The Total Absolute Curvature of Nonclosed Curves in S^2

Kazuyuki Enomoto

Tokyo University of Science

(Joint work with Jin–ichi Itoh)

Let $\Sigma : x(s)$ be a piecewise C^2 curve in the 2–dimensional unit sphere S^2, where s is the arclength parameter. Let $k(s)$ be the oriented geodesic curvature of Σ at $x(s)$. We define the *total absolute curvature* $\tau(\Sigma)$ of Σ by

$$\tau(\Sigma) = \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} |k(s)|\, ds + \sum_{i=1}^{n-1} |\theta_i|,$$

where θ_i is an oriented exterior angle at a nonsmooth point $x(s_i)$. There have been a lot of works done for the study of the total absolute curvature of closed curves, while not much has been studied for nonclosed curves. In this work, we study the total absolute curvature of nonclosed curves.

We first consider the following set of curves. Given points p, q in S^2 and unit tangent vectors X at p, Y at q, let $\mathcal{C}(p, X, q, Y)$ be the set of all piecewise C^2 curves in S^2 whose element $\Sigma : x(s)$ satisfies $x(0) = p$, $x'(0) = X$, $x(L) = q$, $x'(L) = Y$. Let Γ_X (Γ_Y resp.) be the oriented geodesic which is tangent to X at p (to Y at q resp.). Let r be the point where Γ_X first meets Γ_Y. Let $\varphi(X, Y)$ be the oriented angle from Γ_X to Γ_Y at r. Let pr be the oriented geodesic segment from p to r whose length is not greater than π.

Theorem 1. $\inf\{\tau(\Sigma) : \Sigma \in \mathcal{C}(p, X, q, Y)\} = |\varphi(X, Y)|.$
The equality holds if and only if $\Sigma = pr \cup rq$.

We next fix the length L of the curves as well, and define $\mathcal{C}(p, X, q, Y, L)$ as the set of curves in $\mathcal{C}(p, X, q, Y)$ whose length is L.

Theorem 2. Suppose that X and Y head into different hemispheres determined by a great circle passing through p and q and L satisfies $|pq| \leq L \leq |pr| + |rq|$. Then $\inf\{\tau(\Sigma) : \Sigma \in \mathcal{C}(p, X, q, Y, L)\}$ can be completely determined.

It is natural to ask what is $\inf\{\tau(\Sigma) : \Sigma \in \mathcal{C}(p, X, q, Y, L)\}$ in the general setting for p, q, X, Y and L. This includes, for example, the following questions:

What is $\inf\{\tau(\Sigma) : \Sigma \in \mathcal{C}(p, X, q, Y, L)\}$ if
(i) p, q, X, Y are as in Theorem 2 but $L > |pr| + |rq|$?
(ii) both X and Y head into the same side of a great circle passing through p and q ?
At the moment, we are not able to give a complete answer to these questions. The following theorem gives, however, a partial answer.

Theorem 3. $\inf\{\tau(\Sigma) : \Sigma \in \mathcal{C}(p, X, q, Y, L)\}$ is always attained by a curve which consists of small circular arcs and geodesic segments, which has possible singularities at endpoints.

This problem was considered for curves in the Euclidean plane in [**1**].

References

1. Enomoto, K.: The total absolute curvature of nonclosed plane curves of fixed length, *Yokohama Math. J.* **48** (2000), 83–96
2. Enomoto, K. and Itoh, J.: The total absolute curvature of nonclosed curves in S^2, Preprint