

High-dimensional analysis of the piecewise deterministic Markov process for Bayesian inference

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In this talk, we review the recent results of piecewise deterministic Markov processes for Bayesian inference in a high-dimension context. Suppose we wish to sample from a probability distribution

$$\Pi(dx) = \exp(-\mathcal{U}(x))dx \quad (x \in \mathbb{R}^d)$$

where $\mathcal{U} : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuously differentiable function. For the Bayesian context, this probability distribution is the posterior distribution of interest. If we have an i.i.d. sample from Π , we can approximate Π -integral of any function $f(x)$ by the law of large numbers

$$\frac{1}{M} \sum_{m=1}^M f(X_m) \longrightarrow \int_{\mathbb{R}^d} f(x)\Pi(dx). \quad (0.1)$$

In most of the cases, direct i.i.d. sampling is impossible or computationally very expensive. For these cases, the Markov chain Monte Carlo method is useful which originated with the classic paper by Metropolis et al. (1953). The Markov chain Monte Carlo method is designed to construct an ergodic Markov kernel P which is Π -invariant. If a Markov chain X_1, X_2, \dots is generated from the Markov kernel P then the law of large numbers (0.1) is satisfied. The Markov chain Monte Carlo is now a gold standard for Bayesian inference.

Recently, its continuous process version, the Markov **process** Monte Carlo method is of substantial interest for Monte Carlo analysis. Known Markov process Monte Carlo methods rely on an auxiliary variable trick which uses an auxiliary variable v with a probability density ν on Ξ and considers the joint probability distribution

$$\mu := \Pi(dx) \otimes \nu(dv)$$

as an extended target distribution on $\mathcal{Z} = \mathbb{R}^d \times \Xi$. The original target distribution is a marginal distribution of the extended target distribution. Since Brownian motion does not have an absolutely continuous path, we can not simulate processes driven by Brownian motion exactly. For our Monte Carlo analysis, exact sampling is necessary. Therefore, the Markov processes of interest should not have a Brownian part. Known processes consist of a deterministic part and a pure jump part. These processes are known as the **piecewise deterministic Markov processes**.

Here we follow Azaïs et al. (2014) for the expression of the piecewise deterministic Markov processes. The processes are constructed by characteristics $(\phi, \lambda_k, Q_k : l = 1, \dots, K)$. The flow $\phi : \mathcal{Z} \times \mathbb{R} \rightarrow \mathcal{Z}$ is a one-parameter group of homeomorphism, that is, ϕ is continuous, $\phi(\cdot, t)$ is a homeomorphism for each $t \in \mathbb{R}$ and $\phi(\phi(\cdot, s), t) = \phi(\cdot, s + t)$. For each $k = 1, \dots, K$, the jump rate $\lambda_k : \mathcal{Z} \rightarrow \mathbb{R}_+$ determines the jump time of pure jump processes, and Q_k is a Markov kernel on \mathcal{Z} . Let

$$\Lambda_k(z, t) = \int_0^t \lambda_k(\phi(z, s))ds.$$

The Markov process is defined by the following way. Suppose $z(0) = (x(0), t(0)) \in \mathcal{Z}$. Let T_1, \dots, T_K be independent processes with

$$\mathbb{P}(T_k \geq t) = \exp(-\Lambda_k(z, t)).$$

Let $T_* = \min_{k=1, \dots, K} T_k$. If $T_k = T_*$, then Z is generated from $Q_k(\phi(z, T_*), \cdot)$ and set

$$X(t) = \begin{cases} \phi(z(0), t) & \text{for } t < T_* \\ Z & \text{for } t = T_*. \end{cases}$$

After T_* , the process evolves in the same way with starting value Z . There are several choices of characteristics. We introduce three piecewise deterministic Markov processes.

Two popular piecewise deterministic Markov processes use the same flow ϕ defined by

$$\begin{aligned} \frac{dx(t)}{dt} &= v(t) \\ \frac{dv(t)}{dt} &= 0. \end{aligned}$$

The **Zig-Zag sampler** proposed by Bierkens et al. (2018) uses d Markov kernels Q_1, \dots, Q_d with d jump rates $\lambda_1, \dots, \lambda_d$. For each $i = 1, \dots, d$, the Markov kernel is a deterministic kernel Q_i defined by a map

$$(x, v) \mapsto (x, F_i(v))$$

where F_i is an operator which flips the sign of the i -th coordinate of x . The jump rate is defined by

$$\lambda_i((x, v)) = \max\{0, \partial_i \mathcal{H}(x) v_i\}.$$

The **bouncy particle sampler** proposed by Peters and de With (2012), Bouchard-Côté et al. (2017) uses two Markov kernels Q_{bounce} and Q_{ref} with corresponding jump rates λ_{bounce} and λ_{ref} . The kernel Q_{bounce} is a deterministic kernel defined by a map

$$(x, v) \mapsto (x, \kappa(x, v))$$

where

$$\kappa(x, v) = v - 2 \frac{\langle \nabla \mathcal{H}(x), v \rangle}{\|\nabla \mathcal{H}(x)\|^2} \nabla \mathcal{H}(x)$$

and $\lambda_{\text{bounce}}(x, v) = \max\{0, \langle \nabla \mathcal{H}(x), v \rangle\}$. The jump rate λ_{ref} is a positive constant, and Q_{ref} is a μ -invariant Markov kernel. For our analysis, for simplicity, we assume $Q_{\text{ref}}((x, v), d(y, w)) = \nu(dw)$.

We review some recent results on asymptotic properties of the above deterministic Markov processes such as Bierkens et al. (2018), Andrieu et al. (2018), Deligiannidis et al. (2018).

References

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