A $tt^*$-bundle associated with a harmonic map from a Riemann surface into a sphere

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Abstract

A $tt^*$-bundle is constructed by a harmonic map from a Riemann surface into an $n$-dimensional sphere. This $tt^*$-bundle is a high-dimensional analogue of quaternionic line bundle with a Willmore connection. For the construction, a flat connection is decomposed into four parts by a fiberwise complex structure.

1 Introduction

A $tt^*$-bundle is a real vector bundle equipped with a family of flat connections, parametrized by a circle. The present paper delivers a $tt^*$-bundle derived from a harmonic map from a Riemann surface to an $n$-dimensional sphere.

The notion of $tt^*$-bundles is introduced by Schäfer [10] as a simple solution to a generalized version of the equation of topological-antitopological fusion, introduced by Cecotti and Vafa [2], in terms of real differential geometry. A topological-antitopological fusion of a topological field theory model is a special geometry structure on a Frobenius manifold. As a geometric interpretation of a special geometry structure on a quasi-Frobenius manifold, Dubrovin [6] showed that a solution to the equation is locally a pluriharmonic map from an $n$-dimensional quasi-Frobenius manifold to the symmetric space $\text{GL}(n, \mathbb{R})/\text{O}(n)$.

Schäfer [10] showed that an admissible pluriharmonic map from a simply connected complex manifold $M$ to a symmetric space $\text{GL}(r, \mathbb{R})/\text{O}(p, q)$, and that to $\text{SL}(r, \mathbb{R})/\text{SO}(p, q)$ with $p + q = r$, gives rise from a metric $tt^*$-bundle. A harmonic map from a Riemann surface to $\text{SU}(1, 1)/\text{U}(1) \times \text{U}(1) \cong \text{SL}(2, \mathbb{R})/\text{SO}(2)$ is obtained by the generalized Weierstrass representation formula by Dorfmeister, Pedit, and Wu [5]. The Gauss map of a spacelike surface of constant mean curvature in the Minkowski space $\mathbb{R}^{2,1}$ is a harmonic map from a Riemann surface to $\text{SL}(2, \mathbb{R})/\text{SO}(2)$. The Sym-Bobenko formula (Bobenko [1], Dorfmeister and Haak [4]) connects a surface and its Gauss map. Applying these formulae, Dorfmeister, Guest, and Rossman [3] gave the description of the quantum cohomology of $\mathbb{C}P^1$. The quantum cohomology of $\mathbb{C}P^1$ provides a solution to the third Painlevé equation.

A surface of constant mean curvature in $\mathbb{R}^3$ is an interesting research subject in the theory of surfaces. Its Gauss map is a harmonic map from a Riemann
surface to the two-dimensional sphere $S^2$. It is impossible to write $S^2$ as a symmetric space $GL(r, \mathbb{R})/O(p, q)$ or $SL(r, \mathbb{R})/SO(p, q)$. This led the authors to find a $tt^*$-bundle for a harmonic map into $S^2$. The theory of a quaternionic line bundle with a Willmore connection by Ferus, Leschke, Pedit, and Pinkall [8] provides a way to construct a $tt^*$-bundle for a harmonic map from a Riemann surface into $S^2$. This method is extended and a $tt^*$-bundle associated with a harmonic map from a Riemann surface into $S^n$ ($n \geq 2$) is obtained (Theorem 4.1).

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2 $tt^*$-bundles

We recall a $tt^*$-bundle (Schäfer [10]).

Let $M$ be a complex manifold of complex dimension $n$ with a complex structure $J_M$. For a one-form $\omega$ on $M$, we define a one-form $*\omega$ on $M$ by $*\omega := \omega \circ J_M$. Let $E$ be a trivial real vector bundle of rank $n$ over $M$, $\nabla$ a connection on $E$, and $S$ a one-form with values in the real endomorphisms of $E$. A one-form $S$ is considered as a one-form with values in $n$-by-$n$ real matrices. Define a family of connections $\{\nabla^\theta\}_{\theta \in \mathbb{R}}$ on $E$ by

$$\nabla^\theta := \nabla + (\cos \theta) S + (\sin \theta) * S.$$ 

The curvature of $\nabla^\theta$ is

$$d\nabla^\theta \circ \nabla^\theta = d\nabla \circ \nabla + (\cos \theta) d\nabla S + (\sin \theta) d\nabla * S$$

$$+ ((\cos \theta) S + (\sin \theta) * S) \wedge ((\cos \theta) S + (\sin \theta) * S)$$

$$= d\nabla \circ \nabla + (\cos \theta) d\nabla S + (\sin \theta) d\nabla * S$$

$$+ (\cos \theta)^2 S \wedge S + \cos \theta \sin \theta (S \wedge * S + * S \wedge S) + (\sin \theta)^2 * S \wedge * S$$

$$= d\nabla \circ \nabla + (\cos \theta) d\nabla S + (\sin \theta) d\nabla * S$$

$$+ \frac{1 + \cos 2\theta}{2} S \wedge S + \frac{\sin 2\theta}{2} (S \wedge * S + * S \wedge S) + \frac{1 - \cos 2\theta}{2} * S \wedge * S$$

$$= d\nabla \circ \nabla + \frac{1}{2} S \wedge S + \frac{1}{2} * S \wedge * S$$

$$+ (\cos \theta) d\nabla S + (\sin \theta) d\nabla * S$$

$$+ \frac{\cos 2\theta}{2} (S \wedge S - * S \wedge * S) + \frac{\sin 2\theta}{2} (S \wedge * S + * S \wedge S).$$

A vector bundle $E$ with $\nabla$ and $S$ is called a $tt^*$-bundle if $\nabla^\theta$ is flat for all $\theta \in \mathbb{R}$. By the preceding calculation, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if

$$d\nabla \circ \nabla + S \wedge S = 0, \quad d\nabla S = 0, \quad d\nabla * S = 0,$$

$$S \wedge S = * S \wedge * S, \quad S \wedge * S = - * S \wedge S.$$
Indeed,

\[(S \wedge S - *S \wedge *S)(X, Y) = S(X)S(Y) - S(Y)S(X) - S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X)\]

\[= -S(X)S(J^M J^M Y) + S(J^M J^M Y)S(X)\]

\[= -S(X)S(J^M J^M Y) + S(J^M Y)S(J^M X)\]

\[+ S(J^M J^M Y)S(X) - S(J^M X)S(J^M Y)\]

\[= -(S \wedge *S + *S \wedge S)(X, J^M Y)\]

for any tangent vectors \(X, Y\) of \(M\). Hence, \(S \wedge S = *S \wedge *S\) is equivalent to \(S \wedge *S = - *S \wedge S\). Then, a vector bundle \(E\) with \(\nabla\) and \(S\) is a \(tt^*\)-bundle, if and only if

\[d \nabla \circ \nabla + S \wedge S = 0, \quad d \nabla S = 0, \quad d \nabla *S = 0, \quad S \wedge S = *S \wedge *S\]

(see Schäfer [10], Proposition 1).

Assume that \(E\) with \(\nabla\) and \(S\) forms a \(tt^*\)-bundle of even rank. Define \(F\) as the complexification of \(E\), that is, \(F := \mathbb{C} \otimes E\). Denote the complex-linear extensions of \(\nabla\) and \(S\) by the same notations respectively. Define a family of connections \(\{\nabla^\mu\}_{\mu \in \mathbb{C}\setminus \{0\}}\) of \(F\) by

\[\nabla^\mu = \nabla + \frac{1}{\mu} C + \mu \bar{C}, \quad C = \frac{1}{2}(S - i * S).\]  

(1)

Then \(C\) is a \((1, 0)\)-form on \(M\) with values in complex linear endmorphisms of \(F\). The \(tt^*\)-bundle \(E\) with \(\nabla\) and \(S\) is the real part of \(F\) with \(\nabla^\mu\) if and only if \(|\mu| = 1\).

**Proposition 2.1.** For each \(\mu \in \mathbb{C}\setminus \{0\}\), the connection \(\nabla^\mu\) is flat.

**Proof.** As \(E\) with \(\nabla\) and \(S\) is a \(tt^*\)-bundle, it follows that

\[d \nabla^\mu C = 0, \quad d \nabla^\mu \bar{C} = 0,\]

\[C \wedge C = \frac{1}{4}(S \wedge S - i S \wedge *S - i * S \wedge S - *S \wedge *S) = 0,\]

\[C \wedge \bar{C} = \frac{1}{4}(S \wedge S + i S \wedge *S - i * S \wedge S + *S \wedge *S) = \frac{1}{2}(S \wedge S + i S \wedge *S).\]

Then

\[d \nabla^\mu \circ \nabla^\mu = d \nabla \circ \nabla + \left(\frac{1}{\mu} C + \mu \bar{C}\right) \wedge \left(\frac{1}{\mu} C + \mu \bar{C}\right)\]

\[= d \nabla \circ \nabla + C \wedge C + \bar{C} \wedge C\]

\[= d \nabla \circ \nabla + S \wedge S = 0.\]

Hence \(\nabla^\mu\) is flat. \(\square\)
Adding the assumption in Proposition 2.1, we assume that there exists a hermitian pseudo-metric $h$ on $F$, and a metric connection $\nabla$ with respect to $h$, such that

$$h(C(X)a, b) = h(a, \tilde{C}(X)b),$$

where $a, b \in \Gamma(F)$, and $X$ is a vector field of type $(1, 0)$ on $M$. Then $(F, \nabla, C, \tilde{C}, h)$ becomes a harmonic bundle defined in Schäfer [11].

### 3 Decomposition of a connection

We obtain a condition for a map from a Riemann surface into a sphere, to become a harmonic map, by decomposing a flat connection into four parts.

Let $C\ell_n$ be the Clifford algebra associated with $\mathbb{R}^n$ and the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$ (see Lawson and Michelsohn [9]). The Clifford algebra $C\ell_n$ is the algebra generated by an orthonormal basis $e_1, \ldots, e_n$ subject to the relation

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

Then $C\ell_n$ is identified with $\mathbb{R}^{2n}$. The set

$$\{a \in \mathbb{R}^n \subset C\ell_n \mid a^2 = -1\}$$

is an $(n-1)$-dimensional unit sphere $S^{n-1} \subset \mathbb{R}^n \subset C\ell_n \cong \mathbb{R}^{2n}$.

Let $M$ be a Riemann surface with its complex structure $J^M$ and $V$ be the trivial associate bundle of a principal $C\ell_n$-bundle, with right $C\ell_n$ action, over $M$. We denote the set of smooth sections of $V$ by $\Gamma(V)$ and the fiber of $V$ at $p$ by $V_p$. Let $\Omega^m(V)$ be the set of $V$-valued $m$-forms on $M$ for every non-negative integer $m$. Then $\Omega^0(V) = \Gamma(V)$. Let $W$ be another trivial associate bundle of a principal $C\ell_n$-bundle, with right $C\ell_n$ action, over $M$. We denote by $\text{Hom}(V, W)$ the $C\ell_n$-homomorphism bundle from $V$ to $W$. Let $N$ be a smooth section of the Clifford endomorphism bundle $\text{End}(V)$ of $V$ such that $-N_p \circ N_p$ is the identity map $\text{Id}_p$ on $V_p$ for every $p \in M$. The section $N$ is a complex structure at each fiber of $V$. We have a splitting $\text{End}(V) = \text{End}(V)_+ \oplus \text{End}(V)_-$, where

$$\text{End}(V)_+ = \{\xi \in \text{End}(V) : N\xi = \xi N\},$$
$$\text{End}(V)_- = \{\xi \in \text{End}(V) : N\xi = -\xi N\}.$$ 

This splitting induces a decomposition of $\xi \in \text{End}(V)$ into $\xi = \xi_+ + \xi_-$, where $\xi_+ = (\xi - N\xi N)/2 \in \text{End}(V)_+$ and $\xi_- = (\xi + N\xi N)/2 \in \text{End}(V)_-$.

Let $T^*M \otimes_\mathbb{R} V$ be the tensor bundle of the cotangent bundle $T^*M$ of $M$ and $V$ over real numbers. We set $*\omega = \omega \circ J^M$ for every $\omega \in \Omega^1(V)$. We have a splitting $T^*M \otimes_\mathbb{R} V = KV \oplus \bar{KV}$, where

$$KV = \{\eta \in T^*M \otimes_\mathbb{R} V : *\eta = N\eta\}, \quad \bar{KV} = \{\eta \in T^*M \otimes_\mathbb{R} V : *\eta = -N\eta\}.$$  

This splitting induces the type decomposition of $\eta \in T^*M \otimes_\mathbb{R} V$ into $\eta = \eta' + \eta''$, where $\eta' = (\eta - N * \eta)/2 \in KV$ and $\eta'' = (\eta + N * \eta)/2 \in \bar{KV}$. 

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Let $C$ be the right trivial Clifford bundle over $M$ with fiber $C_{\mathbb{R}^n}$. We identify a smooth map $\phi: M \rightarrow C_{\mathbb{R}^n}$ with a smooth section $p \rightarrow (p, \phi(p))$ of $C$. The bundle $\text{End}(C)$ is identified with $C$, by the identification of $\xi_p \in \text{End}(C)_p$ with $P_p \in C_p$ such that $\xi_p(1) = P_p$ for every $p \in M$. We assume that $N$ takes values in $\mathbb{R}^n \subset C_{\mathbb{R}^n}$. Then $N$ is considered as a map from $M$ to $S^{n-1} \subset \mathbb{R}^n$. Then $T^*M \otimes_{\mathbb{R}} C$ decomposes as

$$T^*M \otimes_{\mathbb{R}} C = (KC)_+ \oplus (KC)_- \oplus (\tilde{K}C)_+ \oplus (\tilde{K}C)_-.$$ 

According to this decomposition, a connection $\nabla: \Gamma(C) \rightarrow \Omega^1(C)$ of Clifford bundle $C$ decomposes as

$$\nabla = \partial^\nabla + A^\nabla + \bar{\partial}^\nabla + Q^\nabla,$$

where $\phi$ is any smooth section of $C$. We see that $A^\nabla$ and $Q^\nabla$ are tensorial, that is, $A^\nabla \in \Gamma(\text{Hom}(C, (KC)_-))$ and $Q^\nabla \in \Gamma(\text{Hom}(C, (KC)_-))$. The sections $A^\nabla$ and $Q^\nabla$ are called the Hopf fields of $\nabla'$ and $\nabla''$ respectively.

We denote by $d$ the trivial connection on $C$.

**Lemma 3.1.** A map $N: M \rightarrow S^{n-1} \subset \mathbb{R}^n \subset C_{\mathbb{R}^n}$ is a harmonic map, if and only if $d \ast A^d = 0$.

**Proof.** The Hopf field $A^d$ satisfies the equation

$$A^d \phi = \frac{1}{2} (d' + Jd' J) \phi$$

$$= \frac{1}{4} (d - J \ast d + (d - J \ast d) J) \phi$$

$$= \frac{1}{4} (\{d \phi\} - N \ast (d \phi))$$

$$+ [N(dN) \phi - d \phi] + [\ast (dN) \phi + N \ast d \phi]$$

$$= \frac{1}{4} [N(dN) + \ast (dN)] \phi$$

for every $\phi \in \Gamma(C)$. Hence

$$d \ast A^d = \frac{1}{4} (dN \wedge \ast dN + N d \ast dN).$$

Hence $d \ast A^d = 0$ if and only if

$$dN \wedge \ast dN + N d \ast dN = 0.$$
For an isothermal coordinate \((x, y)\) such that \(x + yi\) is a holomorphic coordinate, a map \(N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n\) is a harmonic map if and only if

\[
\Delta N = -(N_{xx} + N_{yy})dx \wedge dy = |dN|^2 N
\]

(see Eells and Lemaire [7]). We have

\[
d * dN = d * (N_x \, dx + N_y \, dy) = d(-N_x \, dy + N_y \, dx)
\]

\[
= -(N_{xx} + N_{yy})dx \wedge dy = \Delta N,
\]

\[
dN \wedge * dN = (N_x \, dx + N_y \, dy) \wedge (-N_x \, dy + N_y \, dx) = (-N_x^2 - N_y^2)dx \wedge dy
\]

\[
= (|N_x|^2 + |N_y|^2)dx \wedge dy = |dN|^2,
\]

where the Clifford multiplication is used. Hence, \(N\) is a harmonic map if and only if \(d * A^d = 0\).

\[\square\]

4 Harmonic maps into a sphere

We construct a \(tt^*\)-bundle for a harmonic map from a Riemann surface to an \(n\)-dimensional sphere.

Let \(M\) be a Riemann surface with complex structure \(J^M\). For a one-form \(\omega\) on \(M\), define a one-form \(* \omega\) on \(M\) by \(* \omega := \omega \circ J^M\). For one-forms \(\omega\) and \(\eta\) on \(M\) with values in \(C\ell_n\), we have the relation

\[
* \omega \wedge * \eta = \omega \wedge \eta.
\]

Indeed, for a basis \(E_1, E_2\) of a tangent space of \(M\) with \(J^M E_1 = E_2\), we have

\[
(\omega \wedge \eta)(qE_1 + rE_2, sE_1 + tE_2) = (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\eta(E_1)),
\]

\[
(* \omega \wedge * \eta)(qE_1 + rE_2, sE_1 + tE_2) = (\omega \wedge \eta)(qE_2 - rE_1, sE_2 - tE_1)
\]

\[
= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\eta(E_1)),
\]

where \(q, r, s, t \in \mathbb{R}\).

Let \(F := M \times \mathbb{R}^2 \cong M \times C\ell_n\). For a map \(N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n\), define a one-form \(S\) on \(M\) with values in \(C\ell_n\) by

\[
S := \frac{1}{4}(\ast dN + N \, dN).
\]

**Lemma 4.1.** \(N\) is a harmonic map if and only if the one-form \(S\) satisfies \(d * S = 0\).

**Proof.** By Lemma 3.1, we have \(d * dN = N \, dN \wedge * dN\). Hence

\[
4 \, d * S = d((-dN + N \, * dN) = dN \wedge * dN + N \, d + dN
\]

\[
= N(d \wedge dN - N \, dN \wedge * dN) = 0.
\]

Hence Lemma 4.1 holds.
Theorem 4.1. A vector bundle $F$ with $\nabla := d - S$ and $S$ is a $tt^*$-bundle.

Proof of Theorem. We see that

$$4 dS = d * dN + dN \land dN = dN \land dN + N dN \land *dN;$$

$$16 S \land S = (* dN + N dN) \land (* dN + N dN)$$

$$= * dN \land * dN + *dN \land N dN + N dN \land * dN + N dN \land * dN$$

$$= 2(dN \land dN + N dN \land * dN).$$

Hence $dS = 2 S \land S$ holds.

Lemma 4.1 and the direct calculation yield

$$\nabla^0 = d + (\cos \theta - 1) S + (\sin \theta) \ast S,$$

$$d\nabla^0 \circ \nabla^0$$

$$= (\cos \theta - 1) dS + ((\cos \theta - 1) S + (\sin \theta) \ast S) \land ((\cos \theta - 1) S + (\sin \theta) \ast S)$$

$$= (\cos \theta - 1) dS + (\cos \theta - 1) S \land S + (\cos \theta - 1)(\sin \theta) S \land \ast S$$

$$+(\sin \theta)(\cos \theta - 1) S \land S + (\sin \theta)^2 S \land \ast S$$

$$= (\cos \theta - 1) dS - 2(\cos \theta - 1) S \land S = 0.$$

Hence $F$ with $\nabla$ and $S$ is a $tt^*$-bundle. $\square$

For a harmonic map from a Riemann surface to $S^2$, we have two $tt^*$-bundles. One is the $tt^*$-bundle in Theorem 4.1. The other is that in the theory of quaternionic holomorphic line bundles (see [8]). These do not coincide directly as the fiber of the former is $C\ell_3$ and that of the latter is $C\ell_2$.

References


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