BUBBLETONS IN 3-DIMENSIONAL SPACE FORMS VIA THE DPW METHOD

SHIMPEI KOBAYASHI (KOBE UNIVERSITY GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY MASTER 2ND)

1. Introduction

We define three-dimensional space forms as the unique complete simply-connected 3-dimensional Riemann manifolds $\boldsymbol{R}^3, H^3$ and S^3, of constant sectional curvature 0, -1 and 1, respectively. A more concrete description of space forms is given in Section 2.

Bubbletons in $\boldsymbol{R}^3$ have been closely examined in [26], [17] and [24]. In this paper, we analyze bubbleton surfaces in all three space forms $\boldsymbol{R}^3$ and S^3 and H^3, using the DPW method. Bubbleton surfaces are CMC surfaces made from Bäcklund transformations (in Bianchi's sense) of round cylinders. The surface is shaped like a cylinder with attached bubbles, thus it is called a bubbleton. The parallel constant positive Gaussian curvature surface of the bubbleton is well known. It was first found by Sievert [23], thus it is called the Sievert surface.

With respect to the DPW method, the Bäcklund transformation is a dressing action on loop groups and this dressing action is described by elements of the simplest possible type like those of Terng and Uhlenbeck [25]. Using these elements

> we find an explicit immersion formula and solve the period problems for bubbletons in $\boldsymbol{R}^3$ and S^3 and H^3.

More generally, we can do the Bäcklund transformation for any surfaces. So

> we can solve period problems for the Bäcklund transformations of general Delaunay surfaces, which we do here.

In the $\boldsymbol{R}^3$ case, this is also done in [26], [17].

The DPW method was created by Dorfmeister and Pedit and Wu (see [8]) for making CMC surfaces in $\boldsymbol{R}^3$. The DPW method uses loop group theory involving the loop groups $\Lambda SL(2,\boldsymbol{C})$, $\Lambda SU(2)$ and $\Lambda_+ SL(2,\boldsymbol{C})$ to be defined later and is related to the methods of integrable systems. The DPW method also (equivalently) makes extended frames corresponding to harmonic maps from Riemann surfaces to the unit sphere S^2. Using holomorphic 1-forms, the DPW method constructs holomorphic maps to $\Lambda SL(2,\boldsymbol{C})$ and after that constructs extended frames corresponding to harmonic maps. More concretely, one first gives $\Lambda sl(2,\boldsymbol{C})$-matrix-valued holomorphic 1-forms called holomorphic potentials. Next one solves a linear first-order (homogeneous) ordinary differential equation whose coefficient is the above holomorphic potential. The solution of this equation is in $\Lambda SL(2,\boldsymbol{C})$ when the initial condition is chosen in $\Lambda SL(2,\boldsymbol{C})$. We then decompose $\Lambda SL(2,\boldsymbol{C})$ to $\Lambda SU(2) \times \Lambda SL_+(2,\boldsymbol{C})$ via Iwasawa splitting, producing an $\Lambda SU(2)$ element from an $\Lambda SL(2,C)$ element. This element in $\Lambda SU(2)$ is an extended frame of a CMC surface. Finally, the Sym-Bobenko formula produces the CMC immersion. The advantages of this DPW approach are that we can deal with the asymptotic behaviors and period problems for CMC surfaces.

2. Lax pairs for CMC surfaces in space forms

The arguments in this section are similar to arguments in [1] and [19].

2.1. **The space forms.** S^3, resp. H^3, is the unique complete simply connected 3-dimensional Riemannian manifold with constant sectional curvature $+1$, resp. -1.

There are a variety of models for describing S^3 and H^3. S^3 is the unit 3-sphere in $\boldsymbol{R}^4$ with the metric induced by $\boldsymbol{R}^4$, but for viewing graphics of CMC surfaces in S^3, we shall stereographically project S^3 from its north pole to the space $\boldsymbol{R}^3 \cup \{\infty\}$. For H^3 we shall use the Lorentz model:

$$H^3 = \{(t,x,y,z) \in R^{3,1} \,|\, x^2+y^2+z^2-t^2 = -1\,,\, t>0\}$$

with the metric induced by $\boldsymbol{R}^{3,1}$, where $\boldsymbol{R}^{3,1}$ is the 4-dimensional Lorentz space

$$\{(t,x,y,z) \,|\, t,x,y,z \in \boldsymbol{R}\}$$

with the Lorentz metric

$$\langle (t_1,x_1,y_1,z_1),(t_2,x_2,y_2,z_2)\rangle = x_1x_2+y_1y_2+z_1z_2-t_1t_2\ .$$

This metric is not positive definite, but its restriction to the tangent space of H^3 is positive definite. For viewing graphics of CMC surfaces in H^3, we shall use the Poincare model for H^3, which is stereographic projection of the Minkowski model in Lorentz space from the point $(0,0,0,-1)$ to the 3-ball $\{(0,x,y,z) \in R^{3,1} \,|\, x^2+y^2+z^2<1\} \cong \{p=(x,y,z) \in \boldsymbol{R}^3 \,|\, |p|<1\}$.

2.2. **Surfaces in the space forms.** Before we describe the DPW representation for CMC surfaces in Section 3, we show here that CMC surfaces in 3-dimensional space forms are locally equivalent to solutions of a certain kind of Lax pair. Then proving that the DPW recipe gives all CMC surfaces means showing that it gives all possible solutions for this certain kind of Lax pair.

Let M be a Riemann surface and let $f: M \to \mathcal{M}^3$ be a CMC conformal immersion, where $\mathcal{M}^3$ is either $\boldsymbol{R}^3$ or S^3 or H^3. Let Σ be a simply-connected domain in M with conformal coordinate $z=x+iy$ defined on Σ. We can consider the restriction $f|_\Sigma$ of f to Σ, i.e.

$$f = f(z,\bar z): \Sigma \to \mathcal{M}^3 = \boldsymbol{R}^3 \text{ or } S^3 \text{ or } H^3\ .$$

We write f as a function of both z and $\bar z$ to emphasize that f is not holomorphic in z.

Each of the three space forms lies isometrically in a vector space V: V is just $\boldsymbol{R}^3$ in the case $\mathcal{M}^3=\boldsymbol{R}^3$; $V=\boldsymbol{R}^4$ in the case $\mathcal{M}^3=S^3$; and $V=\boldsymbol{R}^{3,1}$ in the case $\mathcal{M}^3=H^3$. Let $\langle\cdot,\cdot\rangle$ be the inner product associated to V, which is the Euclidean inner product in the first two cases, and the Lorentz inner product in the third case. We may also view f as a map into V, i.e.

$$f: \Sigma \to \mathcal{M}^3 \subseteq V = \boldsymbol{R}^3 \text{ or } \boldsymbol{R}^4 \text{ or } \boldsymbol{R}^{3,1}\ .$$

The derivatives $f_x = \partial_x f$ and $f_y=\partial_y f$ are vectors in the tangent space $T_{f(z,\bar z)}V$ of V at $f(z,\bar z)$. Because V is a vector space, V naturally corresponds to $T_{f(z,\bar z)}V$, so f_x and f_y can be viewed as lieing in V itself. So $f_z = (1/2)(f_x - if_y)$ and

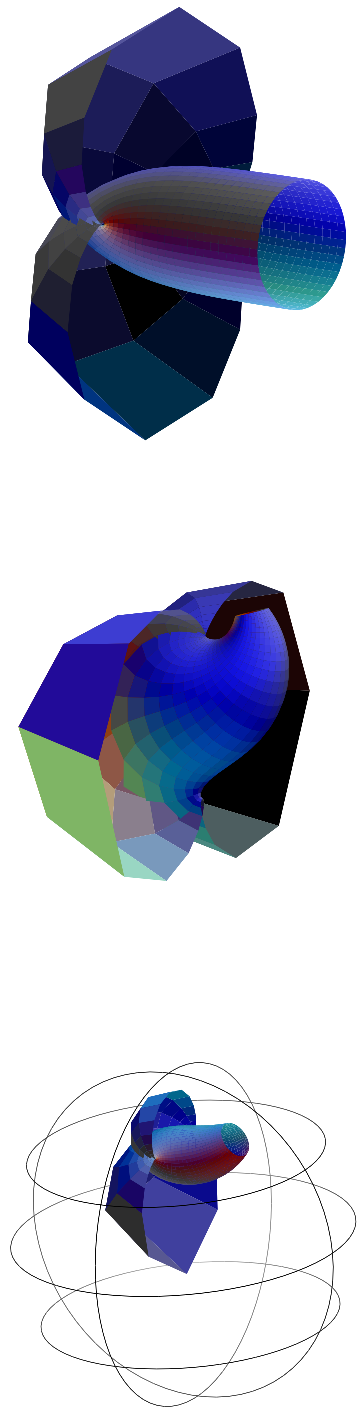

FIGURE 1. CMC bubbletons in R^3, S^3 and H^3. The R^3 bubbleton was first described in [26].

$f_{\bar z} = (1/2)(f_x + i f_y)$ are defined in the complex extension $V_{\boldsymbol{C}} = \{c \cdot v \mid c \in \boldsymbol{C}, v \in V\}$ of V with inner product extended to $\langle c_1 v_1, c_2 v_2 \rangle = c_1 c_2 \langle v_1, v_2 \rangle$ (which we also denote by $\langle \cdot, \cdot \rangle$ and is not a true inner product on $V_{\boldsymbol{C}}$). Since f is conformal, we have

$$\langle f_z, f_z \rangle = \langle f_{\bar z}, f_{\bar z} \rangle = 0 \ , \ \ \langle f_z, f_{\bar z} \rangle = 2e^{2u} \ ,$$

where the right-most equation defines the function $u : \Sigma \to \boldsymbol{R}$.

There is a natural notion of a unit normal vector $N = N(z, \bar z) \in T_{f(z,\bar z)} V \equiv V$ of f, defined by the properties

(1) $\langle N, N \rangle = 1$,
(2) $N \in T_{f(z,\bar z)} \mathcal{M}^3$, and
(3) $\langle N, f_z \rangle = \langle N, f_{\bar z} \rangle = 0$.

In each space form, the mean curvature of f is given by

$$H = \frac{1}{2e^{2u}} \langle f_{z\bar z}, N \rangle \ , \tag{2.1}$$

which is constant, by assumption. We also define the Hopf differential to be

$$Q = \langle f_{zz}, N \rangle \ .$$

Because f exists as a surface in $\mathcal{M}^3$, u and H and Q satisfy the Gauss and Codazzi equations for $\mathcal{M}^3$. For H constant, we will see that the Gauss and Codazzi equations for $\mathcal{M}^3$ remain satisfied when Q is replaced by $\lambda^{-2} Q$ for any $\lambda \in S^1 = \{p \in \boldsymbol{C} \mid |p| = 1\}$. Hence, up to rigid motions, there is a unique surface f_λ with metric determined by u and with mean curvature H and Hopf differential $\lambda^{-2} Q$. (We use the notation f_λ to state that f depends on λ; it does *not* denote the derivative $\partial_\lambda f$.) The surfaces f_λ for $\lambda \in S^1$ form a one-parameter family called the *associate family* of f. The parameter λ is called the *spectral parameter* and is essential to the DPW method.

We remark that in the cases of S^3 and H^3, we will actually be choosing Q so that it differs from the true Hopf differential by a particular constant factor.

2.3. **The vector spaces V in terms of quaternions.** Define the matrices

$$\sigma_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \ , \ \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

We can think of $\mathcal{Q} = \mathrm{span}_{\boldsymbol{R}} \{ i\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3 \}$ as the quaternions because it has the quaternionic algebraic structure.

2.3.1. When $\mathcal{M}^3 = V = \boldsymbol{R}^3$, we associate $\mathcal{M}^3$ with the imaginary quaternions $\mathcal{Q}_{Im} = \mathrm{span}_{\boldsymbol{R}} \{ -i\sigma_1, -i\sigma_2, -i\sigma_3 \} \subseteq \mathcal{Q}$ by the map

$$(x_1, x_2, x_3) \to x_1 \frac{i}{2} \sigma_1 + x_2 \frac{i}{2} \sigma_2 + x_3 \frac{i}{2} \sigma_3 \ .$$

Then for $X, Y \in \mathcal{Q}_{Im}$, the inner product inherited from $\boldsymbol{R}^3$ is

$$\langle X, Y \rangle = -2 \cdot \mathrm{trace}(XY) = +2 \cdot \mathrm{trace}(XY^*) \ , \tag{2.2}$$

where $Y^* := \bar{Y}^t$. Also, any oriented orthonormal basis $\{X, Y, Z\}$ of vectors of $\mathcal{M}^3 \equiv \mathcal{Q}_{Im}$ satisfies

$$X = F \left(\frac{i}{2} \sigma_1 \right) F^{-1} \ , \ \ Y = F \left(\frac{i}{2} \sigma_2 \right) F^{-1} \ , \ \ Z = F \left(\frac{i}{2} \sigma_3 \right) F^{-1} \tag{2.3}$$

for some $F \in \mathrm{SU}(2)$, and this F is unique up to sign. In other words, rotations of $\boldsymbol{R}^3$ fixing the origin are represented in the quaternionic representation $\mathcal{Q}_{Im}$ of $\boldsymbol{R}^3$ by matrices $F \in \mathrm{SU}(2)$.

2.3.2. When $\mathcal{M}^3 = S^3$ and $V = \boldsymbol{R}^4$, we associate V with $\mathcal{Q}$ by the map

$$(x_1, x_2, x_3, x_4) \to x_1 i\sigma_0 + x_2 i\sigma_1 + x_3 i\sigma_2 + x_4 i\sigma_3 \ ,$$

so points $(x_1, x_2, x_3, x_4) \in V = \boldsymbol{R}^4$ are matrices of the form

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ , \tag{2.4}$$

where $a = x_1 + ix_4$ and $b = x_3 + ix_2$. That is, they are matrices X that satisfy

$$X = \sigma_2 \bar{X} \sigma_2 \ . \tag{2.5}$$

The inner product on $\mathcal{Q}$ inherited from V is

$$\langle X, Y \rangle = (1/2) \cdot \mathrm{trace}(XY^*) \ , \tag{2.6}$$

where $Y^* := \bar{Y}^t$. Note that this inner product is the same as in (2.2), up to a factor of 4.

2.3.3. When $\mathcal{M}^3 = H^3$ and $V = \boldsymbol{R}^{3,1}$, we can associate V with the set of self-adjoint matrices $\{X \in M_{2\times 2} \,|\, X^* = X\}$ by the map

$$(x_0, x_1, x_2, x_3) \in R^{3,1} \to X = x_0 i\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \ .$$

One can check that $\sigma_2 X^t \sigma_2 = X^{-1} \det X$ and the inner product inherited from V is

$$\langle X, Y \rangle = (-1/2)\mathrm{trace}(X\sigma_2 Y^t \sigma_2) \ ,$$

so

$$\langle X, X \rangle = -\det X \ ,$$

for self-adjoint matrices X, Y.

2.4. The case $\mathcal{M}^3 = \boldsymbol{R}^3$.

Theorem 2.1. *Let u and Q solve*

$$4u_{z\bar{z}} - Q\bar{Q}e^{-2u} + e^{2u} = 0 \ , \quad Q_{\bar{z}} = 0 \ , \tag{2.7}$$

and let $F(z, \bar{z}, \lambda)$ be a solution, which is in $\mathrm{SU}(2)$ for all $\lambda \in S^1$ and is complex analytic in λ, of the system

$$F_z = FU \ , \quad F_{\bar{z}} = FV \tag{2.8}$$

with

$$U = \frac{1}{2}\begin{pmatrix} u_z & -e^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix} \ , \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u}\lambda \\ e^u\lambda & u_{\bar{z}} \end{pmatrix} \ . \tag{2.9}$$

Define

$$f = \left[\frac{-1}{2} F \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} F^{-1} - i\lambda(\partial_\lambda F) \cdot F^{-1} \right] \Bigg|_{\lambda=1} \ . \tag{2.10}$$

Then f is of the form

$$\frac{-i}{2} \begin{pmatrix} -t & r + is \\ r - is & t \end{pmatrix} \ , \tag{2.11}$$

for reals r, s, t, and

$$(r, s, t)$$

is a conformal parametrization of a CMC $1/2$ surface in $\boldsymbol{R}^3$, parametrized by z. Furthermore, every CMC $1/2$ conformal immersion in $\boldsymbol{R}^3$ can be attained this way.

2.5. The case $\mathcal{M}^3 = S^3$.

Theorem 2.2. *Let u and Q solve (2.7) and let $F_j(z, \bar{z}, \lambda = e^{-i\gamma_j})$, $j = 1, 2$, be two solutions of the system (2.8)-(2.9) such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and $F(z, \bar{z}, \lambda)$ is complex analytic in λ. Define*

$$f = F_1 \begin{pmatrix} \sqrt{e^{i(\gamma_2-\gamma_1)}} & 0 \\ 0 & \sqrt{e^{i(\gamma_1-\gamma_2)}} \end{pmatrix} F_2^{-1} \, . \tag{2.12}$$

Then f is a conformal immersion with CMC $H = \cot(\gamma_1 - \gamma_2)$ into S^3. Conversely, every conformal immersion with CMC $H = \cot(\gamma_1 - \gamma_2)$ into S^3 can be attained this way.

2.6. The case $\mathcal{M}^3 = H^3$, with $H > 1$.

Theorem 2.3. *Let u and Q solve (2.7) and let $F(z, \bar{z}, \lambda = e^{-q/2}e^{-i\psi})$ for some real q be a solution of the system (2.8)-(2.9) such that $F \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and F is complex analytic in λ. Then*

$$f = F \begin{pmatrix} 0 & -ie^{-q/2} \\ ie^{q/2} & 0 \end{pmatrix} \overline{F^{-1}} \sigma_2 \tag{2.13}$$

is a CMC $H = \coth q$ conformal immersion into H^3. Conversely, all CMC $H = \coth q$ conformal immersions into H^3 can be attained this way.

3. The DPW recipe

We saw in Section 2 that finding CMC $H \neq 0$ surfaces in $\boldsymbol{R}^3$ and CMC H surfaces in S^3 and CMC $H > 1$ surfaces in H^3 is equivalent to finding integrable Lax pairs of the form (2.8)-(2.9) and their solutions F. Then the surfaces are found by using the Sym-Bobenko type formulas (2.10) and (2.12) and (2.13). So to prove that the DPW recipe finds all of these types of surfaces, it is sufficient to prove that the DPW recipe produces all integrable Lax pairs of the form (2.8)-(2.9) and all their solutions F. The goal of this section is to show how this is done in [8].

3.1. The loop groups.

Let C_r be the circle of radius $r \leq 1$ centered at the origin in $\boldsymbol{C}$.

Definition 1. For any $r \in (0, 1] \subset \boldsymbol{R}$, we define the following loop groups:

(1) The twisted $sl(2, \boldsymbol{C})$ r-loop algebra is

$$\Lambda_r sl(2, \boldsymbol{C}) = \{A : C_r \rightarrow^{C^\infty} sl(2, \boldsymbol{C}) \mid A(-\lambda) = \sigma_3 A(\lambda) \sigma_3 \ \} \, .$$

(The condition $A(-\lambda) = \sigma_3 A(\lambda)\sigma_3$ is why we call the loop group "twisted".)

(2) The twisted $\mathrm{SL}(2, \boldsymbol{C})$ r-loop group is

$$\Lambda_r \, \mathrm{SL}(2, \boldsymbol{C}) = \{\phi : C_r \rightarrow^{C^\infty} \mathrm{SL}(2, \boldsymbol{C}) \mid \phi(-\lambda) = \sigma_3 \phi(\lambda) \sigma_3 \ \} \, .$$

(3) The twisted SU(2) r-loop group is

$$\Lambda_r \mathrm{SU}(2) = \{F \in \Lambda_r \mathrm{SL}(2,\boldsymbol{C}) \mid F(1/\bar{\lambda})^* = (F(\lambda))^{-1} \ ,$$

$$F = F(\lambda) \text{ extends holomorphically to } \lambda \text{ for } r < |\lambda| < r^{-1}$$

$$\text{and continuously for } r \leq |\lambda| \leq r^{-1}\}$$

$$\cong \{F : C_r \to^{C^\infty} \mathrm{SL}(2,\boldsymbol{C}) \mid F = F(\lambda) \text{ extends holomorphically to}$$

$$\lambda \text{ for } r < |\lambda| \leq 1 \text{ and continously for } r \leq |\lambda| \leq 1 \text{ and } F|_{C_1} \in \mathrm{SU}(2)\ \} \ .$$

When $r = 1$, we may abbreviate $\Lambda_1 \mathrm{SU}(2)$ to $\Lambda \mathrm{SU}(2)$. The condition in $\Lambda \mathrm{SU}(2)$ that F extends holomorphically is vacuous.

(4) The twisted plus r-loop group with $\boldsymbol{R}^+$ constant terms is

$$\Lambda_{+,r,\boldsymbol{R}^+} SL(2,\boldsymbol{C}) = \{B \in \Lambda_r \mathrm{SL}(2,\boldsymbol{C}) \mid B \text{ extends holomorphically to } \lambda \text{ for } |\lambda| < r$$

$$\text{and continuously for } |\lambda| \leq r \ , \text{ and } B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ with } \rho > 0\ \} \ .$$

When $r = 1$, we may abbreviate $\Lambda_{+,1,\boldsymbol{R}^+} SL(2,\boldsymbol{C})$ to $\Lambda_+ SL(2,\boldsymbol{C})$.

(5) The twisted plus r-loop group with general constant terms is

$$\Lambda_{+,r} SL(2,\boldsymbol{C}) = \{B \in \Lambda_r \mathrm{SL}(2,\boldsymbol{C}) \mid B \text{ extends holomorphically to } \lambda \text{ for}$$

$$|\lambda| < r \text{ and continously for } |\lambda| \leq r\ \} \ .$$

(6) The twisted minus r-loop group with id constant terms is

$$\Lambda_{-,r,*} SL(2,\boldsymbol{C}) = \{B \in \Lambda_r \mathrm{SL}(2,\boldsymbol{C}) \mid B \text{ extends holomorphically to } \lambda \text{ for}$$

$$|\lambda| > r \text{ and continously for } |\lambda| \geq r\,, \text{ and } B|_{\lambda=\infty} = \mathrm{id}\ \} \ .$$

3.2. **Iwasawa and Birkhoff splittings.** It is irrelevant how we topologize the loop algebra and loop groups, as long as the smooth loops are contained in the topology, since we will always be staying in the smooth category. However, to state the next two splitting lemmas, we must choose a topology. Let us choose the topology determined by the H^α norm for some $\alpha > 1/2$ (see [20]). With respect to this norm, all of the above smooth loops will have finite norm. (Loops with poles will probably not have finite norm.) We can then extend the above loop groups $\Lambda_r \mathrm{SL}(2,\boldsymbol{C})$, $\Lambda_r \mathrm{SU}(2)$, $\Lambda_{+,r,\boldsymbol{R}^+} SL(2,\boldsymbol{C})$, $\Lambda_{+,r} SL(2,\boldsymbol{C})$ and $\Lambda_{-,r,*} SL(2,\boldsymbol{C})$ to their completions with respect to the H^α norm. Then the notion of diffeomorphisms between these loops groups, and also the notion of smooth (resp. real-analytic, complex-analytic) dependence of the following splittings on z, makes sense.

Lemma 3.1. *(*Iwasawa decomposition*) For any $r \in (0,1]$, we have the following real-analytic diffeomorphism globally defined from $\Lambda_r \mathrm{SL}(2,\boldsymbol{C})$ to $\Lambda_r \mathrm{SU}(2) \times \Lambda_{r,+,\boldsymbol{R}^+} \mathrm{SL}(2,\boldsymbol{C})$: For any $\phi \in \Lambda_r SL(2,\boldsymbol{C})$, there exist unique $F \in \Lambda_r \mathrm{SU}(2)$ and $B \in \Lambda_{+,r} SL(2,\boldsymbol{C})$ so that*

$$\phi = FB \ .$$

We call this r-Iwasawa splitting of ϕ. We $r = 1$, we may call it simply Iwasawa splitting. Because the diffeomorphism is real-analytic, we know that if ϕ depends real-analytically (resp. smoothly) on some parameter z, then F and B do as well.

From now on, whenever we apply these splitting results, it is sufficient to simply check that the loops we are splitting are smooth.

3.3. **The DPW method.** We now describe the DPW method. Let

$$\xi = A(z,\lambda)dz\ ,\quad A(z,\lambda) \in \Lambda sl(2,\boldsymbol{C})\ , \tag{3.1}$$

where $A := A(z,\lambda)$ is holomorphic in both z and λ for $z \in \Sigma$ and $\lambda \in \boldsymbol{C} \setminus \{0\}$. Furthermore, we assume the following:

$$\begin{pmatrix} A \text{ has a pole of order at most 1 at } \lambda = 0, \\ \text{and the upper-right entry of } A \text{ really does have a pole at } \lambda = 0. \end{pmatrix} \tag{3.2}$$

We call ξ a *holomorphic potential*.

In practice, when we wish to make specific examples of CMC surfaces, we will write A in the form

$$A = A_{-1}(z)\lambda^{-1} + A_0(z) + A_1(z)\lambda + A_2(z)\lambda^2 + \dots\ ,$$

where the $A_j = A_j(z) \in M_{2\times 2}$ are holomorphic in $z \in \Sigma$ and do not depend on λ. By (3.2), we must choose A_{-1} so that its upper-right entry is never zero on Σ. Because $A \in \Lambda sl(2,\boldsymbol{C})$, A_j is off-diagonal (resp. diagonal) when j is odd (resp. even). Furthermore, all A_j are traceless. In fact, in all the example we later consider, only finitely many of the A_j will be nonzero.

Let ϕ be the solution to

$$d\phi = \phi\xi\ ,\qquad \phi(z_*) = \mathrm{id}$$

for some base point $z_* \in \Sigma$. Then ϕ is holomorphic and

$$\phi \in \Lambda SL(2,\boldsymbol{C})\ .$$

By Lemma 3.1 above, we can perform r-Iwasawa splitting, and write the result as

$$\phi = FB\ .$$

Proposition 3.2. *Up to a conformal change of the coordinate z, F is a solution to a Lax pair of the form* (2.8)-(2.9)*.*

3.4. **The meaning of dressing and gauging.** Given a solution ϕ to $d\phi = \phi\xi$, if we define

$$\hat{\phi} = h_+(\lambda)\cdot\phi\cdot p_+(z,\bar{z},\lambda)\ ,\qquad h_+, p_+ \in \Lambda_+\,\mathrm{SL}(2,\boldsymbol{C})\ ,$$

then the multiplication on the left by h_+ is a dressing, and the multiplication on the right by p_+ is a gauging. The matrix h_+ cannot depend on z. The matrix p_+ can depend on z, but must have trivial monodromy about all loops in the z-domain.

Note that $\hat{\phi}$ satisfies $d\hat{\phi} = \hat{\phi}\hat{\xi}$, where

$$\hat{\xi} = p_+^{-1}\xi p_+ + p_+^{-1}dp_+\ .$$

Hence,

> the dressing h_+ does not change the potential ξ, and changes only the resulting surface.

Furthermore, if we look at the Iwasawa splittings $\phi = FB$ and $\hat{\phi} = \hat{F}\hat{B}$, then the change $F \to \hat{F}$ is affected only by h_+, and is independent of p_+, hence

> the gauging p_+ does not change the surface, and changes only the potential ξ.

To see how the surface is changed by h_+, one must Iwasawa split h_+F into $h_+F = \tilde{F}\tilde{B}$, and then $\hat{F}$ equals $\tilde{F}$, so the change in the frame is not trivial to understand, hence the change in the surface is also not trivial to understand.

However, it is easier to understand how the monodromy matrices of ϕ and $\hat{\phi}$ are related by h_+, and this is often just the information we need, because we are interested in getting the monodromy matrices into SU(2) so we can solve period problems. Define M and $\hat{M}$ by

$$\phi \to M\phi \,, \qquad \hat{\phi} \to \hat{M}\hat{\phi}$$

as one travels about some loop in the z-domain. Then it is simple (i.e. Iwasawa splitting is not required) to check that

$$\hat{M} = h_+ M h_+^{-1} \,.$$

3.5. **Period problems in S^3 and H^3.** In the case of $\boldsymbol{R}^3$, we have a six real dimensional period problem for each homology class of loops, as in [15]. If M is defined so that $\phi \to M \cdot \phi$ about a loop on the Riemann surface (with local coordinate z), then the Sym-Bobenko formula implies that the immersion changes as

$$f \to \left[M f M^{-1} - i\lambda(\partial_\lambda M) M^{-1} \right]_{\lambda=1}$$

as one travels about the loop. M is independent of z, but not of λ. Supposing that we already know $M \in SU(2)$ for all $\lambda \in S^1$, then for the surface to be well defined about the loop we need to know that

$$M|_{\lambda=1} = \pm id \quad \text{and} \quad \partial_\lambda M|_{\lambda=1} = 0 \,, \tag{3.3}$$

that is, we need to get $(M, \partial_\lambda M)|_{\lambda=1}$ to be the identity element (up to sign) in $SU(2) \times su(2)$. Since the dimension of the space $SU(2) \times su(2)$ is six, the period problem is six dimensional.

For the cases of S^3 and H^3, we check here that again the period problem is six dimensional.

H^3 **case,** $H > 1$**:** About a loop we have $F \to M \cdot F$, and assume that we already know M is unitary on S^1 (i.e. $(M(\lambda)^*)^{-1} = M(1/\bar{\lambda})$ for all $\lambda \in \boldsymbol{C} \setminus \{0\}$). About the loop, the immersion changes as

$$f \to \left[M f \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \overline{M}^{-1} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right]_{\lambda = e^{q/2}} \,.$$

So for the surface to be well defined about the loop, we need

$$M|_{\lambda=e^{q/2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm id \,, \tag{3.4}$$

that is, we need the identity element (up to sign) in $SL(2, \boldsymbol{C})$. (Note that even though M is unitary on S^1, we can only consider the problem in $SL(2, \boldsymbol{C})$, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C}) \,,$$

because $\lambda = e^{q/2}$ does not lie on S^1.) Since $SL(2, \boldsymbol{C})$ is six dimensional, so is the period problem.

S^3 **case:** Again assume M_1, M_2, defined by $F_j \to M_j \cdot F_j$ as we travel about the loop, are unitary on S^1. The pair $(\lambda_1, \lambda_2) = (1, e^{2i\psi})$ implies

$$f = F_1 \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} F_2^{-1} .$$

So when we travel about the loop, we have

$$f \to M_1 f M_2^{-1} .$$

To close the surface about this loop, we need

$$M_1 = M_2 = \pm \mathrm{id} \ . \tag{3.5}$$

Note that M_j are in SU(2), since $|\lambda_j| = 1$. So we need the identity element (up to sign) in $SU(2) \times SU(2)$. As $SU(2) \times SU(2)$ is six dimensional, so is the period problem.

4. Surfaces of Revolution

4.1. **Cylinders via DPW.** Define

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \frac{dz}{z} ,$$

for the complex variable $z \in \boldsymbol{C}$ and $\lambda \in S^1$ and $a \in \boldsymbol{R}$.

When $\mathcal{M}^3 = \boldsymbol{R}^3$, we choose $a = 1/4$. When $\mathcal{M}^3 = H^3$, we choose $\lambda = e^{q/2}$ for $q \in \boldsymbol{R}^+$ and $a = 1/(4\cosh(q/2))$, so $\lambda > 1$ and the resulting surface has mean curvature $H = \coth q > 1$. When $\mathcal{M}^3 = S^3$, we choose $\lambda_1 = e^{i\gamma}$ and $\lambda_2 = e^{-i\gamma}$ for $\gamma \in (0, \pi/4]$ and $a = 1/(4\cos\gamma)$, so the resulting surface has mean curvature $H = \cot(2\gamma)$.

In each of these three space forms, the conditions (3.3), (3.4) and (3.5) are satisfied, respectively. Hence in all three cases we have produced surfaces that are homeomorphically cylinders.

4.2. **Delaunay surfaces via DPW.** Delaunay surfaces via DPW in $\boldsymbol{R}^3$ are described in detail in [17].

Define

$$\xi = D\frac{dz}{z} , \qquad \text{where } D = \begin{pmatrix} r & s\lambda^{-1} + t\lambda \\ s\lambda + t\lambda^{-1} & -r \end{pmatrix} ,$$

with $r, s, t \in \boldsymbol{R}$.

- When $\mathcal{M}^3 = \boldsymbol{R}^3$, (3.3) is satisfied if
$$r^2 + (s+t)^2 = 1/4 ,$$
so we impose this condition when $\mathcal{M}^3 = \boldsymbol{R}^3$.
- When $\mathcal{M}^3 = H^3$, (3.4) is satisfied if
$$r^2 + (s+t)^2 + 4st\sinh^2(\frac{q}{2}) = 1/4 ,$$
so we impose this when $\mathcal{M}^3 = H^3$.
- When $\mathcal{M}^3 = S^3$, (3.5) is satisfied if
$$r^2 + (s+t)^2 - 4st\sin^2(\gamma) = 1/4 ,$$
so we impose this when $\mathcal{M}^3 = S^3$.

5. Bubbletons

5.1. **Bubbletons via DPW.** Let $\mathcal{R}$ be the Riemann surface $S^2\backslash\{p_1,p_2\}$ with the standard holomorphic structure. Using stereographic projection, we can denote $\mathcal{R}=\boldsymbol{C}\cup\{\infty\}\backslash\{p_1,p_2\}$. And using a Moebius transformation, we can transform $\mathcal{R}$ to $\boldsymbol{C}^*=\boldsymbol{C}\backslash\{0\}$. Stereographic projection and Moebius transformations preserve the holomorphic structure of the Riemann surface. So we need only consider $\mathcal{R}=\boldsymbol{C}^*=\boldsymbol{C}\backslash\{0\}$.

Let $\phi(z,\lambda)$ be a solution of $d\phi=\phi\xi$ with some initial condition $\phi(z_*,\lambda)$ at $z=z_*$ and let $\phi=F\cdot B$ be the r-Iwasawa splitting of ϕ, where $\xi=A(z,\lambda)dz$ and $A(z,\lambda)\in\Lambda_r sl(2,\boldsymbol{C})$ for some $r\in(0,1]$. Let f be as in the Sym-Bobenko formula (2.10) or (2.12) or (2.13), respectively, made from the extended frame F. We assume that the monodromy M_ϕ of ϕ (associated to a counterclockwise loop around $z=0$) is in $\Lambda_r SU(2)$ and M_ϕ satisfies one of the closing conditions (3.3) or (3.4) or (3.5), respectively. Thus f is well-defined on $\mathcal{R}$.

Consider the dressing $\phi\to\tilde{\phi}:=h\cdot\phi$, where h is the matrix

$$h=\begin{pmatrix}\sqrt{\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2}} & 0\\ 0 & \sqrt{\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2}}\end{pmatrix},\ \alpha\in\boldsymbol{C}^*.$$

Let $\tilde{\phi}=\tilde{F}\cdot\tilde{B}$ be the r-Iwasawa splitting of $\tilde{\phi}$ and let $\tilde{f}$ be the Sym-Bobenko formula (2.10) or (2.12) or (2.13), respectively, made from the extended frame $\tilde{F}$. Note that if $|\alpha|<r$ or $r^{-1}<|\alpha|$, then $h\in\Lambda_r SU(2)$. So the surface $\tilde{f}$ differs from f by only a rigid motion. Therefore we assume $r<|\alpha|<1$.

Lemma 5.1. *If $hM_\phi h^{-1}$ in $\Lambda_r SU(2)$, then $\tilde{F}$ changes to $(hM_\phi h^{-1})\cdot\tilde{F}$ when one travels a counterclockwise loop around $z=0$. Hence the monodromy of $\tilde{F}$ is $hM_\phi h^{-1}$.*

Noting the previous lemma, we define the bubbleton surfaces.

Definition 2. Let $f,\tilde{f}:\mathcal{R}\longrightarrow\boldsymbol{R}^3$ or H^3 or S^3 be CMC immersions derived from the above solutions ϕ and $\tilde{\phi}$. Then $\tilde{f}$ is a bubbleton surface of f if $hM_\phi h^{-1}\in\Lambda_r SU(2)$.

Lemma 5.2. *The bubbleton $\tilde{f}$ satisfies the closing condition: that is, it is well-defined on $\mathcal{R}$.*

Theorem 5.3. *There exist cylinder bubbleton and Delaunay bubbleton surfaces for all three space forms.*

5.2. **Computing the change of frame for the simple type dressing.** Now we do the story of the Bäcklund transformation in the sense of Terng and Uhlenbeck (see [25]). This will lead to explicit parametrization of the cylinder bubbletons in all three space forms.

Let ϕ be a solution of $d\phi=\phi\xi$ with the some initial condition $\phi(z_*,\lambda)$ and let $\phi=F\cdot B$ be the r-Iwasaswa splitting. In this section, the situation and the assumptions are the same as in Section 5. We consider $\boldsymbol{C}^2$ with inner product $\langle,\rangle$ and e_1,e_2 forming the orthonormal basis

$$e_1=\begin{pmatrix}1\\0\end{pmatrix},\qquad e_2=\begin{pmatrix}0\\1\end{pmatrix}$$

of $\boldsymbol{C}^2$. We define two subspace V_1, V_2 spaned by v_1, v_2.:

$$V_1 := \{a \cdot v_1 | v_1 = \begin{pmatrix} \bar{A} \\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B} \end{pmatrix}, a \in \boldsymbol{C}\}, \qquad V_2 := \{a \cdot v_2 | v_2 = \begin{pmatrix} -\lambda\alpha^{-1}B \\ A \end{pmatrix}, a \in \boldsymbol{C}\}$$

where

$$F|_{\lambda=\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We define projections $\pi_1, \pi_2, \tilde{\pi}_1, \tilde{\pi}_2$ and linear combinations $h, \tilde{h}$ of these projections.

$$\begin{cases} \pi_1 & := \text{ orthogonal projection to } e_1 \\ \pi_2 & := \text{ orthogonal projection to } e_2 \\ h & := f^{-1/2}\pi_1 + f^{1/2}\pi_2 \end{cases} \qquad \begin{cases} \tilde{\pi}_1 & := \text{projection to } V_1 \text{ parallel to } V_2 \\ \tilde{\pi}_2 & := \text{projection to } V_2 \text{ parallel to } V_1 \\ \tilde{h} & := f^{-1/2}\tilde{\pi}_1 + f^{1/2}\tilde{\pi}_2 \end{cases}$$

where

$$f = \frac{\lambda^2 - \alpha^2}{1 - \bar{\alpha}^2\lambda^2}.$$

Note that in general $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are non-orthogonal projections.

We now define a matrix $\mathcal{C} \in \Lambda_r \mathrm{SU}(2)$:

$$\mathcal{C} := \frac{-ie^{i\theta}}{\sqrt{|T|^2 + 1}} \begin{pmatrix} 1 & T\lambda \\ \bar{T}\lambda^{-1} & -1 \end{pmatrix} \quad ,$$

$$\text{where } T = \frac{\bar{\alpha}^{-1}A\bar{B}(1 + \bar{\alpha}^2)}{|A|^2 - \frac{\bar{\alpha}^2}{|\alpha|^2}|B|^2} \text{ and } \theta = arg\big(|A|^2 - \frac{\bar{\alpha}^2}{|\alpha|^2}|B|^2\big) \quad .$$

Theorem 5.4. *Let ϕ be a solution of $d\phi = \phi\xi$ on $\mathcal{R}$ and let $\phi = FB$ be the r-Iwasawa splitting of ϕ. We assume that the monodromy M_ϕ of ϕ is in $\Lambda_r \mathrm{SU}(2)$ and is $\pm id$ at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$. We do the dressing $\phi \to h\cdot\phi$, then $h\phi = (hF\tilde{h}^{-1}\mathcal{C}^{-1})(\mathcal{C}\tilde{h}B)$ is r-Iwasawa splitting of $h\cdot\phi$, i.e. $hF\tilde{h}^{-1}\mathcal{C}^{-1} \in \Lambda_r \mathrm{SU}(2)$ and $\mathcal{C}\tilde{h}B \in \Lambda_{r+} \mathrm{SL}(2, \boldsymbol{C})$, where $h, \tilde{h}, \mathcal{C}$ are defined as above.*

Theorem 5.4 has the following corollary:

Corollary 5.5. *We have explicit parametrizations for cylinder bubbletons in all three space forms using the Sym-Bobenko formulas* (2.10), (2.12) *and* (2.13).

5.3. **Equivalence of the simple type dressing and Bianchi's Bäcklund transformation on the cylinder.** In this section we prove the equivalence of the simple type dressing and Bianchi's Bäcklund transformation in R^3 in the case of the cylinder. Bianchi's Bäcklund transformation is described in [24]. Actually, in the cylinder case, we can show that the metric, the Hopf differential and mean curvature of Bianchi's Bäcklund trasformation are the same as those resulting from the simple type dressing. In a general setting, Fran Burstall [5] has proven that equivalence of the simple type dressing and Darboux transformation of CMC surfaces. This implies the equivalence of the simple type dressing and Bianchi's Bäcklund transformation, because Udo Hertrich-Jeromin and Franz Pedit [7] have proven that the equivalence of Darboux transformation of CMC surfaces and Bianchi's Bäcklund transformation of CMC surfaces. Thus what we are proving here is only a special case of something that has been recently proven by Fran Burstall. But we include a proof here, because our proof is more direct and tailored to the case for which we need it.

Theorem 5.6. *Bianchi's Bäcklund transformation of the cylinder and the simple type dressing of the cylinder are the same surface.*

5.4. **Parallel surfaces of the bubbletons.** CMC surfaces have parallel CMC surfaces. In this section, we prove that the parallel surfaces of the bubbletons are the same surface as the original bubbletons. First we derive a result on parallel CMC surfaces that can be found in [2]:

Theorem 5.7. *Let f be a confromal CMC surface defined by the Sym-Bobenko formula* (2.10) *on a simply-connected domain $D \subseteq \boldsymbol{R}^2$. Then*

$$f^* = \left[\frac{1}{2} F \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} F^{-1} - i\lambda(\partial_\lambda F) \cdot F^{-1}\right]\Bigg|_{\lambda=1} .$$

is a conformal parametrization of another CMC surface defined for $(x, y) \in D$. We denote the metric, the mean curvature and the Hopf differential of f^* by $2e^{2u^*}(dx^2 + dy^2)$, H^* and Q^*, respectively. Then the conformal factor of $2e^{2u^*}(dx^2 + dy^2)$, H^* and Q^* have the following forms:

$$2e^{2u^*} = 2e^{-2u}|Q|^2 \ ,$$

$$H^* = H \ ,$$

$$Q^* = Q \ .$$

Here $2e^{2u}(dx^2 + dy^2)$, H and Q are the metric, the mean curvature and the Hopf differential of the CMC surface f, respectively. We call f^* the parallel surface *of f.*

Theorem 5.8. *The parallel surface of a cylinder bubbleton is the same surface as the original cylinder bubbleton, up to a rigid motion.*

References

[1] A. I. Bobenko, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys 46:4, 1-45 (1991).

[2] A. I. Bobenko, *Surfaces in terms of 2 by 2 matrices. Old and new integrable cases* , Harmonic maps and integrable systems, 83–127, Aspects Math., E23, Vieweg, Braunschweig, 1994.

[3] A. I. Bobenko and U. Eitner *Painlevé Equation in Differential Geometry of Surfaces*, Springer-Verlag, 2001

[4] A. I. Bobenko and Its, *The Painleve III equation and the Iwasawa decomposition*, Manuscripta Math. 87, 369-377 (1995).

[5] F. Burstall, *Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems*, to appear in "Integrable systems, Geometry and Topology", International Press.

[6] F. Burstall, F. Pedit. *Dressing orbits of harmonic maps*, Duke Math. J. 80 (1995), no. 2, 353–382.

[7] U. Hertrich-Jeromin, F. Pedit, *Remarks on the Darboux transform of isothermic surfaces*, Doc. Math. 2 (1997), 313–333 (electronic).

[8] J. Dorfmeister, F. Pedit and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6(4)** (1998), 633–668.

[9] J. Dorfmeister, G. Haak, *Meromorphic potentials and smooth surfaces of constant mean curvature*, Math. Z., **224** (1997), 603–640.

[10] J. Dorfmeister, G. Haak, *On symmetries of constant mean curvature surfaces. I. General Theory*, Tohoku Math. J., **50** (1998), 437–454.

[11] J. Dorfmeister, G. Haak, *On symmetries of constant mean curvature surfaces. II. Symmetries in a Weierstrass-type representation*, Int. J. Math. Game Theory Algebra **10** (2000), 121–146.
[12] J. Dorfmeister, G. Haak, *On constant mean curvature surfaces with periodic metric*, Pacific J. Math. **182** (1998), 229–287.
[13] J. Dorfmeister, G. Haak, *Investigation and application of the dressing action on surfaces of constant mean curvature*, Q. J. Math. **51** (2000), 57–73.
[14] J. Dorfmeister, G. Haak, *Construction of non-simply connected CMC surfaces via dressing*, submitted.
[15] J. Dorfmeister, H. Wu, *Construction of constant mean curvature trinoids from holomorphic potentials*, preprint.
[16] M. A. Guest, *Harmonic maps, loop groups, and integrable systems*, London Mathematical Society Student Texts 38, Cambridge University Press 1997.
[17] M. Kilian, Constant mean curvature cylinders, doctoral thesis, Univ. of Massachusetts (Amherst), U.S.A., September (2000).
[18] M. Kilian, I. McIntosh, and N. Schmitt, *New constant mean curvature surfaces*, preprint, http://www.gang.umass.edu/preprint/authors/Kilian_M.html.
[19] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Annals of Math. 130, 407-451 (1989).
[20] A. Pressley and G. Segal, *Loop Groups*, Oxford Mathematical Monographs, Oxford University Press 1986.
[21] N. Schmitt, *New constant mean curvature trinoids*, preprint, http://www.gang.umass.edu/preprint/authors/Schmitt_N.html.
[22] N. Schmitt, *CMCLab software*, Center for Geometry, Analysis, Numerics and Graphics(GANG), University of Massachusetts at Amherst. http://www.gang.umass.edu/cmc/
[23] H. Sievert, *Über die Zentralflächen der Enneperschen Flächen konstanten Krümmungsmasses*, Diss. Tübingen (1886).
[24] I. Sterling and H. C. Wente, *Existence and classification of constant mean curvature multibubbletons of finite and infinite type*, Indiana U. Math. J. 42(4), 1239-1266 (1993).
[25] Terng and K. Uhlenbeck, *Bäcklund transformations and loop group actions*, Comm. Pure and Appl. Math, Vol. L3, 1-75(2000).
[26] H. C. Wente, *Constant Mean Curvature Immersions of Enneper Type*, Memoir's of A.M.S. 478, 1-76 (1992).