18.784 Homework Set 7
Due Friday, March 19, 2010.

Part I (AF-AZ, 3/12/10)

1. Write elements of $MP_2(\mathbb{Z})$ as $(\binom{ab}{cd}, \pm\sqrt{cz+d})$. Show that the multiplication rule
$$(M_1, \phi_1)(M_2, \phi_2) = (M_1 M_2, \phi_1(M_2 z)\phi_2 z)$$
yields a group law.

2. Let M be an even positive definite integral lattice. Check that $\Theta_M(z+1) = \rho_M(T)\Theta_M(z)$, where $\Theta_M : \mathbb{H} \to \mathbb{C}[M^\vee/M]$ is defined by $\Theta_M(z) = \sum_{\gamma \in M^\vee/M} e_\gamma \theta_{M+\gamma}(z)$.

Part II (SK-WS, 3/15/10)

1. Show that the points of $\mathbb{H}/\Gamma_1(N)$ parametrize pairs (E, P), where E is a complex elliptic curve, and P is a point in E of order N.

2. Let m and N be relatively prime positive integers. Given two pairs (E, P) and (E', P'), where P and P' have order N in E and E', respectively, we define a degree m isogeny from (E, P) to (E', P') to be a degree m map $\phi : E \to E'$ such that $\phi(P) = P'$. If E is given by the lattice $\Lambda = \langle 1, z\rangle$ and P is given by the coset $z/N + \Lambda$, describe all of the possible (E', P') arising from isogenies of degree m from (E, P) for the case m is prime.

Part III (MD-BW, 3/17/10)

1. Let f be an elliptic function of order N with respect to a lattice $\Omega \subset \mathbb{C}$, i.e., a meromorphic function on $\mathbb{C}$ that is invariant under translation by elements of Ω, such that the equation $f(z) = \infty$ has N solutions in each fundamental parallelogram (counted with multiplicity). Let C be a contour around the boundary of a fundamental parallelogram (chosen so that the contour does not hit any zeroes or poles of f). Show that:
$$\frac{1}{2\pi i}\oint_C f(z)dz = 0$$
$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = 0$$

2. Let (x_1, y_1) and (x_2, y_2) be distinct points in the intersection of a line and the curve $y^2 = 4x^3 - g^2 x - g_3$. Find a formula for the third point of intersection as a rational function of x_1, x_2, y_1, and y_2.