Existentially closed models of the theory of differential fields with a cyclic automorphism

Makoto Yanagawa

University of Tsukuba

September 15, 2014
Motivation

Let $C$ be any field and choose an arbitrary element $q \in C \setminus \{0, 1\}$. Let $\mathbb{K}_0$ denote the prime field included in $C$, and set $\mathbb{K} = \mathbb{K}_0(q)$, the subfield of $C$ generated by $q$ over $\mathbb{K}_0$.

**Definition.** The $q$-integer, the $q$-factorial and $q$-binomial, respectively, denotes

\[
[k]_q = \frac{q^k - 1}{q - 1}, \quad [0]_q = 0, \\
[k]_q! = [k]_q[k - 1]_q \cdots [1]_q, \quad [0]_q! = 1, \\
\binom{m}{n}_q = \frac{[m]_q!}{[n]_q![m - n]_q!},
\]

where $k, m, n \in \mathbb{N}$ with $m \geq n$. 

Existentially closed models of the theory of differential fields with a cyclic automorphism

Makoto Yanagawa
Suppose that $R$ is a field containing $\mathbb{k}(t)$ and $\sigma_q : R \to R$ is a ring automorphism such that it is an extension of the $q$-difference operator $f(t) \mapsto f(qt)$ on $\mathbb{k}(t)$.

**Definition (C. Hardouin).** We say that a sequence $\delta_R^* = (\delta_R^{(k)})_{k \in \mathbb{N}}$ of maps on $R$ is an iterative $q$-difference operator on $R$ if it satisfies the following condition:

1. $\delta_R^{(0)} = \mathrm{id}_R$,
2. $\delta_R^{(1)} = \frac{1}{(q-1)t}(\sigma_q - \mathrm{id}_R)$,
3. $\delta_R^{(k)}(x + y) = \delta_R^{(k)}(x) + \delta_R^{(k)}(y)$, $x, y \in R$,
4. $\delta_R^{(k)}(xy) = \sum_{i+j=k} \sigma_q^i \circ \delta_R^{(i)}(x) \delta_R^{(j)}(y)$, $x, y \in R$,
5. $\delta_R^{(i)} \circ \delta_R^{(j)} = \binom{i+j}{i}_q \delta_R^{(i+j)}$
Remark. Assume that $q$ is not a root of unity. Then,

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1} \neq 0$$

for all $k > 0$. If $\delta^*_R = (\delta^{(k)}_R)_{k \in \mathbb{N}}$ is an iterative $q$-difference operator on $R$, conditions 1, 2 and 5 above require

$$\delta^{(1)}_R = \frac{1}{(q - 1)t}(\sigma_q - \text{id}_R), \quad \delta^{(k)}_R = \frac{1}{[k]_q!}(\delta^{(1)}_R)^k, \quad k \in \mathbb{N}.$$ 

Conversely, if we define $\delta^{(k)}_R$ by above, then $\delta^*_R = (\delta^{(k)}_R)_{k \in \mathbb{N}}$ forms an iterative $q$-difference operator on $R$. Therefore under the assumption, an iterative $q$-difference ring is nothing but a difference field $(R, \sigma_q)$. 

Existentially closed models of the theory of differential fields with a cyclic automorphism

Makoto Yanagawa
From now on, we assume $q$ is a root of unity of order $N > 1$.

**Fact (Masuoka and Y., 2013).**

1. For any iterative $q$-difference field $(R, (\delta_R^{(k)})_{k \in \mathbb{N}})$, the $q$-difference operator $\sigma_q$ on $R$ is of order $N$, that is $\sigma_R^N = \text{id}_R$.
2. There is the smallest iterative $q$-difference field $\mathbb{k}(t)$. 
Theorem (Masuoka and Y., 2013). There is a functor

\[ \mathcal{F} : \{ \text{IQD-fields} \} \to \{ \text{models of } DF_\sigma \} \]

and satisfies the following properties:

1. \( \mathcal{F} \) is a strictly embedding,
2. for any model \((R, \sigma)\) of \(DF_\sigma\) there is \(\mathcal{F}^{-1}(R)\) whenever \(R \supset \mathcal{F}(\mathbb{k}(t))\) and \(\sigma^N = \text{id}_R\), and
3. \(\mathcal{F}\) has a good property for model theory.
\[ DF_{C_N} \text{ denotes the theory } DF_{\sigma} \cup \{ \forall x (\sigma^N(x) = x) \}. \]

**Corollary.** Suppose that \( q \) is a root of unity of order \( N > 1 \). Then the theory \( lqDF \) and the theory \( DF_{C_N} \cup \text{Diag}(\mathcal{F}(\mathbb{k}(t))) \) have same model theoretical property.

To study \( lqDF \), first, to study about \( DF_{C_N} \).

**Question.** Does the theory \( DF_{C_N} \) admit a model companion?
Introduction

Definition. Let $K$ be a field and $\delta$ be an additive map on $K$. We say that $(K, \delta)$ is a differential field if $\delta$ satisfies the Leibnitz rule:

$$\delta(ab) = a\delta(b) + \delta(a)b, \quad \text{for all } a, b \in K.$$ 

The language of differential fields, denoted by $L_\delta$, is the language of rings with a new unary function symbol $\delta$. $DF$ denotes the theory of differential fields (of characteristic 0) in the language $L_\delta$. 

Existentially closed models of the theory of differential fields with a cyclic automorphism

Makoto Yanagawa
Suppose that $T$ is a theory in a language $L$. $L_\sigma$ denotes the language $L$ with a new unary function symbol $\sigma$. We consider the theory

$$T_\sigma = T \cup \text{“}\sigma \text{ is an automorphism”}.\]

**Example.** Let $K = \mathbb{Q}(X)$, $\delta = \frac{d}{dX}$, and $\sigma(X) = X + 1$. Then

- $K \models Fld$,
- $(K, \delta) \models DF$,
- $(K, \sigma) \models Fld_\sigma$, and
- $(K, \delta, \sigma) \models DF_\sigma$.\]
Model companion

Let $T$ be a theory in a language $L$.

A model $M$ of $T$ is existentially closed if for any extension $N \models T$ of $M$ and quantifier-free formula $\varphi(x)$ over $M$,

$$\text{if } N \models \exists x \varphi(x) \text{ then } M \models \exists \varphi(x).$$

**Definition.** Suppose that $T$ is a $\forall \exists$-theory. We say that $T$ admits a model companion if the class

$$\mathcal{K} = \{ M \models T \mid M \text{ is existentially closed.} \}$$

is elementary.
Fact.

1. (Tarski) \( Fld \) admits a model companion. \( \rightarrow \) \( ACF \).
2. (Robinson) \( DF \) admits a model companion. \( \rightarrow \) \( DCF \)
3. (Macintyre) \( Fld_\sigma \) admits a model companion. \( \rightarrow \) \( ACFA \).
4. (Hrushovski) \( DF_\sigma \) admits a model companion. \( \rightarrow \) \( DCFA \)

Example. The theory of groups does not admit model companion.
Remark. The automorphism $\sigma$ of any model of ACFA (or DCFA) does not have finite order.

Suppose that $(K, \sigma) \models ACFA$. Let $n(> 0)$ be a natural number. We define $\tilde{\sigma}$ on $K(X_1, \ldots, X_{n+1})$ by

$$\tilde{\sigma}(X_i) = X_{i+1} \quad (i < n),$$
$$\tilde{\sigma}(X_{n+1}) = X_1,$$
$$\tilde{\sigma}|_K = \sigma.$$

Then $(K(X_1, \ldots, X_{n+1}), \tilde{\sigma})$ is a model of $Fld_\sigma$ extending $(K, \sigma)$ and $K(X_1, \ldots, X_{n+1}) \models \exists x \bigwedge_{0<i<n+1} \sigma^i(x) \neq x$. 

Existentially closed models of the theory of differential fields with a cyclic automorphism

Makoto Yanagawa
Some results
First approach

We want to construct an existentially closed model of $DF_{C_N}$, where $DF_{C_N}$ is $DF_{\sigma} \cup \{\forall x(\sigma^N(x) = x)\}$.

Let $(K, \delta, \sigma)$ be a model of $DCFA$.

We consider the fixed field

$$K^{\langle \sigma^N \rangle} := \{a \in K \mid \sigma^N(a) = a\}$$

of $\sigma^N$ in $K$. Then $(K^{\langle \sigma^N \rangle}, \delta|_{K^{\langle \sigma^N \rangle}}, \sigma|_{K^{\langle \sigma^N \rangle}})$ is naturally a model of $DF_{C_N}$, since $\sigma \circ \delta = \delta \circ \sigma$ in $K$. 

Existentially closed models of the theory of differential fields with a cyclic automorphism

Makoto Yanagawa
Question. Is \((K^{\langle \sigma^N \rangle}, \delta|_{K^{\langle \sigma^N \rangle}}, \sigma|_{K^{\langle \sigma^N \rangle}})\) an existentially closed model of \(DF_{C_N}\)?

Answer. I don’t know, but it has the property close to existentially closed.
Suppose that \((F, \delta, \sigma)\) is an extension of \((K^{\langle \sigma^N \rangle}, \delta|_{K^{\langle \sigma^N \rangle}}, \sigma|_{K^{\langle \sigma^N \rangle}})\) and \(\varphi(x)\) is a quantifier-free formula over \(K^{\langle \sigma^N \rangle}\) such that

\[
(F, \delta', \sigma') \models \exists x \varphi(x).
\]
We can define derivation and difference on $K \otimes_{K^{\langle \sigma^N \rangle}} F$. We assume $K \otimes_{K^{\langle \sigma^N \rangle}} F$ is an integral domain. Then there is a model $K'$ of $DF_\sigma$ such that

$$K \subseteq K' \quad \text{and} \quad K' \models \exists x (\varphi(x) \land \sigma^N(x) = x).$$

Since $K$ is existentially closed, $K \models \exists x (\varphi(x) \land \sigma^N(x) = x)$. This means $K^{\langle \sigma^N \rangle} \models \exists x \varphi(x)$.

**Problem.** Is $K \otimes_{K^{\langle \sigma^N \rangle}} F$ always an integral domain?

In my opinion, there is little possibility that the answer is yes.
Second approach

We will modify N. Sjögren’s argument in “The Model Theory of Field with a Group action”.
In this paper, he showed the following theorem:

**Theorem (N. Sjögren, 2005).** $\text{Fld}_{C_N}$ admits a model companion.
More precisely, a model \((K, \sigma)\) of \(Fld_{C_N}\) is existentially closed iff it holds the following properties

1. \(K\) and \(K^{\langle \sigma \rangle}\) are pseudo-algebraically closed,
2. \(\text{Gal}(K \cap (K^{\langle \sigma \rangle})^{alg} / K^{\langle \sigma \rangle}) \simeq C_N(\simeq \mathbb{Z}/N\mathbb{Z}),\)
3. \(\text{Gal}((K^{\langle \sigma \rangle})^{alg} / K^{\langle \sigma \rangle}) \simeq \text{Gal}(K^{alg} / K) \simeq \mathbb{Z}_N\)

**Remark.** To prove \(\Rightarrow\), We need some knowledge of pro-finite group. To prove \(\Leftarrow\), on the other hand, the following lemma is essentially:

**Lemma.** The above conditions 2 and 3 imply that \((K, \sigma)\) has no algebraic \(C_N\)-field extension.
(Sketch of proof of $\iff$)

- Suppose $K'$ be a extension of $K$ and $\varphi(x) := "f(x) = 0"$ is an $L_\sigma(K)$-formula such that $K' \models f(a) = 0$ for some $a \in K'$.

- Since $\sigma$ is of order $N$, there is finite tuple $b \in K$ such that $K = K^{\langle \sigma \rangle}(b)$ and $K' = K'^{\langle \sigma \rangle}(b)$

- We write $a_i = \sum_j c_{ij} b_j$, and consider $V = V(c/K)$.

- Since $K$ has no algebraic $G$-extension, $V$ is absolutely irreducible, so $V$ has $K$-rational point $c'$.

- We put $a'_i = \sum_j c'_{ij} b_j \in K$, then $K \models f(a') = 0$. 

Existentially closed models of the theory of differential fields with a cyclic automorphism  

Makoto Yanagawa
By modifying the Sjögren’s proof, we can prove the following result.

**Lemma.** Suppose that \((K, \delta, \sigma)\) is an existentially closed model of \(DF_{C_N}\). Then the following properties hold:

1. \(K\) and \(K^{\langle \sigma \rangle}\) are pseudo-differentially closed,
2. \(C_K = \{a \in K \mid \delta(a) = 0\}\) and \(C_{K^{\langle \sigma \rangle}} = (C_K)^{\langle \sigma \rangle}\) are pseudo-algebraically closed.
Comments.

1. If the property “one has no differentially algebraic $C_N$-field extension” is possible to imply from the common first-order property among existentially closed models of $DF_{C_N}$, then (I think that) it is possible to prove that $DF_{C_N}$ admits a model companion.

2. To do this, we need more knowledge of differential Galois group. (However, I luck it now...)
References

