A SPACE OF MINIMAL TORI WITH ONE END AND CYCLIC
SYMMETRY

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Abstract. We will explicitly give the defining equation of the moduli space of symmetric minimal tori with one end.

1. Introduction

The 2-dimensional manifold $M$ of a complete minimal surface $X: M \to \mathbb{R}^3$ of finite total curvature is conformally equivalent to a compact Riemann surface $\bar{M}$ with finite number of points $\{p_1, \ldots, p_r\}$ removed ([3]). The removed points are called the ends of the surface. The Weierstrass representation $(g, \phi)$ of $X$ is a pair consisting of a meromorphic function $g$ on $M$ and a holomorphic one-form $\phi$ on $M$ such that

$$X(p) = \text{Re} \int^p (\Phi_1, \Phi_2, \Phi_3),$$

$$\Phi_1 = \frac{1}{2}(1 - g^2)\phi, \quad \Phi_2 = \frac{\sqrt{-1}}{2}(1 + g^2)\phi, \quad \Phi_3 = g\phi.$$ 

The one-forms $\Phi_1, \Phi_2,$ and $\Phi_3$ are holomorphic on $M$ and extend meromorphically to $\bar{M}$. They have poles at the ends of highest order greater than or equal to 2 and have no real periods on $M$. The one-form $g^2\phi$ must be nonzero holomorphic one form at a pole of $g$ on $M$. The meromorphic function $g$ is the stereographic projection of the Gauss map of $X$. The total curvature of $X$ is $-4\pi$ times the degree of $g$.

When the total curvature of $X$ is $-4\pi m$ and $X$ has a unique end ($m = 1, 2, \ldots$), the branch order of $g$ at the end is equal to or smaller than $m - 1$. We will assume that $g$ is maximally branched at the end, too. Since the Enneper surface of total curvature $-4\pi$ has this property, we will call this surface a minimal surface of Enneper type. The Chen-Gackstatter surface of genus one with total curvature $-8\pi$ has a unique end of Enneper type. But it is not a minimal surface of Enneper type.

In the case of minimal surfaces of Enneper type with total curvature $-4\pi m$, we may assume that the Gauss map $g$ has a unique pole at the end since the degree of $g$ is $m$ and the order of pole at the end is $m$. Then the meromorphic one-form $\phi$ has a unique pole at the end or a holomorphic one-form on $\bar{M}$. The degree of meromorphic or holomorphic one-form on a compact Riemann surface of genus $s$ is $2s - 2$ by the Riemann-Roch theorem ([1]). Hence, in the first case, $\bar{M}$ is the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. In the second case, $\bar{M}$ is a conformal torus. Although the classification in the case of genus zero is merely elementary algebraic

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exercise, that in the case of genus one is more involved since the moduli space of
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to the arising moduli space, where two minimal surfaces are identified if they are
axis followed by a reflection about a horizontal plane ([2]). This paper is devoted
by the moduli space of the surfaces with total curvature

We will add the assumption that $M$ is a conformal torus and that $X(M)$ has
symmetry $R$ such that $X(M)$ is invariant under a rotation of $90^\circ$ around a vertical
axis followed by a reflection about a horizontal plane ([2]). This paper is devoted
to the arising moduli space, where two minimal surfaces are identified if they are
congruent up to orientation-preserving isometries of $\mathbb{R}^3$. The moduli space is filtered
by the moduli space of the surfaces with total curvature $-4\pi(2n+3)$ which contains
a $2n + 1$ dimensional smooth subset $(n = 0, 1, 2, \ldots)$.

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2. The theorem

We will consider a conformal torus as a two-sheeted holomorphic branched covering
of a Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ ([1]). Then any conformal tori is written as

\begin{equation}
M_a := \{(w, z) \in \mathbb{C} \times \mathbb{C} \mid w^2 = (z - a)(z^2 - 1)\} \cup \{\infty, \infty\},
\end{equation}

where $a \in \mathbb{C} \setminus \{-1, 1\}$. The map $z: \tilde{M}_a \to \mathbb{C}P^1$ is the holomorphic branched
covering map. The points

$P_0 = z^{-1}(a), \ P_1 = z^{-1}(1), \ P_2 = z^{-1}(-1), \ P_3 = z^{-1}(\infty) = (\infty, \infty)$

are the branch points of $z$ with branch number 1.

The functions $z$ and $w$ are meromorphic on $\tilde{M}_a$ whose divisors are

$$(z) = 2 \cdot P_0 - 2 \cdot P_3, \quad (w) = \sum_{i=0}^{2} 1 \cdot P_i - 3 \cdot P_3.$$

The one-form $dz/w$ is holomorphic on $\tilde{M}_a$. The complex vector space of meromorphic
functions with a unique pole at $P_3$ is spanned by the meromorphic functions $z^i$ and $z^i w$ which are linearly independent over $\mathbb{C} (i = 0, 1, \ldots)$.

Let $M_a := \tilde{M}_a \setminus \{P_3\}$. If $(g, \phi)$ is the Weierstrass representation of a minimal
surface $X: M_a \to \mathbb{R}^3$ of Enneper type such that $g$ has a unique pole at $P_3$, then
$g = Q(z)$ or $Q(z)w$ and $\phi = A dz/w$, where $Q(z)$ is a polynomial of $z$ and $A \in \mathbb{C} \setminus \{0\}$. The symmetry $R$ induces a conformal automorphism $J$ of $\tilde{M}_a$ such that
$J(P_3) = P_3$ and $R \circ X = X \circ J$ ([2]).

**Lemma 2.1.** The conformal automorphism $J$ is holomorphic.

**Proof.** Let $p \in M_a$ be a fixed point of $J$. Then $(R \circ X)(p) = (X \circ J)(p) = X(p)$.
Since $X(p)$ is a fixed point of $R$, the fixed point of $J$ on $M_a$ is nonempty and finite.
Hence $J$ preserves the orientation of $M_a$.

Hence $z \circ J = Bz + C$ ($B \neq 0$). Since the holomorphic automorphism of $\tilde{M}_a$
preserves the branch point set $\{P_0, P_1, P_2, P_3\}$ of $z$, we have $a = 0$ and $z \circ J = -z$.
Thus $w \circ J = iw$.

Let $\gamma$ be a closed curve in $M_0$ such that $z(\gamma)$ is a loop winding once around $-1$
and $0$ and leaves $1$ in the non-bounded component of $\mathbb{C} \setminus z(\gamma)$, $R_n(z) = \int_\gamma z^n w \, dz \in \mathbb{R} \setminus \{0\}$, $S := \int_\gamma dz/w \in \mathbb{R} \setminus \{0\}$, and
$F_n(c) = \sum_{j=0}^{n} R_j c_j^2 + 2 \sum_{0 \leq j < k \leq n} R_{2j} + 2k c_j c_k$. 

\end{proof}
Thus the period condition is reduced to

$$\text{plane}$$

the surfaces cut by the plane

Remark 2.4. Figure 1 shows the surface in Corollary 2.3. The left of Figure 2 shows the surfaces cut by the plane \( \{x_1 = 0\} \) and the right shows the surfaces cut by the plane \( \{x_3 = 0\} \).

**Theorem 2.2.** The moduli space can be identified with \( \bigcup_{n=0}^{\infty} M_n \), where \( M_n \) is the moduli space of the surfaces with total curvature \(-4\pi(2n + 3)\) defined by

$$M_n := \{(A, c) \in \mathbb{R} \times \mathbb{C}^{n+1} | A > 0, c_n \neq 0, S - F_n(c) = 0\},$$

which contains a real \((2n + 1)\)-dimensional smooth subset \((n = 0, 1, 2, \ldots)\).

**Proof.** Let \( N \) is the Gauss map and \( g = \pi \circ N \), where \( \pi \) is the stereographic projection. Since \( R \) is orientation-reversing, \( N \circ J = -R \circ N \) \((2), (2.1)\). Hence \( g \circ J = \pi \circ I \circ R \circ N \) by the symmetry of the surface, where \( I(x) = -x \) for \( x \in \mathbb{R}^3 \).

Thus \( g \circ J = ig \). Thus \( g = Q(z)w \), where \( Q(-z) = Q(z) \). Hence the total curvature of the surfaces are \(-4\pi(2n + 3) \((n = 0, 1, 2, \ldots)\).

The one-form \( \Phi_i \) does not have residue at the end since it has a unique pole \((i = 1, 2, 3)\). The first homology \( H^1(M_n) \) of \( M_n \) is generated by \( \{\gamma, J\gamma\} \). Since \( \Phi_3 = Q(z)dz, \Phi_3 \) is exact. Let \( g = \sum_{j=0}^{\infty} c_j z^{2j}w \) and \( \phi = Adz/w \), where \( c_j, A \in \mathbb{C} \) \((j = 0, 1, 2, \ldots, n)\). Since \( J^*\phi = -\phi \) and \( g \circ J = ig \), we have

$$\begin{align*}
\text{Re} \int_{\gamma} (1 - g^2)\phi &= -\text{Re} \int_{J(\gamma)} \sqrt{-1}(1 + g^2)\phi \\
\text{Re} \int_{\gamma} \sqrt{-1}(1 + g^2)\phi &= \text{Re} \int_{J(\gamma)} (1 - g^2)\phi.
\end{align*}$$

Thus the period condition is reduced to

$$\int_{\gamma} \phi - \int_{\gamma} g^2\phi = 0,$$

that is

$$S\bar{A} - AF_n(c) = 0 \quad (n = 0, 1, 2, \ldots).$$

Hence the set \( \bigcup_{n=0}^{\infty} N_n \), \( N_n := \{(A, c_0, \ldots, c_n) \in \mathbb{C}^{n+2} | A \neq 0, c_n \neq 0, S - AF_n(c) = 0\} \) contains all the surfaces up to rigid motion in \( \mathbb{R}^3 \). The action of the group of rotation around vertical axis with angle \( t \) induces the action of the group \( S^1 = \{e^{it} | t \in \mathbb{R}\} \) on \( N_n \) as \( e^{it} \cdot (A, c) := (Ae^{-it}e^{it}c), \) where \( t \in \mathbb{R}, e^{it}c := (e^{it}c_0, e^{it}c_1, \ldots, e^{it}c_n), \) and \( (A, c) \in N_n \). Hence the moduli space can be identified with \( \bigcup_{n=0}^{\infty} \mathcal{M}_n \). Each \( \mathcal{M}_n \) has an element \( \pm p_n := (A, 0, \ldots, 0, \pm \sqrt{S/R_{n^2}}) \) \((A > 0)\).

Since \( \partial \Phi_n/\partial c_n \)(p_n) = \(2R_{4n}\sqrt{S/R_{4n}} \neq 0\), \( \mathcal{M}_n \) contains \((2n + 1)\) dimensional smooth subset around \( \pm p_n \).

**Corollary 2.3.** There exists a unique complete minimal tori of Enneper type with total curvature \(-12\pi \) having symmetry \( R \) up to homotheties and isometries of \( \mathbb{R}^3 \).

**Proof.** The solution of \( S - F_0(c) = 0 \) is \( \pm \sqrt{S/R_0} \). The surface corresponding to \((g, \phi) = (-\sqrt{S/R_0}w, Adz/w) \) \((A > 0)\) is the reflection of the surface corresponding to \((g, \phi) = (\sqrt{S/R_0}w, Adz/w) \) about the horizontal plane. Since \( A \) is the parameter of homotheties, the corollary holds.
Figure 1. The surface of genus one with one end of total curvature $-12\pi$.

Figure 2. The surfaces cut by \{x_1 = 0\} (left) and by \{x_3 = 0\} (right).

References


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