QUOTIENTS OF QUATERNIONIC HOLOMORPHIC SECTIONS

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Abstract. A surface is represented as a quotient of two quaternionic holomorphic sections. Utilizing these quotients, we explain a correspondence between super-conformal surfaces and complex holomorphic null curves.

1. Introduction

The theory of quaternionic analysis on a Riemann surface by Pedit and Pinkall [3] draws an analogy between complex holomorphic functions on a Riemann surface and weakly-conformal immersions from a Riemann surface to the Euclidean four-space. A complex holomorphic section of a trivial complex line bundle over a Riemann surface is a holomorphic function. A quaternionic holomorphic section of a trivial complex quaternionic line bundle over a Riemann surface is a weakly-conformal immersion. A quotient of two complex holomorphic sections of a complex line bundle is a complex holomorphic function on a Riemann surface. Similarly, a quotient of two quaternionic holomorphic sections of a complex quaternionic line bundle is a weakly-conformal immersion from a Riemann surface to the Euclidean four-space.

A quotient of two complex holomorphic sections of a complex line bundle is a complex holomorphic function on a Riemann surface. Similarly, a quotient of two quaternionic holomorphic sections of a complex quaternionic line bundle is a weakly-conformal immersion from a Riemann surface by [3, p. 395, Example].

When we consider holomorphic one-forms, we have a similar situation. A quotient of two complex holomorphic one-forms is a complex holomorphic function on a Riemann surface. A quotient of quaternionic holomorphic one-forms is a weakly-conformal immersion. We will review quotients of quaternionic holomorphic one-forms and updated the correspondence between super-conformal surfaces and complex holomorphic null curves obtained in [4].

2. Surfaces and Quaternions

In this section, we review quaternionic analysis on a Riemann surface by Pedit and Pinkall [3].

2.1. Conformal structures. Let $N$ be the Riemannian manifold and $g_0$ the Riemannian metric of $N$. We consider the conformal class $c$ of Riemannian metrics on $N$ where $g_0$ belongs to:

$$c = \{ g = \lambda g_0 \mid \lambda : N \to \mathbb{R}, \lambda > 0 \}.$$
We denote by $\mathcal{C}$ the set of diffeomorphisms from $N$ to $N$ preserving the conformal structure $c$. The set $\mathcal{C}$ becomes a Lie group in a standard way. An action of $\mathcal{C}$ on $N$ is naturally defined. Then $\mathcal{C}$ becomes a Lie transformation group of $N$. An element of $\mathcal{C}$ is called a conformal transformation of $N$.

2.2. Surfaces. We recall the description of surfaces in terms of quaternions. Let $E^4$ be the Euclidean four-space, $g_0$ the Riemannian metric of $E^4$ and $c$ the conformal structure of $E^4$ represented by $g_0$. We consider a two-dimensional oriented manifold $M$ and an immersion $f: M \to E^4$. Then a conformal structure of $M$ is induced from $c$ by $f$. The conformal structure $c$ and the orientation of $M$ determines a complex structure of $M$ such that $(v, J^M v)$ is a positive orthogonal basis of a tangent space of $M$ for every non-zero tangent vector $v$ of $M$.

We relax the condition for $f$. Let $M = (M, J^M)$ be a Riemann surface with complex structure $J^M$. We assume that $f: (M, J^M) \to E^4$ is a branched immersion such that $J^M$ is orthogonal with respect to the metric induced from $g_0$ by $f$ at every immersed point. Then $f$ is a weakly-conformal immersion. We call $f$ a surface.

Let $\mathbb{H}$ be the quaternions. The Euclidean four-space $E^4$ is identified with $\mathbb{H}$ in a natural manner. Then $f: (M, J^M) \to \mathbb{H}$ is an immersed surface if and only if there exist smooth maps $N, R: M \to \text{Im} \mathbb{H}$ such that $* df := df \circ J^M = N df = - df R$. We see that $N^2 = R^2 = -1$. The maps $N$ and $R$ are called the left normal vector and the right normal vector of $f$. If $f$ is branched, then $N$ and $R$ are not defined at branch points.

If $N = i$, then $f$ is a complex holomorphic map from $M$ to $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$. Similarly, if $R = -i$, then $f$ is a complex holomorphic map from $M$ to $\mathbb{C}^2 \cong \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{H}$. The map $f: M \to \mathbb{C} \subset \mathbb{H}$ is a complex holomorphic function if and only if $N = -R = i$. Hence we can consider a surface in $\mathbb{H}$ as an analogue of a complex holomorphic function.

2.3. Quaternionic holomorphic structures of surfaces. We introduce the terminology of vector bundles and consider equations which characterize holomorphic sections.

For a vector bundle $V$ over a Riemann surface, we denote by $Ω^n(V)$ the set of smooth sections of $V \otimes \bigwedge^n TM$ ($n = 0, 1, 2$).

Let $H$ be the right trivial quaternionic line bundle over $M$. A smooth map $\phi: M \to H$ is considered as a smooth section of $H$. We fix a smooth map $N: M \to \text{Im} \mathbb{H}$ with $N^2 = -1$. We define $D^N: Ω^0(H) \to Ω^1(H)$ by

$$D^N\phi = \frac{1}{2}(d\phi + N * d\phi).$$

If $D^N\phi = 0$, then $\phi$ is a constant map or a surface with left normal vector $N$. We call $D^N$ the quaternionic holomorphic structure of a surface with left normal vector $N$. We see that

$$D^N(\phi \lambda) = (D^N\phi)\lambda + \frac{1}{2}(\phi d\lambda + N\phi * d\lambda) = (D^N\phi)\lambda + \phi D^{-1} N\phi \lambda$$

$$* D^N\phi = -ND^N\phi$$

for every $\lambda: M \to \mathbb{H}$.

Let $C$ be the trivial complex line bundle over $M$. If $N = i$, then $D^i|_{Ω^0(C)}$ is a complex holomorphic structure of $C$. Indeed,

$$D^i\phi = \frac{1}{2}(d\phi + i * d\phi) = \partial\phi$$
for every $\phi \in \Omega^0(C)$. If $\bar{\partial}\phi = 0$, then $\phi$ is a complex holomorphic function. We have

$$\bar{\partial}(\phi\lambda) = \bar{\partial}\lambda + \frac{1}{2}(\phi\,d\lambda + i\phi\,d\lambda) = (\bar{\partial}\phi)\lambda + \phi\,\bar{\partial}\lambda, \quad \ast\bar{\partial}\phi = -i\,\bar{\partial}\phi$$

for every $\phi \in \Omega^0(C)$ and every $\lambda: M \to \mathbb{C}$.

3. Quotients of holomorphic sections

We review quotients of quaternionic holomorphic sections and update the correspondence between super-conformal surfaces and complex holomorphic null curves.

3.1. Quotients of surfaces. We assume that $\phi \in \Omega^0(H)$ is nowhere vanishing and $\lambda: M \to \mathbb{H}$ is a smooth map. If $\phi$ and $\phi\lambda$ are surfaces with left normal vector $N$, then the map $\lambda: M \to \mathbb{H}$ is a surface with left normal vector $\phi^{-1}N\phi$. Indeed, $D^N(\phi\lambda) = \phi D^{\phi^{-1}N}\phi\lambda = 0$. Since $\lambda = \phi^{-1}(\phi\lambda)$, we may say that a quotient of two surfaces with the same left normal vector is a surface.

This is an analogue of the fact that if $\xi$ and $\eta$ are complex holomorphic functions, then $\xi^{-1}\eta$ is a complex holomorphic function.

3.2. Quotients of quaternionic holomorphic one-forms. We say that a quaternionic-valued one-form $\omega$ is quaternionic holomorphic with respect to $N$ if $d\omega = 0$ and $\ast\omega = \Omega\omega$. If $\omega$ is nowhere vanishing, then there exists a map $R: M \to \text{Im} \mathbb{H}$ such that $N\omega = -\omega R$. By the definition, $R^2 = -1$. We assume that $\omega$ is a nowhere-vanishing, quaternionic holomorphic one-form with respect to $N$ such that $\ast\omega = \Omega\omega = -\omega R$. Let $\lambda: M \to \mathbb{H}$ be a smooth branched immersion such that $\omega\lambda$ is closed. Then

$$d(\omega\lambda) = -\omega \wedge d\lambda = 0.$$

In a similar way to the proof of Proposition 16 in [1], we see that $\lambda$ is a surface with its left normal vector $-R$.

If $\omega$ is a complex holomorphic one-form, then $N = -R = i$. Let $\lambda: M \to \mathbb{C}$ be a smooth map such that $\omega\lambda$ is closed. Then $d(\omega\lambda) = -\omega \wedge d\lambda = 0$. This shows that $d\lambda$ is a complex holomorphic one-form. Hence $\lambda$ is a complex holomorphic function.

3.3. The Weierstrass representation. Let $N, R: M \to \text{Im} \mathbb{H}$ be maps which are the left normal vector and the right normal vector of a surface $f: (M, J^M) \to \mathbb{H}$ respectively. Let $\lambda$ be a smooth branched immersion. If $df\lambda$ is closed, then $\lambda$ is a surface with left normal vector $-R$. If $df\lambda$ is exact, then there exists a surface $g: (M, J^M) \to \mathbb{H}$ with left normal vector $N$ such that $dg = df\lambda$. The equation $dg = df\lambda$ is considered as a Weierstrass representation of $g$. The relation between left normal vectors and right normal vectors are listed in Table 1.

<table>
<thead>
<tr>
<th>left normal</th>
<th>right normal</th>
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<tbody>
<tr>
<td>$f$</td>
<td>$N_f$</td>
</tr>
<tr>
<td>$g$</td>
<td>$N_f$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-R_f$</td>
</tr>
</tbody>
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Table 1. Left normal vectors and right normal vectors
3.4. Super-conformal surfaces and complex holomorphic null curves. We recall a characterization of super-conformal surfaces.

Let \( f : (M, J^M) \to \mathbb{H} \) be a surface. We assume that there exist maps \( N, R : M \to \mathbb{H} \) such that \( N \) and \( R \) are the left normal vector and the right normal vector of \( f \) respectively. If \( *dN + N\,dN = 0 \) or \( *dR + RdR = 0 \), then \( f \) is super-conformal by \cite[Theorem 5]{1}. When we replace \( N \) and \( R \) to \( -N \) and \( -R \) respectively, then we have the equations \( *dN + N\,dN = 0 \) and \( *dR + RdR = 0 \). If \( *dN + N\,dN = 0 \) or \( *dR + RdR = 0 \), then \( f \) is minimal by \cite[Proposition 8]{1}. The equation \( *dN + N\,dN = 0 \) implies the equation \( *dR + R\,dR = 0 \) and vise versa. Hence, if the left normal vector of a minimal surface \( f : (M, J^M) \to \mathbb{H} \) is the same as that of a minimal surface \( g : (M, J^M) \to \mathbb{H} \), then the quaternionic-valued function \( \lambda \) defined by \( dg = df \lambda \) is a super-conformal surface by Table 1.

A combination of two minimal surfaces is a complex holomorphic map. A complex holomorphic curve \( \psi : M \to \mathbb{C}^4 \) is called null if \( \sum_{n=0}^4 \partial_\psi_n \otimes \partial_\psi_n = 0 \). We identify \( \mathbb{C}^4 \) with \( \mathbb{C} \otimes \mathbb{H} \). Let \( f = \text{Re} \, \psi : M \to \mathbb{H} \) and \( g = \text{Im} \, \psi : M \to \mathbb{H} \). Then \( \psi \) is null if and only if \( f \) and \( g \) are minimal surfaces such that \( *df = -dg \). The minimal surface \( f \) has the same left normal vector and the same right normal vector as the minimal surface \( g \) has.

Let \( g_0 + ig_1 : M \to \mathbb{C}^4 \) be a complex holomorphic null curve with minimal surfaces \( g_0 \) and \( g_1 \). Then the map \( \lambda \) defined by \( dg_1 = d\bar{g}_0 \lambda \) is a super-conformal surface. It is not trivial whether we can construct a complex holomorphic null curve from a given super-conformal surface. The following theorem is an answer to this problem.

**Theorem 1.** Let \( N : M \to \text{Im} \, \mathbb{H} \) be a map which is the left normal vector of a super-conformal surface \( f : M \to \mathbb{H} \). We define a map \( g_0 : M \setminus \{p \mid (dN)_p = 0\} \to \mathbb{H} \) and \( g_1 : M \setminus \{p \mid (dN)_p = 0\} \to \mathbb{C}^4 \) by \( df = dN \, g_0 \) and \( g_1 = N \, g_0 - f \). Then \( g_0 + g_1 : M \setminus \{p \mid (dN)_p = 0\} \to \mathbb{C}^4 \) is a complex holomorphic null curve such that \( *dg_0 = -N \, dg_0 \).

Conversely, let \( g_0 + ig_1 : M \to \mathbb{C}^4 \) be a complex holomorphic null curve such that \( *dg_0 = -N \, dg_0 \). Then \( f = N \, g_0 - g_1 : M \to \mathbb{H} \) is a super-conformal surface with \( *df = N \, df \).

This is a variant of \cite[Theorem 1]{4}. Similar result is obtained in Dajczer and Tojeiro \cite{2}.

**Proof.** Let \( f : M \to \mathbb{H} \) be a super-conformal surface with \( *df = N \, df \). Then \( N \) is a surface with left normal vector \( N \) and right normal vector \( -N \). A quaternionic-valued function \( g_0 \) defined by \( df = dN \, g_0 \) is a minimal surface with left normal vector \( -N \). The domain of \( g_0 \) is \( M \setminus \{p \mid (dN)_p = 0\} \). Since \( dg_1 = N \, dg_0 = -* \, dg_0 \), the map \( g_1 \) is a minimal surface and the map \( g_0 + g_1 \) is a complex holomorphic null curve.

Let \( g_0 + g_1 : M \to \mathbb{C}^4 \) be a complex holomorphic null curve such that \( *dg_0 = -N \, dg_0 \). Since \( *dg_0 = -dg_1 \), we have \( df = dN \, g_0 + N \, dg_0 - dg_1 = dN \, g_0 \). The map satisfies the equation \( *dN = N \, dN \). Hence \( f \) is a surface with left normal vector \( N \). Hence \( f \) is a super-conformal surface. \( \square \)

**References**


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