b; \boldsymbol{y}, z \rangle \leftrightarrow \langle \boldsymbol{x}; \mu|_{a=p+1} \,|\, \boldsymbol{r}, a = b \triangleleft a; \boldsymbol{y}, z \triangleleft a \rangle,$$

we have

$$\langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, ab^{-1} \rangle \overset{\text{(ST2)}}{\leftrightarrow} \langle \boldsymbol{x} \,|\, \overline{\boldsymbol{r}}, b^{-1}a \rangle.$$

The invariance under the transformation (S7) follows from (ST4). □

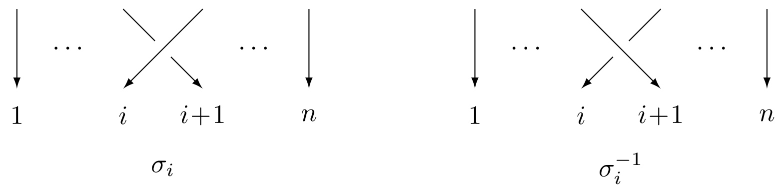

Figure 14: Artin's genelators

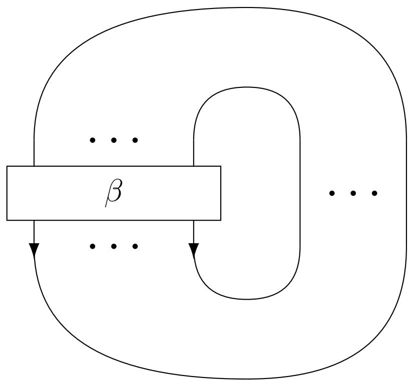

Figure 15: The closure $\hat{\beta}$ of β

6 Shade quandle presentations for closed braids

In this section, we give shade quandle presentations for closed braids with a braid group action and also give an explicit formula of shade quandle presentations for torus links.

Let B_n be the braid group of degree n. It is known that the group B_n has the following presentation:

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j|>1) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (i=1,\ldots,n-2) \end{matrix} \right\rangle$$

We call the generators $\sigma_1, \ldots, \sigma_{n-1}$ *Artin's generator* (see Figure 14).

Let $S = \{x_1, \ldots, x_n\}$ be a set. The braid group B_n acts on $F_{\mathsf{Qnd}}(S)$ from the left by

$$\sigma_i x_j = \begin{cases} x_{i+1} & (j=i), \\ x_i \triangleleft x_{i+1} & (j=i+1), \\ x_j & (j \neq i, i+1). \end{cases}$$

For a braid $\beta \in B_n$, its *closure* $\hat{\beta}$ is the oriented link depicted in Figure 15. We then have a presentation of the fundamental quandle $Q(\hat{\beta})$:

$$\langle x_1, \ldots, x_n \,|\, x_1 = \beta x_1, \ldots, x_n = \beta x_n \rangle.$$

We define a shade quandle presentation associated to a braid $\beta \in B_n$ as follow: We set $y_1 := 1$ and $y_i := x_1 \cdots x_{i-1} \in \mathrm{As}(Q(\hat{\beta}))$ for $i = 2, \ldots, n$. Let $\mu : \mathrm{Orb}(Q(\hat{\beta})) \to \mathbb{Z}$ be the map defined by (3.1) for a diagram depicted in Figure 15. We then define

$$\widetilde{Q}(\beta) := \langle x_1, \ldots, x_n; \mu \,|\, x_1 = \beta x_1, \ldots, x_n = \beta x_n; y_1, \ldots, y_n \rangle.$$

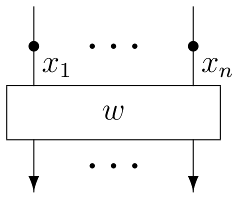

Figure 16: $w_\bullet$

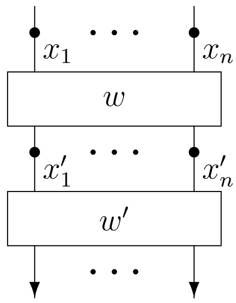

Figure 17: $w_\bullet w'_\bullet$

Let w be a word of $\{\sigma_1, \dots, \sigma_{n-1}\}$. We use the same symbol w to represent the n-strands braid represented by w. We denote by $D_{w_\bullet}$ the diagram of the closure of $w_\bullet$, where $w_\bullet$ is the diagram obtained by adding bivalent vertices to w as shown in Figure 16. In a similar manner, we define $D_{w_\bullet w'_\bullet}$ for words w, w' of $\{\sigma_1, \dots, \sigma_{n-1}\}$ (see Figure 17). We recall that we extended the definition of $\widetilde{Q}(D)$ to oriented link diagrams D with bivalent vertices in Section 4.

Proposition 6.1. *For a braid $\beta \in B_n$, we have $\widetilde{Q}(D_{\beta_\bullet}) \sim \widetilde{Q}(\beta)$.*

Proof. For words w, w' of $\{\sigma_1, \dots, \sigma_{n-1}\}$, let $c_1, \dots, c_n, c'_1, \dots, c'_n$ be the bivalent vertices that are the initial points of the arcs $x_1, \dots, x_n, x'_1, \dots, x'_n$ in Figure 17, and let $c_{n+1}, \dots, c_m$ and $c'_{n+1}, \dots, c'_{m'}$ be the crossings of w and w', respectively. We denote by x_i and x'_i the arcs starting from c_i and c'_i for $i > n$, respectively. We then set

$$\widetilde{Q}(D_{w_\bullet w'_\bullet}) = \left\langle x_1, \dots, x_m, x'_1, \dots, x'_{m'}; \mu \;\middle|\; \begin{matrix} r_1, \dots, r_n, \boldsymbol{r}, r'_1, \dots, r'_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, \boldsymbol{y}, y'_1, \dots, y'_n, \boldsymbol{y}' \end{matrix} \right\rangle,$$

where $y'_1 := 1$ and $y'_i := x'_1 \cdots x'_{i-1}$. It is sufficient to show the following claim, since we have

$$\begin{aligned}
\widetilde{Q}(D_{w_\bullet}) &\sim \widetilde{Q}(D_{w_\bullet \emptyset_\bullet}) \\
&= \left\langle \begin{matrix} x_1, \dots, x_n, \\ x'_1, \dots, x'_n; \mu \end{matrix} \;\middle|\; \begin{matrix} x_1 = x'_1, \dots, x_n = x'_n, \\ x'_1 = wx_1, \dots, x'_n = wx_n; y_1, \dots, y_n, y_1, \dots, y_n \end{matrix} \right\rangle \\
&\sim \left\langle \begin{matrix} x_1, \dots, x_n, \\ x'_1, \dots, x'_n; \mu \end{matrix} \;\middle|\; \begin{matrix} x_1 = wx_1, \dots, x_n = wx_n, \\ x'_1 = wx_1, \dots, x'_n = wx_n; y_1, \dots, y_n, 0, \dots, 0 \end{matrix} \right\rangle \\
&\sim \langle x_1, \dots, x_n; \mu \,|\, x_1 = wx_1, \dots, x_n = wx_n; y_1, \dots, y_n \rangle \\
&= \widetilde{Q}(w).
\end{aligned}$$

Claim. Let w be a word of $\{\sigma_1, \dots, \sigma_{n-1}\}$. We have

$$\widetilde{Q}(D_{w_\bullet w'_\bullet}) \sim \left\langle \begin{matrix} x_1, \dots, x_n, \\ x'_1, \dots, x'_{m'}; \mu \end{matrix} \;\middle|\; \begin{matrix} r_1, \dots, r_n, x'_1 = wx_1, \dots, x'_n = wx_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, y'_1, \dots, y'_n, \boldsymbol{y}' \end{matrix} \right\rangle,$$

for any word w' of $\{\sigma_1, \dots, \sigma_{n-1}\}$.

We show the claim by induction on the length of the word w. Obviously, the claim is true for the empty word $\emptyset$. We assume that the claim is true for

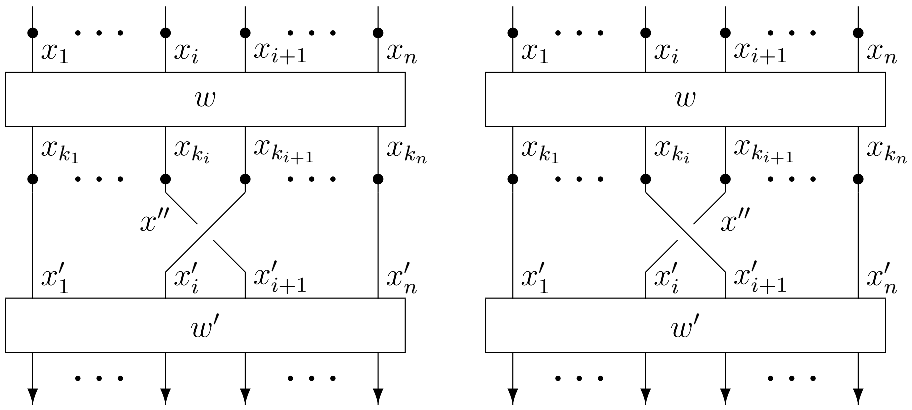

Figure 18: $w_\bullet(\sigma_i w')_\bullet$ and $w_\bullet(\sigma_i^{-1} w')_\bullet$

words w of length l. We denote the sequence $x_1, \dots, x_k$ by $\boldsymbol{x}_k$. We also denote the sequence $x'_1, \dots, x'_{m'}$ by $\boldsymbol{x}'$.

We will show the claim for a word $w\sigma_i$. Let c_{m+1} be the crossing corresponding to σ_i. See the left picture in Figure 18. By the induction hypothesis, we have

$$
\begin{aligned}
&\widetilde{Q}(D_{w_\bullet(\sigma_i w')_\bullet}) \\
&= \left\langle \begin{array}{l} \boldsymbol{x}_m, x'', \\ \boldsymbol{x}'; \mu \end{array} \middle| \begin{array}{l} r_1, \dots, r_n, \boldsymbol{r}, x'' = x_{k_i}, x'_1 = x_{k_1}, \dots, x'_{i-1} = x_{k_{i-1}}, x'_i = x_{k_{i+1}}, \\ x'_{i+1} = x'' \triangleleft x'_i, x'_{i+2} = x_{k_{i+2}}, \dots, x'_n = x_{k_n}, \boldsymbol{r}'; \\ y_1, \dots, y_n, \boldsymbol{y}, y'_i, y'_1, \dots, y'_{i-1}, y'_i x'', y'_{i+1}, \dots, y'_n, \boldsymbol{y}' \end{array} \right\rangle \\
&\sim \left\langle \begin{array}{l} \boldsymbol{x}_n, x'', \\ \boldsymbol{x}'; \mu \end{array} \middle| \begin{array}{l} r_1, \dots, r_n, x'' = wx_i, x'_1 = wx_1, \dots, x'_{i-1} = wx_{i-1}, x'_i = wx_{i+1}, \\ x'_{i+1} = x'' \triangleleft x'_i, x'_{i+2} = wx_{i+2}, \dots, x'_n = wx_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, y'_i, y'_1, \dots, y'_{i-1}, y'_i x'', y'_{i+1}, \dots, y'_n, \boldsymbol{y}' \end{array} \right\rangle .
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\widetilde{Q}(D_{(w\sigma_i)_\bullet w'_\bullet}) \\
&\sim \widetilde{Q}(D_{w_\bullet(\sigma_i w')_\bullet}) \\
&\sim \left\langle \begin{array}{l} \boldsymbol{x}_n, x'', \\ \boldsymbol{x}'; \mu \end{array} \middle| \begin{array}{l} r_1, \dots, r_n, x'' = wx_i, x'_1 = wx_1, \dots, x'_{i-1} = wx_{i-1}, x'_i = wx_{i+1}, \\ x'_{i+1} = (wx_i) \triangleleft x'_i, x'_{i+2} = wx_{i+2}, \dots, x'_n = wx_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, 0, y'_1, \dots, y'_{i-1}, y'_i(wx_i), y'_{i+1}, \dots, y'_n, \boldsymbol{y}' \end{array} \right\rangle \\
&\sim \left\langle \begin{array}{l} \boldsymbol{x}_n, \\ \boldsymbol{x}'; \mu \end{array} \middle| \begin{array}{l} r_1, \dots, r_n, x'_1 = wx_1, \dots, x'_{i-1} = wx_{i-1}, x'_i = wx_{i+1}, \\ x'_{i+1} = (wx_i) \triangleleft x'_i, x'_{i+2} = wx_{i+2}, \dots, x'_n = wx_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, y'_1, \dots, y'_{i-1}, y'_i(wx_i), y'_{i+1}, \dots, y'_n, \boldsymbol{y}' \end{array} \right\rangle \\
&\sim \left\langle \begin{array}{l} \boldsymbol{x}_n, \\ \boldsymbol{x}'; \mu \end{array} \middle| \begin{array}{l} r_1, \dots, r_n, x'_1 = wx_1, \dots, x'_{i-1} = wx_{i-1}, x'_i = wx_{i+1}, \\ x'_{i+1} = (wx_i) \triangleleft (wx_{i+1}), x'_{i+2} = wx_{i+2}, \dots, x'_n = wx_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, y'_1, \dots, y'_{i-1}, y'_i, y'_{i+1}, \dots, y'_n, \boldsymbol{y}' \end{array} \right\rangle \\
&= \left\langle \begin{array}{l} \boldsymbol{x}_n, \\ \boldsymbol{x}'; \mu \end{array} \middle| \begin{array}{l} r_1, \dots, r_n, x'_1 = (w\sigma_i)x_1, \dots, x'_n = (w\sigma_i)x_n, \boldsymbol{r}'; \\ y_1, \dots, y_n, y'_1, \dots, y'_n, \boldsymbol{y}' \end{array} \right\rangle .
\end{aligned}
$$

We will show the claim for a word $w\sigma_i^{-1}$. Let c_{m+1} be the crossing corresponding to σ_i^{-1}. See the right picture in Figure 18. By the induction hypothesis, we have

$$\begin{aligned}
&\widetilde{Q}(D_{w_\bullet(\sigma_i^{-1}w')_\bullet})\\
&= \left\langle \begin{matrix}\boldsymbol{x}_m, x'',\\ \boldsymbol{x}'; \mu\end{matrix} \,\middle|\, \begin{matrix} r_1, \ldots, r_n, \boldsymbol{r}, x'' = x_{k_{i+1}}, x'_1 = x_{k_1}, \ldots, x'_{i-1} = x_{k_{i-1}},\\ x'_i = x'' \lhd^{-1} x'_{i+1}, x'_{i+1} = x_{k_i}, x'_{i+2} = x_{k_{i+2}}, \ldots, x'_n = x_{k_n}, \boldsymbol{r}';\\ y_1, \ldots, y_n, \boldsymbol{y}, y'_i x'_{i+1}, y'_1, \ldots, y'_i, y'_i, y'_{i+2}, \ldots, y'_n, \boldsymbol{y}'\end{matrix} \right\rangle\\
&\sim \left\langle \begin{matrix}\boldsymbol{x}_n, x'',\\ \boldsymbol{x}'; \mu\end{matrix} \,\middle|\, \begin{matrix} r_1, \ldots, r_n, x'' = wx_{i+1}, x'_1 = wx_1, \ldots, x'_{i-1} = wx_{i-1},\\ x'_i = x'' \lhd^{-1} x'_{i+1}, x'_{i+1} = wx_i, x'_{i+2} = wx_{i+2}, \ldots, x'_n = wx_n, \boldsymbol{r}';\\ y_1, \ldots, y_n, y'_i x'_{i+1}, y'_1, \ldots, y'_i, y'_i, y'_{i+2}, \ldots, y'_n, \boldsymbol{y}'\end{matrix} \right\rangle.
\end{aligned}$$

We then have

$$\begin{aligned}
&\widetilde{Q}(D_{(w\sigma_i^{-1})_\bullet w'_\bullet})\\
&\sim \widetilde{Q}(D_{w_\bullet(\sigma_i^{-1}w')_\bullet})\\
&\sim \left\langle \begin{matrix}\boldsymbol{x}_n, x'',\\ \boldsymbol{x}'; \mu\end{matrix} \,\middle|\, \begin{matrix} r_1, \ldots, r_n, x'' = wx_{i+1}, x'_1 = wx_1, \ldots, x'_{i-1} = wx_{i-1},\\ x'_i = (wx_{i+1}) \lhd^{-1} x'_{i+1},\\ x'_{i+1} = wx_i, x'_{i+2} = wx_{i+2}, \ldots, x'_n = wx_n, \boldsymbol{r}';\\ y_1, \ldots, y_n, 0, y'_1, \ldots, y'_i, y'_i, y'_{i+2}, \ldots, y'_n, \boldsymbol{y}'\end{matrix} \right\rangle\\
&\sim \left\langle \begin{matrix}\boldsymbol{x}_n,\\ \boldsymbol{x}'; \mu\end{matrix} \,\middle|\, \begin{matrix} r_1, \ldots, r_n, x'_1 = wx_1, \ldots, x'_{i-1} = wx_{i-1}, x'_i = (wx_{i+1}) \lhd^{-1} x'_{i+1},\\ x'_{i+1} = wx_i, x'_{i+2} = wx_{i+2}, \ldots, x'_n = wx_n, \boldsymbol{r}';\\ y_1, \ldots, y_n, y'_1, \ldots, y'_i, y'_i, y'_{i+2}, \ldots, y'_n, \boldsymbol{y}'\end{matrix} \right\rangle\\
&\sim \left\langle \begin{matrix}\boldsymbol{x}_n,\\ \boldsymbol{x}'; \mu\end{matrix} \,\middle|\, \begin{matrix} r_1, \ldots, r_n, x'_1 = wx_1, \ldots, x'_{i-1} = wx_{i-1}, x'_i = (wx_{i+1}) \lhd^{-1} (wx_i),\\ x'_{i+1} = wx_i, x'_{i+2} = wx_{i+2}, \ldots, x'_n = wx_n, \boldsymbol{r}';\\ y_1, \ldots, y_n, y'_1, \ldots, y'_i, y'_{i+1}, y'_{i+2}, \ldots, y'_n, \boldsymbol{y}'\end{matrix} \right\rangle\\
&= \left\langle \begin{matrix}\boldsymbol{x}_n,\\ \boldsymbol{x}'; \mu\end{matrix} \,\middle|\, \begin{matrix} r_1, \ldots, r_n, x'_1 = (w\sigma_i^{-1})x_1, \ldots, x'_n = (w\sigma_i^{-1})x_n, \boldsymbol{r}';\\ y_1, \ldots, y_n, y'_1, \ldots, y'_n, \boldsymbol{y}'\end{matrix} \right\rangle.
\end{aligned}$$

This completes the proof. □

We denote by $\lfloor x \rfloor$ the maximum integer less than x, where we remark that $\lfloor x \rfloor = x - 1$ for any integer $x \in \mathbb{Z}$. For $a \in Q$ and $w \in F_{\mathsf{Grp}}(Q)$, we define $a \lhd w$ by $a \lhd 1 = a$ and

$$a \lhd xy = (a \lhd x) \lhd y \quad (x, y \in F_{\mathsf{Grp}}(Q)).$$

We note that $a \lhd b^{-1} = a \lhd^{-1} b$ and $a \lhd (b \lhd c) = a \lhd (c^{-1}bc)$. As a corollary of Proposition 6.1, we obtain a shade quandle presentation for an (n, m)-torus link. We set $\overline{i} := i - n\lfloor i/n \rfloor$ for $i \in \mathbb{Z}_{>0}$. That is, $\overline{i + kn} = i$ for any $i \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}$. We denote the sequence $x_1, \ldots, x_n$ by $\boldsymbol{x}$. We set $\overline{\boldsymbol{x}} := x_1 \cdots x_n$.

Corollary 6.2. *Let $\beta := (\sigma_1 \cdots \sigma_{n-1})^m \in B_n$, where m is a positive integer. Under the same conditions as Proposition 6.1, we have*

$$\widetilde{Q}(\beta_\bullet) \sim \left\langle \boldsymbol{x}; \mu \,\middle|\, \begin{matrix} x_1 = x_{\overline{m+1}} \lhd \overline{\boldsymbol{x}}^{\lfloor (m+1)/n \rfloor}, \ldots, x_n = x_{\overline{m+n}} \lhd \overline{\boldsymbol{x}}^{\lfloor (m+n)/n \rfloor};\\ y_1, \ldots, y_n \end{matrix} \right\rangle.$$

Proof. It is sufficient to show that for each $i \in \{1, \ldots, n\}$,

$$(\sigma_1 \cdots \sigma_{n-1})^m x_i = x_{\overline{m+i}} \lhd \overline{\boldsymbol{x}}^{\lfloor (m+i)/n \rfloor}$$

for any positive integer m. We show the equality by induction on m.

First, for $i < n$,

$$\begin{aligned}(\sigma_1 \cdots \sigma_{n-1})x_i &= (\sigma_1 \cdots \sigma_{i-1}\sigma_i \cdots \sigma_{n-1})x_i \\ &= (\sigma_1 \cdots \sigma_{i-1}\sigma_i)x_i \\ &= (\sigma_1 \cdots \sigma_{i-1})x_{i+1} \\ &= x_{i+1} = x_{\overline{1+i}} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (1+i)/n \rfloor},\end{aligned}$$

where we note that $\overline{1+i} = i+1$ and $\lfloor (1+i)/n \rfloor = 0$. For $i = n$,

$$\begin{aligned}(\sigma_1 \cdots \sigma_{n-1})x_n &= (\sigma_1 \cdots \sigma_{n-2})(x_{n-1} \triangleleft x_n) \\ &= (\sigma_1 \cdots \sigma_{n-2})x_{n-1} \triangleleft (\sigma_1 \cdots \sigma_{n-2})x_n \\ &= ((\sigma_1 \cdots \sigma_{n-3})(x_{n-2} \triangleleft x_{n-1})) \triangleleft x_n \\ &\cdots \\ &= (\cdots((x_1 \triangleleft x_2) \triangleleft x_3) \triangleleft \cdots) \triangleleft x_n \\ &= (\cdots(((x_1 \triangleleft x_1) \triangleleft x_2) \triangleleft x_3) \triangleleft \cdots) \triangleleft x_n \\ &= x_1 \triangleleft \overline{\boldsymbol{x}} = x_{\overline{1+n}} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (1+n)/n \rfloor},\end{aligned}$$

where we note that $\overline{1+n} = 1$ and $\lfloor (1+n)/n \rfloor = 1$.

Next, for $i \leq n$, by the induction hypothesis, we have

$$\begin{aligned}(\sigma_1 \cdots \sigma_{n-1})^m x_i &= (\sigma_1 \cdots \sigma_{n-1})((\sigma_1 \cdots \sigma_{n-1})^{m-1} x_i) \\ &= (\sigma_1 \cdots \sigma_{n-1})(x_{\overline{m-1+i}} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (m-1+i)/n \rfloor}) \\ &= (\sigma_1 \cdots \sigma_{n-1})x_{\overline{m-1+i}} \triangleleft (\sigma_1 \cdots \sigma_{n-1})\overline{\boldsymbol{x}}^{\lfloor (m-1+i)/n \rfloor} \\ &= (\sigma_1 \cdots \sigma_{n-1})x_{\overline{m-1+i}} \triangleleft ((\sigma_1 \cdots \sigma_{n-1})\overline{\boldsymbol{x}})^{\lfloor (m-1+i)/n \rfloor} \\ &= (\sigma_1 \cdots \sigma_{n-1})x_{\overline{m-1+i}} \triangleleft (x_2 \cdots x_n(x_1 \triangleleft \overline{\boldsymbol{x}}))^{\lfloor (m-1+i)/n \rfloor} \\ &= (\sigma_1 \cdots \sigma_{n-1})x_{\overline{m-1+i}} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (m-1+i)/n \rfloor}. \qquad (6.1)\end{aligned}$$

If $\overline{m-1+i} < n$, then we obtain

$$(6.1) = x_{\overline{m-1+i}+1} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (m-1+i)/n \rfloor} = x_{\overline{m+i}} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (m+i)/n \rfloor}.$$

If $\overline{m-1+i} = n$, then we obtain

$$(6.1) = (x_1 \triangleleft \overline{\boldsymbol{x}}) \triangleleft \overline{\boldsymbol{x}}^{\lfloor (m+i)/n \rfloor - 1} = x_{\overline{m+i}} \triangleleft \overline{\boldsymbol{x}}^{\lfloor (m+i)/n \rfloor}.$$

This completes the proof. □

7 Triples of matrices

We recall the definitions of an Alexander pair and relation maps and introduce three matrices obtained from a shade quandle presentation. We also recall the equivalence relation on triples of matrices and show that the equivalent shade quandle presentations induce the equivalent triples of matrices.

Let Q be a quandle, and let R be a unital ring. The pair $f = (f_1, f_2)$ of maps $f_1, f_2 : Q \times Q \to R$ is an *Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a,a) + f_2(a,a) = 1$.
- For any $a, b \in Q$, $f_1(a,b)$ is invertible.
- For any $a, b, c \in Q$,

$$
\begin{aligned}
&f_1(a \triangleleft b, c) f_1(a,b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a,c), \\
&f_1(a \triangleleft b, c) f_2(a,b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b,c), \text{ and} \\
&f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a,c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b,c).
\end{aligned}
$$

An *f-column relation map* $f_{\mathrm{col}} : Q \to R$ is a map satisfying

$$f_{\mathrm{col}}(a \triangleleft b) = f_1(a,b) f_{\mathrm{col}}(a) + f_2(a,b) f_{\mathrm{col}}(b)$$

for any $a, b \in Q$. Let Y be a Q-set. An *f-row relation map* $f_{\mathrm{row}} : Y \times Q \to R$ is a map satisfying

$$
\begin{aligned}
&f_{\mathrm{row}}(y, a) = f_{\mathrm{row}}(y \triangleleft b, a \triangleleft b) f_1(a,b), \text{ and} \\
&f_{\mathrm{row}}(y \triangleleft a, b) = f_{\mathrm{row}}(y, b) + f_{\mathrm{row}}(y \triangleleft b, a \triangleleft b) f_2(a,b)
\end{aligned}
$$

for any $a, b \in Q$ and $y \in Y$. For a presentation $Q \cong \langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle$, we denote by $\mathrm{pr} : F_{\mathsf{Qnd}}(\boldsymbol{x}) \to Q$ the canonical projection. The *f-derivative* $\frac{\partial_f}{\partial x_j} : F_{\mathsf{Qnd}}(\boldsymbol{x}) \to R$ is a map satisfying

$$\frac{\partial_f}{\partial x_j}(a \triangleleft b) = f_1(\mathrm{pr}(a), \mathrm{pr}(b)) \frac{\partial_f}{\partial x_j}(a) + f_2(\mathrm{pr}(a), \mathrm{pr}(b)) \frac{\partial_f}{\partial x_j}(b), \quad \frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}$$

for any $a, b \in F_{\mathsf{Qnd}}(\boldsymbol{x})$, where δ_{ij} is the Kronecker delta. Hereafter, we omit the canonical projection pr as $f_1(a,b)$.

We define $\frac{\partial_f}{\partial x_j}(a = b) := \frac{\partial_f}{\partial x_j}(a) - \frac{\partial_f}{\partial x_j}(b)$. We extend $f_{\mathrm{row}} : Y \times Q \to R$ to $f_{\mathrm{row}} : \mathbb{Z}[Y] \times Q \to R$ linearly and define $f_{\mathrm{row}}(y, a = b) := f_{\mathrm{row}}(y, a)$.

Definition 7.1. Let $\widetilde{Q} = \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle = \langle x_1, \ldots, x_n; \mu \,|\, r_1, \ldots, r_m; y_1, \ldots, y_m \rangle$ be a shade quandle presentation. Set $Q := \langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle$. Let $f = (f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$, $f_{\mathrm{col}} : Q \to R$ an f-column relation map, and $f_{\mathrm{row}} : \mathrm{As}(Q) \times Q \to R$ an f-row relation map. For the orbit decomposition $\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle = \bigsqcup_{i=1}^{l} \mathrm{orb}(z_i)$, we set $\omega_i := f_1(z_i, z_i)$, $p_i := \mu(\mathrm{orb}(z_i))$ and $\omega(\mu) := \omega_1^{p_1} \cdots \omega_l^{p_l}$. We then define

$$
\begin{aligned}
\widetilde{B}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; \boldsymbol{f}_{\mathrm{row}}) &:= \begin{pmatrix} f_{\mathrm{row}}(y_1, r_1) & \cdots & f_{\mathrm{row}}(y_m, r_m) & 0 \end{pmatrix}, \\
\widetilde{A}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; f_1, f_2) &:= \begin{pmatrix} \frac{\partial_f r_1}{\partial x_1} & \cdots & \frac{\partial_f r_1}{\partial x_n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial_f r_m}{\partial x_1} & \cdots & \frac{\partial_f r_m}{\partial x_n} & 0 \\ 0 & \cdots & 0 & \omega(\mu)^{-1} \end{pmatrix}, \\
\widetilde{C}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; \boldsymbol{f}_{\mathrm{col}}) &:= \begin{pmatrix} f_{\mathrm{col}}(x_1) \\ \vdots \\ f_{\mathrm{col}}(x_n) \\ 0 \end{pmatrix}.
\end{aligned}
$$

Let R be a unital ring. We denote by $R^\times$ the group of units of R. We denote by $GL(n;R)$ the set of $n \times n$ invertible matrices over R. We define $P_{ij}, E_{ij}(r), E_i(u) \in GL(n;R)$ by

$$\begin{aligned} P_{ij} &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_j, \boldsymbol{e}_{i+1}, \dots, \boldsymbol{e}_{j-1}, \boldsymbol{e}_i, \boldsymbol{e}_{j+1}, \dots, \boldsymbol{e}_n), \\ E_{ij}(r) &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_{j-1}, \boldsymbol{e}_j + r\boldsymbol{e}_i, \boldsymbol{e}_{j+1}, \dots, \boldsymbol{e}_n) \quad (i \neq j), \\ E_i(u) &= (\boldsymbol{e}_1, \dots, \boldsymbol{e}_{i-1}, u\boldsymbol{e}_i, \boldsymbol{e}_{i+1}, \dots, \boldsymbol{e}_n) \end{aligned}$$

for $r \in R$ and $u \in R^\times$, where $\boldsymbol{e}_i$ is the unit column vector whose components are all 0, except the ith component that equals 1.

Definition 7.2 ([9])**.** We write $(B, A, C) \sim (B', A', C')$ if they are related by a finite sequence of the following transformations:

- $(B, A, C) \leftrightarrow (BE_{ij}(r)^{-1}, E_{ij}(r)A, C) \quad (r \in R)$,
- $(B, A, C) \leftrightarrow (B, AE_{ij}(r), E_{ij}(r)^{-1}C) \quad (r \in R)$,
- $(B, A, C) \leftrightarrow (BE_i(u), E_i(u)^{-1}AE_j(u), E_j(u)^{-1}C) \quad (u \in R^\times)$,
- $(B, A, C) \leftrightarrow \left(\begin{pmatrix} B & \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}, \begin{pmatrix} C \\ \boldsymbol{0} \end{pmatrix} \right)$.

We remark that we have

$$\begin{aligned} (B, A, C) &\sim (BP_{ij}E_j(-1), E_j(-1)P_{ij}A, C), \\ (B, A, C) &\sim (B, AP_{ij}E_j(-1), E_j(-1)P_{ij}C), \\ (B, A, C) &\sim (BP_{ij}, P_{ij}AP_{kl}, P_{kl}C) \end{aligned}$$

as we see in [9]. Using triples of matrices obtained from link diagrams, we have Alexander type invariants such as the Alexander polynomial [1], the Conway polynomial [3], the twisted Alexander polynomial [12, 15], quandle twisted Alexander invariants [7, 8, 9], quandle 2-cocycle invariants [2], and so on. The invariances of Alexander type invariants are verified via the equivalence relations on triple of matrices. For further details, we refer the reader to [9].

Proposition 7.3. *If* $\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle \sim \langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle$*, then we have*

$$\begin{aligned} &(\widetilde{B}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; \boldsymbol{f}_{\mathrm{row}}), \widetilde{A}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; f_1, f_2), \widetilde{C}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; \boldsymbol{f}_{\mathrm{col}})) \\ &\sim (\widetilde{B}(\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle; \boldsymbol{f}_{\mathrm{row}}), \widetilde{A}(\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle; f_1, f_2), \widetilde{C}(\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle; \boldsymbol{f}_{\mathrm{col}})). \end{aligned}$$

Proof. It is sufficient to show the equivalence for the transformations (S1)–(S7). We set

$$\begin{aligned} A &:= \widetilde{A}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; f_1, f_2), & A' &:= \widetilde{A}(\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle; f_1, f_2), \\ B &:= \widetilde{B}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; \boldsymbol{f}_{\mathrm{row}}), & B' &:= \widetilde{B}(\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle; \boldsymbol{f}_{\mathrm{row}}), \\ C &:= \widetilde{C}(\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle; \boldsymbol{f}_{\mathrm{col}}), & C' &:= \widetilde{C}(\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle; \boldsymbol{f}_{\mathrm{col}}). \end{aligned}$$

We denote by $\boldsymbol{a}_i$ the i-th row vector of A and denote by a_{ij} the (i, j) entry of A. We set $\frac{\boldsymbol{\partial_f} r}{\boldsymbol{\partial x}} := \begin{pmatrix} \frac{\partial_f r}{\partial x_1} \\ \vdots \\ \frac{\partial_f r}{\partial x_n} \end{pmatrix}$.

For the presentations

$$\begin{aligned}\langle \boldsymbol{x};\mu\,|\,\boldsymbol{r};\boldsymbol{y}\rangle &= \langle x_1,\dots,x_i,\dots,x_j,\dots,x_n;\mu\,|\,r_1,\dots,r_k,\dots,r_l,\dots,r_m;\\ &\qquad\qquad\qquad\qquad\qquad y_1,\dots,y_k,\dots,y_l,\dots,y_m\rangle,\\ \langle \boldsymbol{x}';\mu'\,|\,\boldsymbol{r}';\boldsymbol{y}'\rangle &= \langle x_1,\dots,x_j,\dots,x_i,\dots,x_n;\mu\,|\,r_1,\dots,r_l,\dots,r_k,\dots,r_m;\\ &\qquad\qquad\qquad\qquad\qquad y_1,\dots,y_l,\dots,y_k,\dots,y_m\rangle,\end{aligned}$$

we have

$$(B,A,C)\sim(BP_{kl},P_{kl}AP_{ij},P_{ij}C)=(B',A',C').$$

For the presentations

$$\begin{aligned}\langle \boldsymbol{x};\mu\,|\,\boldsymbol{r};\boldsymbol{y}\rangle &= \langle x_1,\dots,x_n;\mu\,|\,r_1,\dots,r_i,\dots,r_j,\dots,r_m;\\ &\qquad\qquad\qquad\qquad y_1,\dots,y_i,\dots,y_j,\dots,y_m\rangle,\\ \langle \boldsymbol{x}';\mu'\,|\,\boldsymbol{r}';\boldsymbol{y}'\rangle &= \langle x_1,\dots,x_n;\mu\,|\,r_1,\dots,r_j,\dots,-r_i,\dots,r_m;\\ &\qquad\qquad\qquad\qquad y_1,\dots,y_j,\dots,-y_i,\dots,y_m\rangle,\end{aligned}$$

we have

$$(B,A,C)\sim(BP_{ij}E_i(-1),E_i(-1)P_{ij}A,C)=(B',A',C').$$

For the presentations

$$\begin{aligned}\langle \boldsymbol{x};\mu\,|\,\boldsymbol{r};\boldsymbol{y}\rangle &= \langle \boldsymbol{x};\mu\,|\,\boldsymbol{r},c=a_1\triangleleft^{\varepsilon}b,a_1=a_2;\boldsymbol{y},z,w\rangle,\\ \langle \boldsymbol{x}';\mu'\,|\,\boldsymbol{r}';\boldsymbol{y}'\rangle &= \langle \boldsymbol{x};\mu\,|\,\boldsymbol{r},c=a_2\triangleleft^{\varepsilon}b,a_1=a_2;\boldsymbol{y},z,w-z\triangleleft^{-\varepsilon}b\rangle,\end{aligned}$$

we have

$$\begin{aligned}(B,A,C)&=(B,\begin{pmatrix}\frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0\\ \vdots & \vdots\\ \frac{\boldsymbol{\partial_f} r_{n-2}}{\boldsymbol{\partial x}} & 0\\ \frac{\boldsymbol{\partial_f} c}{\boldsymbol{\partial x}}-\alpha\frac{\boldsymbol{\partial_f} a_1}{\boldsymbol{\partial x}}-\beta\frac{\boldsymbol{\partial_f} b}{\boldsymbol{\partial x}} & 0\\ \frac{\boldsymbol{\partial_f} a_1}{\boldsymbol{\partial x}}-\frac{\boldsymbol{\partial_f} a_2}{\boldsymbol{\partial x}} & 0\\ \mathbf{0} & \omega(\mu)^{-1}\end{pmatrix},C)\\ &\sim(B',\begin{pmatrix}\frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0\\ \vdots & \vdots\\ \frac{\boldsymbol{\partial_f} r_{n-2}}{\boldsymbol{\partial x}} & 0\\ \frac{\boldsymbol{\partial_f} c}{\boldsymbol{\partial x}}-\alpha\frac{\boldsymbol{\partial_f} a_2}{\boldsymbol{\partial x}}-\beta\frac{\boldsymbol{\partial_f} b}{\boldsymbol{\partial x}} & 0\\ \frac{\boldsymbol{\partial_f} a_1}{\boldsymbol{\partial x}}-\frac{\boldsymbol{\partial_f} a_2}{\boldsymbol{\partial x}} & 0\\ \mathbf{0} & \omega(\mu)^{-1}\end{pmatrix},C)=(B',A',C'),\end{aligned}$$

where

$$\alpha=\begin{cases}f_1(a_1,b) & \text{if } \varepsilon=1,\\ 1 & \text{if } \varepsilon=0,\\ f_1(a_1\triangleleft^{-1}b,b)^{-1} & \text{if } \varepsilon=-1,\end{cases}$$

$$\beta=\begin{cases}f_2(a_1,b) & \text{if } \varepsilon=1,\\ 0 & \text{if } \varepsilon=0,\\ -f_1(a_1\triangleleft^{-1}b,b)^{-1}f_2(a_1\triangleleft^{-1}b,b) & \text{if } \varepsilon=-1,\end{cases}$$

$A' = E_{n-1,n}(\alpha)A$ and $B' = BE_{n-1,n}(\alpha)^{-1}$.

For the presentations

$$\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y}\rangle = \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \triangleleft b_1, b_1 = b_2; \boldsymbol{y}, z, w\rangle,$$
$$\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}'\rangle = \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \triangleleft b_2, b_1 = b_2; \boldsymbol{y}, z, w + z \triangleleft^{-1} b_2 - (z \triangleleft c) \triangleleft^{-1} b_2\rangle,$$

we have

$$(B, A, C) = (B, \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_{n-2}}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} c}{\boldsymbol{\partial x}} - \alpha \frac{\boldsymbol{\partial_f} a}{\boldsymbol{\partial x}} - \beta \frac{\boldsymbol{\partial_f} b_1}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} b_1}{\boldsymbol{\partial x}} - \frac{\boldsymbol{\partial_f} b_2}{\boldsymbol{\partial x}} & 0 \\ \boldsymbol{0} & \omega(\mu)^{-1} \end{pmatrix}, C)$$
$$\sim (B', \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_{n-2}}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} c}{\boldsymbol{\partial x}} - \alpha \frac{\boldsymbol{\partial_f} a}{\boldsymbol{\partial x}} - \beta \frac{\boldsymbol{\partial_f} b_2}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} b_1}{\boldsymbol{\partial x}} - \frac{\boldsymbol{\partial_f} b_2}{\boldsymbol{\partial x}} & 0 \\ \boldsymbol{0} & \omega(\mu)^{-1} \end{pmatrix}, C) = (B', A', C'),$$

where $\alpha = f_1(a, b_1)$, $\beta = f_2(a, b_1)$, $A' = E_{n-1,n}(\beta)A$ and $B' = BE_{n-1,n}(\beta)^{-1}$.

For the presentations

$$\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y}\rangle = \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \triangleleft^{-1} b_1, b_1 = b_2; \boldsymbol{y}, z, w\rangle,$$
$$\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}'\rangle = \langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}, c = a \triangleleft^{-1} b_2, b_1 = b_2; \boldsymbol{y}, z, w - z + z \triangleleft c\rangle,$$

we have

$$(B, A, C) = (B, \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_{n-2}}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} c}{\boldsymbol{\partial x}} - \alpha \frac{\boldsymbol{\partial_f} a}{\boldsymbol{\partial x}} - \beta \frac{\boldsymbol{\partial_f} b_1}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} b_1}{\boldsymbol{\partial x}} - \frac{\boldsymbol{\partial_f} b_2}{\boldsymbol{\partial x}} & 0 \\ \boldsymbol{0} & \omega(\mu)^{-1} \end{pmatrix}, C)$$
$$\sim (B', \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_{n-2}}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} c}{\boldsymbol{\partial x}} - \alpha \frac{\boldsymbol{\partial_f} a}{\boldsymbol{\partial x}} - \beta \frac{\boldsymbol{\partial_f} b_2}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} b_1}{\boldsymbol{\partial x}} - \frac{\boldsymbol{\partial_f} b_2}{\boldsymbol{\partial x}} & 0 \\ \boldsymbol{0} & \omega(\mu)^{-1} \end{pmatrix}, C) = (B', A', C'),$$

where $\alpha = f_1(a \triangleleft^{-1} b_1, b_1)^{-1}$, $\beta = -f_1(a \triangleleft^{-1} b_1, b_1)^{-1} f_2(a \triangleleft^{-1} b_1, b_1)$, $A' = E_{n-1,n}(\beta)A$ and $B' = BE_{n-1,n}(\beta)^{-1}$.

For the presentations

$$\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle = \langle \boldsymbol{x}; \mu|_{a=p} \,|\, \boldsymbol{r}, a=b; \boldsymbol{y}, z \rangle,$$
$$\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle = \langle \boldsymbol{x}; \mu|_{a=p+1} \,|\, \boldsymbol{r}, a=b \triangleleft a; \boldsymbol{y}, z \triangleleft a \rangle,$$

we have

$$(B,A,C) = (B, \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_{n-1}}{\boldsymbol{\partial x}} & 0 \\ \frac{\boldsymbol{\partial_f} a}{\boldsymbol{\partial x}} - \frac{\boldsymbol{\partial_f} b}{\boldsymbol{\partial x}} & 0 \\ \mathbf{0} & \omega(\mu|_{a=p})^{-1} \end{pmatrix}, C)$$
$$\sim (B', \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_{n-1}}{\boldsymbol{\partial x}} & 0 \\ \alpha\frac{\boldsymbol{\partial_f} a}{\boldsymbol{\partial x}} - \alpha\frac{\boldsymbol{\partial_f} b}{\boldsymbol{\partial x}} & 0 \\ \mathbf{0} & \omega(\mu|_{a=p+1})^{-1} \end{pmatrix}, C') = (B', A', C'),$$

where $\alpha = f_1(a,a)$, $A' = E_n(\alpha) A E_{n+1}(\alpha^{-1})$, $B' = B E_n(\alpha^{-1})$ and $C' = E_{n+1}(\alpha) C$.

For the presentations

$$\langle \boldsymbol{x}; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle = \langle x_1, \dots, x_n; \mu \,|\, \boldsymbol{r}; \boldsymbol{y} \rangle,$$
$$\langle \boldsymbol{x}'; \mu' \,|\, \boldsymbol{r}'; \boldsymbol{y}' \rangle = \langle x_1, \dots, x_n, x_{n+1}; \mu \,|\, \boldsymbol{r}, x_{n+1} = w; \boldsymbol{y}, 0 \rangle,$$

we have

$$(B,A,C) = (B, \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 \\ \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_n}{\boldsymbol{\partial x}} & 0 \\ \mathbf{0} & \omega(\mu)^{-1} \end{pmatrix}, C)$$
$$\sim (\begin{pmatrix} B & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_n}{\boldsymbol{\partial x}} & 0 & 0 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & \omega(\mu)^{-1} \end{pmatrix}, \begin{pmatrix} C \\ \mathbf{0} \end{pmatrix})$$
$$\sim (\begin{pmatrix} B & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \frac{\boldsymbol{\partial_f} r_1}{\boldsymbol{\partial x}} & 0 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\boldsymbol{\partial_f} r_n}{\boldsymbol{\partial x}} & 0 & 0 \\ -\frac{\boldsymbol{\partial_f} w}{\boldsymbol{\partial x}} & 1 & 0 \\ \mathbf{0} & 0 & \omega(\mu)^{-1} \end{pmatrix}, C') = (B', A', C'),$$

where $C' = E_{n+1,1}(\frac{\partial_f w}{\partial x_1}) \cdots E_{n+1,n}(\frac{\partial_f w}{\partial x_n}) \begin{pmatrix} C \\ \mathbf{0} \end{pmatrix}$. □

Acknowledgments

The first author was supported by JSPS KAKENHI Grant Number 21K03217. The second author was supported by JSPS KAKENHI Grant Number 22K13917. The third author was supported by JSPS KAKENHI Grant Number 21K03233. The fourth author was supported by JSPS KAKENHI Grant Number 21J21482.

References

[1] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[2] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003) 3947–3989.

[3] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, (1970) 329–358.

[4] R. Fenn and C. Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992), no. 4, 343–406.

[5] R. H. Fox, *Free differential calculus. I. Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560.

[6] R. H. Fox, *Free differential calculus. II. The isomorphism problem of groups*, Ann. of Math. (2) **59** (1954), 196–210.

[7] A. Ishii and K. Oshiro, *Derivatives with Alexander pairs for quandles*, Fund. Math. **259** (2022), no. 1, 1–31.

[8] A. Ishii and K. Oshiro, *Quandle twisted Alexander invariants*, Osaka J. Math. **59** (2022), no. 3, 683–702.

[9] A. Ishii and K. Oshiro, *Normalized quandle twisted Alexander invariants*, to appear in Internat. J. Math.

[10] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg. **23** (1982) 37–65.

[11] T. Kitayama, *Normalization of twisted Alexander invariants*, Internat. J. Math. **26** (2015), 1550077, 21 pp.

[12] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), 361–380.

[13] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119** (**161**) (1982) 78–88.

[14] H. Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys. **19** (1908), no.1, 1–118.

[15] M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology **33** (1994), no. 2, 241–256.