Second order asymptotics for Brownian motion among heavy tailed Poissonian potentials

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Motivation

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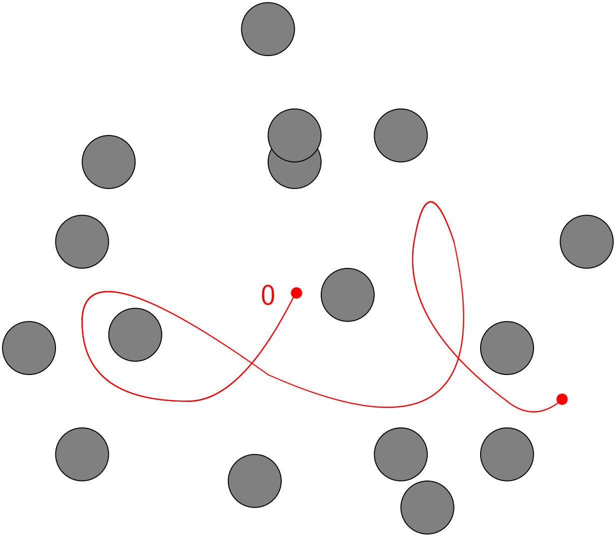
$\longrightarrow$ kill the Brownian motion by a random potential and condition to survive.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1. Setting

- $\left(\{w_t\}_{t\geq 0}, P_x\right)$: $\kappa\Delta$-Brownian motion on $\mathbb{R}^d$
- $\left(\omega = \sum_i \delta_{\omega_i}, \mathbb{P}\right)$: Poisson point process on $\mathbb{R}^d$ with unit intensity

Poisson point process with unit intensity is a random collection of points satisfying

1. If $A \cap B = \emptyset$, then $\omega(A)$ and $\omega(B)$ are independent.
2. $\mathbb{P}(\omega(A) = k) = e^{-|A|}\dfrac{|A|^k}{k!}$. $\left(\mathbb{P}(\omega(A) = 0) = e^{-|A|}\right)$.

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Potential

For an integrable and bounded function v,

$$V(x,\omega) := \sum_i v(x-\omega_i).$$

(Typically $v(x) = 1_{B(0,1)}(x)$ or $|x|^{-\alpha} \wedge 1$ with $\alpha > d$.)

Path measures

We define two measures using the random potential $V(x, \omega)$.
The first one is the quenched path measure:

$$Q_{T,\omega}(\,\cdot\,) = \frac{\exp\left\{-\int_0^T V(w_s, \omega)\, ds\right\} P_0(\,\cdot\,)}{E_0\left[\exp\left\{-\int_0^T V(w_s, \omega)\, ds\right\}\right]}.$$

The configuration is fixed and Brownian motion tries to avoid ω_i's.

The second is the annealed path measure:

$$Q_T(\,\cdot\,) = \frac{\exp\left\{-\int_0^T V(w_s,\omega)\,ds\right\}\mathbb{P}\otimes P_0(\,\cdot\,)}{\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^T V(w_s,\omega)\,ds\right\}\right]}.$$

The configuration is not fixed and hence Brownian motion and ω_i's try to avoid each other.

2. Heuristics

Variational principle

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^T V(w_s, \omega)\, ds \right\} \right]$$
$$\overset{\log}{\sim} \exp \left\{ - \inf_{(w,\omega)} \{\text{energy} + \text{entropy}\} \right\} \quad (T \to \infty),$$

where

$$\begin{cases} \text{energy} & = \int_0^T V(w_s, \omega)\, ds, \\ \text{entropy} & = -\log(\text{“probability” of the sample } (w, \omega)) \\ & = \text{Ent}(\omega) + \text{Ent}(w). \end{cases}$$

$\longrightarrow$ only miniimizers are observed under the conditional measure.

3. Light tailed case

Donsker and Varadhan (1975)

When $v(x) = o(|x|^{-d-2})$ as $|x| \to \infty$,

$$\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^T V(w_s, \omega)\, ds \right\} \right] = \exp\left\{ -c(d,\kappa) T^{\frac{d}{d+2}} (1 + o(1)) \right\} \quad (T \to \infty),$$

where

$$\begin{cases} c(d,\kappa) := \inf_{U \subset \mathbb{R}^d : \mathrm{open}} \{ |U| + \kappa \lambda_1(U) \}, \\ \lambda_1(U) : \text{ Dirichlet smallest eigenvalue of } -\Delta \text{ in } U. \end{cases}$$

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Remark

The minimizer of $\inf \{|U| + \kappa\lambda_1(U)\}$ is $B(x, R_0)$.

Suppose $v = 1_{B(0,1)}$. If $\omega(U) = 0$ and $w_{[0,T]} \subset U$, then

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The entropy of this strategy is

$$\begin{aligned} \text{entropy} &= -\log\{\mathbb{P}(\omega(U) = 0) P_0(w_{[0,T]} \subset U)\} \\ &\sim |U| + \kappa\lambda_1(U) T. \end{aligned} \tag{1}$$

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1. $\mathbb{P}(\omega(U) = 0) = e^{-|U|}$ (by definition).
2. $P_0(w_{[0,T]} \subset U) \overset{\log}{\sim} \exp\{-\kappa\lambda_1(U)T\}$ (the Kac formula).

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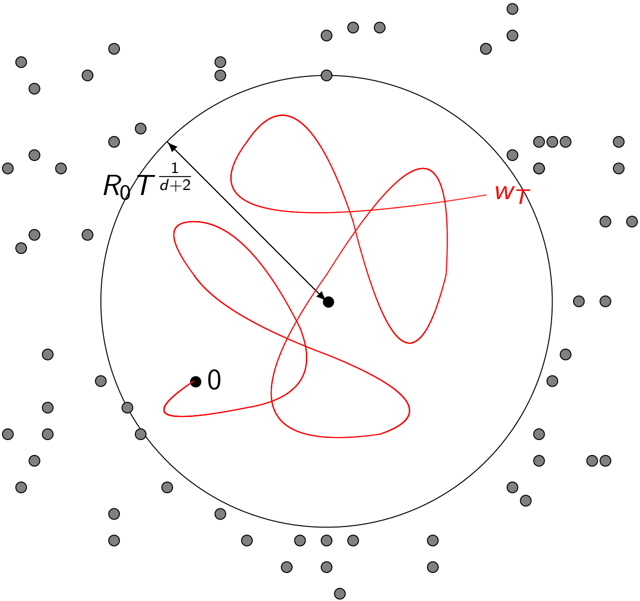
Using the scaling $U \to T^{\frac{1}{d+2}} U'$, we get

$$|U| + \kappa\lambda_1(U)T = T^{\frac{d}{d+2}} \left\{|U'| + \kappa\lambda_1(U')\right\}.$$

The right-hand side of (1) is minimized by $U = B(x, R_0 T^{\frac{1}{d+2}})$.



$R_0 T^{\frac{1}{d+2}}$
w_T
0

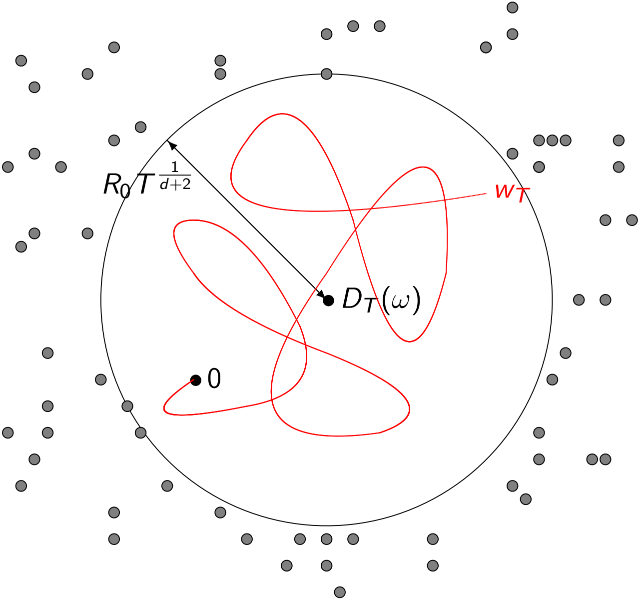
Schmock (1990), Sznitman (1991), and Povel (1999)

When v has a compact support, there exist $\delta(T) \to 0$ $(T \to \infty)$ and

$$D_T(\omega) \in B\big(0, T^{\frac{1}{d+2}}(R_0 + \delta(T))\big)$$

such that

$$Q_T\left(w_{[0,\,T]} \subset B\big(D_T(\omega), T^{\frac{1}{d+2}}(R_0 + \delta(T))\big)\right) \xrightarrow{T\to\infty} 1.$$



$R_0 T^{\frac{1}{d+2}}$
w_T
$D_T(\omega)$
0

4. Heavy tailed case

Pastur (1977)

When $v(x) \sim |x|^{-\alpha}$ $(d < \alpha < d + 2)$,

$$\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^T V(w_s, \omega)\, ds \right\} \right]$$
$$= \exp\left\{ -a_1 T^{\frac{d}{\alpha}} (1 + o(1)) \right\} \quad (T \to \infty),$$

where

$$a_1 := |\partial B(0,1)| \Gamma\left(1 + \frac{d}{\alpha}\right).$$

Note that a_1 is independent of κ. In fact, Pastur proved

$$
\begin{aligned}
&\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^T V(w_s,\omega)\, ds \right\} \right] \\
&\overset{\log}{\sim} \mathbb{E}\left[\exp\left\{ -V(0,\omega)\, T \right\} \right] \\
&= \exp\left\{ -a_1 T^{\frac{d}{\alpha}} (1 + o(1)) \right\}.
\end{aligned}
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This means that $w = 0$ is the best strategy but that the entropy $\mathrm{Ent}(w = 0)$ is negligible.

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&= \exp\left\{ -a_1\, T^{\frac{d}{\alpha}}(1 + o(1)) \right\}.
\end{aligned}$$

This means that $w = 0$ is the best strategy but that the entropy $\mathrm{Ent}(w = 0)$ is negligible.

Moreover, the variational principle holds for the second line:

$$a_1\, T^{\frac{d}{\alpha}} = \inf_{\omega,\, w \equiv 0} \{\mathrm{energy} + \mathrm{Ent}(\omega)\}\,.$$

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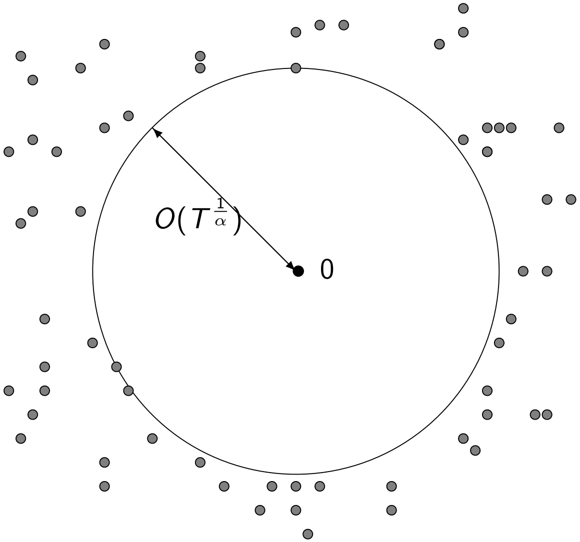
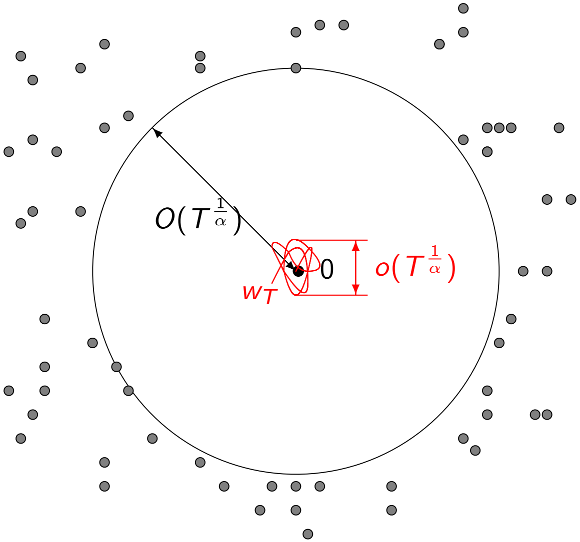


$$\inf_{\omega,\, w\equiv 0} \{\text{energy} + \text{Ent}(\omega)\}\,.$$



Problem: What is the correct scale for the Brownian motion?

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$\longrightarrow$ We need to look at lower order terms.

Theorem (F. in preparation)

When $v(x) = |x|^{-\alpha} \wedge 1$ *with* $d < \alpha < d + 2$,

$$\begin{aligned} &\mathbb{E} \otimes E_0 \left[\exp \left\{ - \int_0^T V(w_s, \omega)\, ds \right\} \right] \\ &= \exp \left\{ -a_1 T^{\frac{d}{\alpha}} - (a_2 + o(1))\, T^{\frac{\alpha+d-2}{2\alpha}} \right\} \end{aligned}$$

as $T \to \infty$, *where*

$$\begin{cases} a_2 = \inf_{\|\phi\|_2 = 1} \left\{ \int \kappa |\nabla \phi(x)|^2 + \tilde{c}(d,\alpha) |x|^2 \phi(x)^2 \, dx \right\}, \\ \tilde{c}(d,\alpha) = \frac{1}{2} |\partial B(0,1)| \Gamma \left(\frac{\alpha+2}{d} \right) \end{cases}$$

This theorem indicates that

$$\mathrm{Ent}(w) \asymp T^{\frac{\alpha+d-2}{2\alpha}}.$$

We expect that this entropy measures the cost for Brownian motion to stay in a small region.

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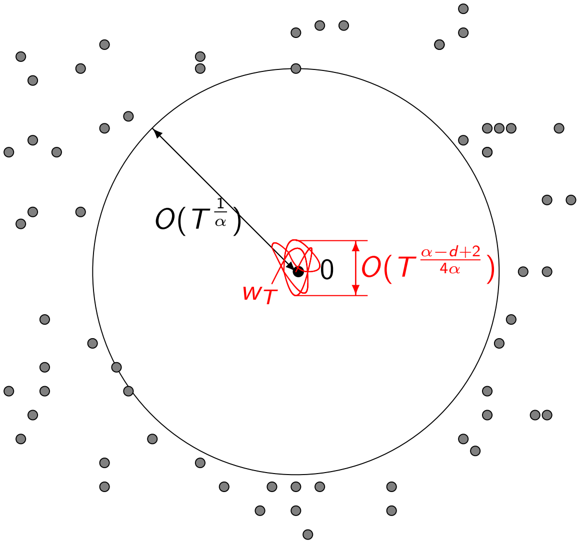
By the Brownian scaling, we know that

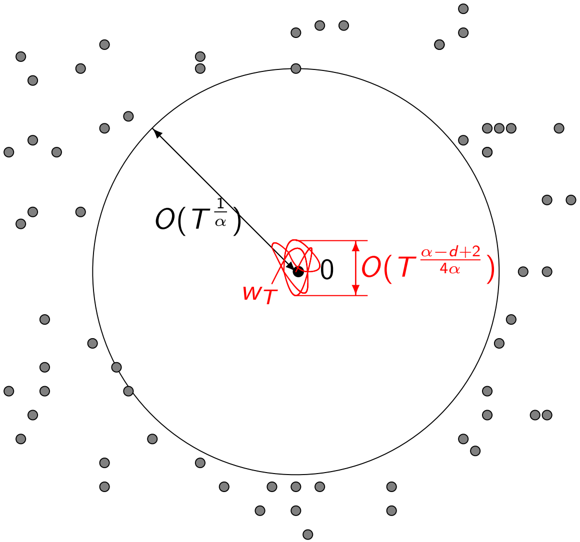
$$\log P_0(w_{[0,T]} \subset B(0,R)) \asymp -TR^{-2}$$

and hence

$$R_{\mathrm{correct}} \asymp T^{\frac{\alpha-d+2}{4\alpha}}.$$



$O(T^{\frac{1}{\alpha}})$
0
$O(T^{\frac{\alpha-d+2}{4\alpha}})$
w_T



$$a_2 = \inf_{\|\phi\|_2=1} \left\{ \int \kappa |\nabla \phi(x)|^2 + \tilde{c}(d,\alpha)|x|^2 \phi(x)^2 \, dx \right\}$$

Thank you!

Theorem (F. in preparation)

Suppose $v(x) = |x|^{-\alpha} \wedge 1$ *with* $d < \alpha < d+2$. *Then for* $\mathbb{P}$*-almost every* ω,

$$E_0\left[\exp\left\{-\int_0^T V_\omega(w_s)\,ds\right\}\right] = \exp\left\{-q_1 T(\log T)^{-\frac{\alpha-d}{d}} - (q_2 + o(1))\,T(\log T)^{-\frac{\alpha-d+2}{2d}}\right\}$$

as $T \to \infty$, *where*

$$\begin{cases} q_1 = \dfrac{d}{\alpha}\left(\dfrac{\alpha-d}{\alpha d}\right)^{\frac{\alpha-d}{d}} a_1^{\frac{\alpha}{d}}, \\ q_2 = \left(\dfrac{\alpha-d}{\alpha d}a_1\right)^{\frac{\alpha-d+2}{2d}} a_2. \end{cases}$$

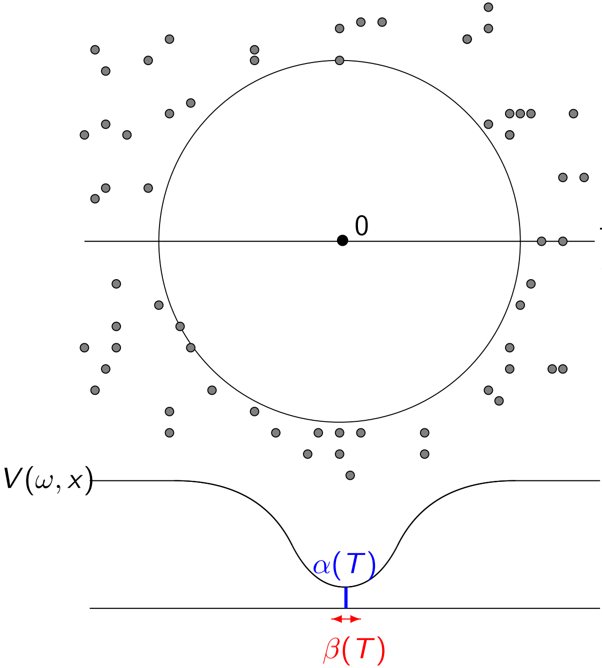
Theorem (F. in preparation)

Suppose $v(x) = |x|^{-\alpha} \wedge 1$ *with* $d < \alpha < d + 2$. *Then,*

$$N(\lambda) = \exp\left\{-\ell_1 \lambda^{-\frac{d}{\alpha-d}} - (\ell_2 + o(1))\lambda^{-\frac{\alpha+d-2}{2(\alpha-d)}}\right\}$$

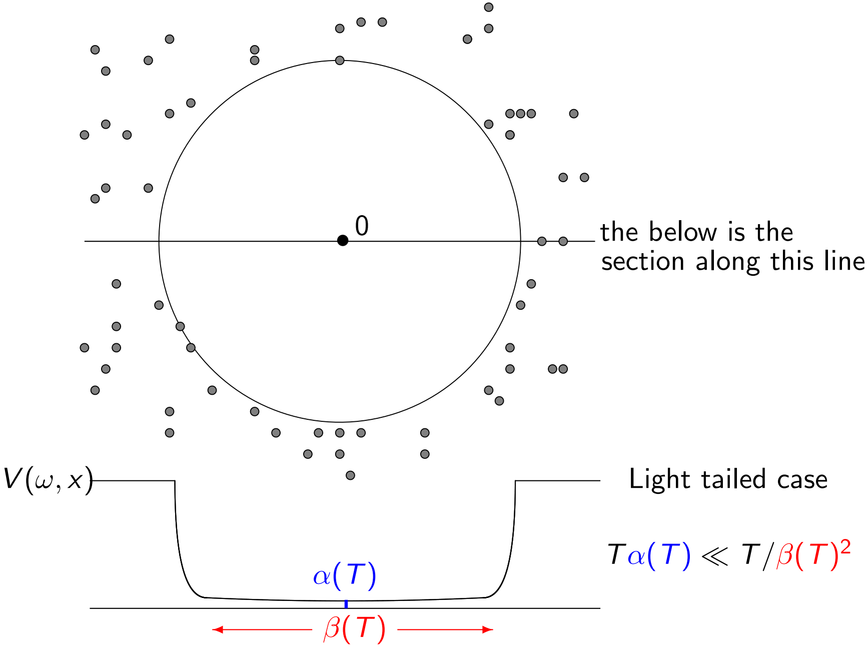
as $\lambda \downarrow 0$, *where*

$$\begin{cases} \ell_1 := \dfrac{\alpha - d}{\alpha}\left(\dfrac{d}{\alpha}\right)^{\frac{d}{\alpha-d}} a_1^{\frac{\alpha}{\alpha-d}}, \\ \ell_2 := a_2\left(\dfrac{da_1}{\alpha}\right)^{\frac{\alpha+d-2}{2(\alpha-d)}}. \end{cases}$$



Heavy tailed case

$T\alpha(T) \gg T/\beta(T)^2$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the below is the
section along this line
$V(\omega, x)$
Light tailed case
$T\alpha(T) \ll T/\beta(T)^2$
$\alpha(T)$
$\beta(T)$