Slowdown estimates for the biased random walk in spatially inhomogeneous holding times

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Work in progress with Amir Dembo (Stanford university and Kyoto university) and Naoki Kubota (Nihon university)

Biased random walk with "random" holding times

- $(\{\mu(x)\}_{z\in\mathbb{Z}}, \mathbb{P})$: IID, positive, mean 1,
$$\mathbb{P}(\mu(x) > r) = C\exp\{-r^{\alpha}\}, \quad \alpha > 0.$$
- $(\{X_t\}_{t\geq 0}, \{P_x^{\mu}\}_{x\in\mathbb{Z}})$: jumps $x \to x \pm 1$ with rate $\frac{1\pm\theta}{2\mu(x)}$.

Law of large numbers holds: $\mathbb{P}P_0^{\mu}\left(\lim_{t\to\infty}\frac{1}{t}X_t = \theta\right) = 1$.

Large deviation principle (LDP) holds as a special case of Dembo, Gantert and Zeitouni 2004.
But when μ is unbounded, the rate function is zero on $[0,\theta]$: for $v < \theta$,
$$P_0^{\mu}(X_t \leq vt) = \exp\{o(t)\}.$$

Biased random walk with "random" holding times

- $(\{\mu(x)\}_{z\in\mathbb{Z}}, \mathbb{P})$: IID, positive, mean 1,

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Law of large numbers holds: $\mathbb{P}P_0^\mu\left(\lim_{t\to\infty}\frac{1}{t}X_t = \theta\right) = 1$.

- Annealed lower bound: $\forall v < \theta$,

$$\mathbb{P}P_0^\mu(Xt \leq vt) \geq \mathbb{P}\left(\mu(0) > t^{\frac{1}{\alpha+1}}, P_0^\mu(\tau_1(0) \geq t)\right)$$
$$\geq \exp\left\{-ct^{\frac{\alpha}{\alpha+1}}\right\}.$$

- Quenched lower bound: $\forall v < \theta$, $\mathbb{P}$-a.s.,

$$\max_{0\leq x\leq vt-1}\mu(x) \geq (1-o(1))(\log t)^{\frac{1}{\alpha}},$$
$$\Rightarrow P_0^\mu(X_t \leq vt) \geq \exp\left\{-ct/(\log t)^{\frac{1}{\alpha}}\right\}.$$

Toward the upper bound: time change

Goal: Matching upper bound:

$$\mathbb{P}P_0^\mu(Xt \le vt) \le \exp\left\{-ct^{\frac{\alpha}{\alpha+1}}\right\},$$
$$P_0^\mu(X_t \le vt) \le \exp\left\{-ct/(\log t)^{\frac{1}{\alpha}}\right\}.$$

- $(\{S_t\}_{t\ge 0}, \{P_x\}_{x\in\mathbb{Z}})$: biased RW with unit jump rate,

$$A_\mu(t) = \int_0^t \mu(S_u)\mathrm{d}u \quad \Rightarrow \quad X_t = S_{A_\mu^{-1}(t)}.$$

It suffices to study this additive functional:

$$\begin{aligned} P_0^\mu(X_t \le (v-\epsilon)t) &\le P_0^\mu(H(vt) \ge t) + \exp\{-c_\epsilon t\} \\ &\sim P_0(H(vt) \ge A_\mu^{-1}(t)) \\ &= P_0(A_\mu(H(vt)) \ge t). \end{aligned}$$

Upper bound: moment generating function

- $\mathcal{G}_t^1 = \left\{\mu\colon \max_{-\epsilon t < x < vt} \mu(x) \le M(t)\right\}$,
  $(M(t) = t^{\frac{1}{\alpha+1}}$ or $(1+\epsilon)(\log t)^{\frac{1}{\alpha}})$,
- $\mathcal{G}_t^2 = \left\{\mu\colon \sum_{-\epsilon t < x < vt} \mu(x) < (v+2\epsilon)t\right\}$,
- $f_\epsilon(x,y) = E_x\left[\exp\left\{\frac{\epsilon}{M(t)}\int_0^{H(y)\wedge H(-\epsilon t)} \mu(S_u)\mathrm{d}u\right\}\right]$, $(x<y)$.

Trivial bound: $f_\epsilon(x,y) \le E_x[\exp\{\epsilon H(y)\}] \le (1+\delta(\epsilon))^{y-x}$.

This bound is useless as it is but noting that

$$(\mathcal{L}_\theta + \epsilon M(t)^{-1}\mu) f_\epsilon(\cdot, y) = 0, \ldots$$

Upper bound: moment generating function

$$f_\epsilon(y-1,y) = 1 + \frac{\epsilon}{M(t)}(-\mathcal{L}_\theta|_{(-\epsilon t,y)})^{-1}(\mu f_\epsilon(\cdot,y))$$
$$\le 1 + \frac{\epsilon}{M(t)} \sum_{-\epsilon t<z<y} G^\theta_{(-\epsilon t,y)}(y-1,z)\mu(z)(1+\delta(\epsilon))^{y-z}.$$

Since $G^\theta_{(-\epsilon t,y)}(y-1,z) \le c(\frac{1-\theta}{1+\theta})^{y-z}$,

$$\log E_0\left[\exp\left\{\frac{\epsilon}{M(t)}\int_0^{H(vt)\wedge H(-\epsilon t)} \mu(S_u)\mathrm{d}u\right\}\right]$$
$$= \sum_{1\le y\le vt} \log f_{\epsilon,t}(y-1,y)$$

Upper bound: moment generating function

$$
\begin{aligned}
f_\epsilon(y-1,y) &= 1 + \frac{\epsilon}{M(t)}(-\mathcal{L}_\theta|_{(-\epsilon t,y)})^{-1}(\mu f_\epsilon(\cdot,y)) \\
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\end{aligned}
$$

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$$
\begin{aligned}
&\log E_0\left[\exp\left\{\frac{\epsilon}{M(t)}\int_0^{H(vt)\wedge H(-\epsilon t)} \mu(S_u)\mathrm{d}u\right\}\right] \\
&\le \sum_{1\le y\le vt} \log\left(1+\frac{\epsilon}{M(t)}\sum_{z\in(-\epsilon t,y)} G^\theta_{(-\epsilon t,y)}(y-1,z)\mu(z)(1+\delta(\epsilon))^{y-z}\right)
\end{aligned}
$$

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$$\le \frac{\epsilon}{M(t)} \sum_{z\in(-\epsilon t,vt)} \mu(z) \sum_{y>z} G^\theta_{(-\epsilon t,y)}(y-1,z)(1+\delta(\epsilon))^{y-z}$$

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$$\leq \frac{\epsilon}{M(t)} \sum_{z\in(-\epsilon t,vt)} \mu(z)\sum_{y>z} G^\theta_{(-\infty,0)}(-1,z-y)(1+\delta(\epsilon))^{y-z}$$

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$$\leq \frac{\epsilon}{M(t)} \sum_{z\in(-\epsilon t,vt)} \mu(z)\left(\frac{1}{\theta}+o(1)\right), \epsilon \to 0.$$

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$$\leq \frac{\epsilon t}{M(t)}\left(\frac{v}{\theta}+o(1)\right), \epsilon \to 0.$$

Now we have

$$E_0\left[\exp\left\{\frac{\epsilon}{M(t)}A_\mu(H(vt)\wedge H(-\epsilon t))\right\}\right] \le \exp\left\{\frac{\epsilon t}{M(t)}\left(\frac{v}{\theta}+o(1)\right)\right\}$$

and by Chebyshev's inequality

$$\begin{aligned} &P_0^\mu(H(vt)\wedge H(-\epsilon t)\ge t) \\ &\quad = P_0(A_\mu(H(vt)\wedge H(-\epsilon t))\ge t) \\ &\quad \le \exp\left\{\frac{\epsilon t}{M(t)}\left(-1+\left(\frac{v}{\theta}+o(1)\right)\right)\right\} \end{aligned}$$

We can drop $H(-\epsilon t)$ since $P_0(H(-\epsilon t)<H(vt))\le \exp\{-ct\}$. $\square$

RWRE with uniformly positive drift

Reviewing the argument, what we needed was

$$E_{z-1}[\exp\{\epsilon H(z)\}] \leq 1 + o(1) \text{ as } \epsilon \to 0 \text{ and}$$

$$\sum_{y\in(-\epsilon t, vt)} \sum_{z\in(-\epsilon t, y)} G^{\theta}_{(-\epsilon t, y)}(y-1, z)\mu(z)(1+\delta(\epsilon))^{y-z} \sim \frac{v}{\theta} t.$$

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$$E^{\omega}_{z-1}[\exp\{\epsilon H(z)\}] \leq 1 + o(1) \text{ as } \epsilon \to 0 \text{ and}$$

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This can be checked even for RWRE if $\operatorname{essinf} \omega > 1/2$.
($\Rightarrow$ dominated convergence)

Reminder:

- $(\{\omega(x)\}_{x\in\mathbb{Z}}, \alpha)$: random variables in [0,1].
- $P^{\omega}(X_{n+1} = x+1 | X_n = x) = \omega(x)$,
  $P^{\omega}(X_{n+1} = x-1 | X_n = x) = 1 - \omega(x)$

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Quenched:

$$\sum_{y \in (-\epsilon t, vt)} \sum_{z \in (-\epsilon t, y)} G^{\omega}_{(-\epsilon t, y)}(y-1, z) \sim E^{\omega}_{-\epsilon t}[H(vt)],$$

$$\sum_{z \in (-\epsilon t, vt)} (\mu(z) - 1) \sum_{y \in (z, vt)} G^{\omega}_{(-\epsilon t, y)}(y-1, z)$$

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Annealed:

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RWRE with uniformly positive drift

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This can be checked even for RWRE if $\operatorname{essinf} \omega > 1/2$.
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Annealed: good control on the deviations from

$$\sum_{y\in(-\epsilon t, vt)} \sum_{z\in(-\epsilon t, y)} G^{\omega}_{(-\epsilon t, y)}(y-1, z) \sim \frac{v}{v_\alpha} t,$$

$$\sum_{z\in(-\epsilon t, vt)} (\mu(z)-1) \sum_{y\in(z, vt)} G^{\omega}_{(-\epsilon t, y)}(y-1, z) = o(t).$$

Large deviations for random walk in random environment

- Greven and den Hollander 1994: Quenched LDP for 1-dim. random walk in IID environment, via environment viewed from the walker. Rate functions look like:

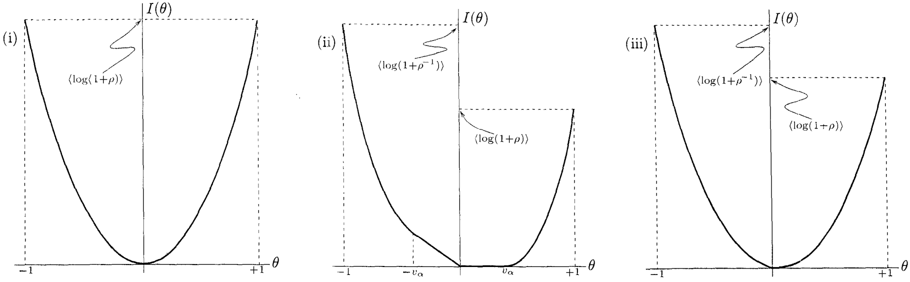

FIG. 11. (i) recurrent; (ii) transient: positive speed; (iii) transient: zero speed

Figures copied from "Large deviations" by Frank den Hollander.

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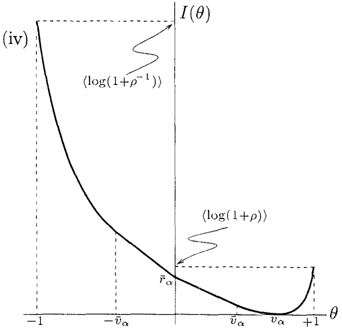

FIG. 14. (iv) transient: uniformly positive drifts (see Fig. 11)

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Large deviations for random walk in random environment

- Comets, Gantert and Zeitouni 2000: LDP for 1-dim. RWRE, quenched under ergodicity and annealed under a mixing condition, via hitting time decomposition.

$$I_\alpha^{\mathrm{a}}(x) = \inf_\beta \{ I_\beta^{\mathrm{q}}(x) + H(\beta|\alpha) \}.$$

The idea is to use

$$P_0^\omega(X_n \approx vn) \approx P_0^\omega(H(vn) \approx n)$$

and

$$\begin{aligned} E_0^\omega[\exp\{\lambda H(vn)\}] &= \prod_{x=1}^{vn} E_{x-1}^\omega[\exp\{\lambda H(x)\}] \\ &= \prod_{x=1}^{vn} E_{-1}^{\theta_x \omega}[\exp\{\lambda H(0)\}]. \end{aligned}$$

Large deviations for random walk in random environment

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$$I_\alpha^{\mathrm{a}}(x) = \inf_\beta \{ I_\beta^{\mathrm{q}}(x) + H(\beta|\alpha) \}.$$

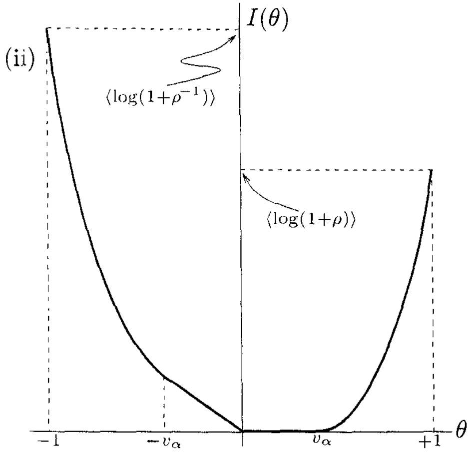
- Dembo, Gantert and Zeitouni 2004: LDP for 1-dim. RWRE with holding times, via hitting time decomposition.

$\tau_k(x)$: k-th holding time at x, the law depends on x.

- Multi-dimension: Zerner 2000 (Quenched LDP for IID), Varadhan 2006 (Quenched LDP for ergodic, Annealed LDP), Rosenbluth, Yilmaz, Rassoul-Agha & Seppäläinen, etc.

Slowdown estimates for RWRE ($I(v) = 0$ for $v \in [0, v_\alpha]$)

When essinf $\omega \leq 1/2$ and $\langle \rho \rangle := \alpha \left[\frac{1-\omega}{\omega} \right] < 1$,



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- Dembo, Peres and Zeitouni 1996: Annealed slowdown estimates for 1-dim. random walk in IID environment.
  1. essinf $\omega < 1/2 <$ esssup ω and $\langle \rho^s \rangle = 1$:
  $$\alpha \times P_0^\omega(X_n \le vn) = n^{-s+1+o(1)},$$
  2. essinf $\omega = 1/2$ and $\alpha(\omega = 1/2) > 0$:
  $$\alpha \times P_0^\omega(X_n \le vn) \asymp \exp\{-n^{1/3}\}.$$

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- Refinements: Pisztora, Povel and Zeitouni, Ahn and Peterson.
- Variant: Fribergh, Gantert and Popov.

Slowdown by holding times

Dembo, Gantert and Zeitouni 2004 (Annals of Probability):

"Subexponential decay of slowdown probabilities is possible ... it seems that the techniques of these papers can be extended to the RWREH. The possible subexponential rates of decay for the RWREH are influenced by the tails of the holding time distribution, and hence not limited to those present in the RWRE model."

Remark

When the holding times has "polynomial tail" or essinf $\omega < 1/2$*, then it is indeed possible to follow Dembo, Peres and Zeitouni 1996 to obtain annealed result and then Gantert and Zeitouni 1998 to transfer the annealed results to quenched results.*

What remains?

1. essinf $\omega = 1/2$ and $\mathbb{P}(\mu(x) > r) = C \exp\{-r^\alpha\}$.
2. Non-exponential holding times.

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1. essinf $\omega = 1/2$ and $\mathbb{P}(\mu(x) > r) = C \exp\{-r^{\alpha}\}$.
2. Non-exponential holding times.

One possible difficulty is unboundedness of $E^{\omega}_{z-1}[\exp\{\epsilon H(z)\}]$.

Old approach for annealed slowdown

Back to biased random walk in "random" holding times.

Our old approach covered only stretched exponential tail case

$$\mathbb{P}(\mu(x) > r) = C \exp\{-r^\alpha\}, \quad \alpha \leq 1$$

but it did not rely on $E_{z-1}[\exp\{\epsilon H(z)\}] < \infty$.

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but it did not rely on $E_{z-1}[\exp\{\epsilon H(z)\}] < \infty$.

Remember that we never worked with $\mathbb{E}E^{\mu}_{z-1}[\exp\{\epsilon H(z)\}]$.
We considered “good” events instead.

We can also consider a good event for the discrete time RW $Z_n = X_{J(n)}$.

Old approach for annealed slowdown

Let $\ell_{vt}(x)$ denote the number of visits of Z to x until $H(vt)$. Also $(\{(\tau_k(x))\colon k \geq 1, x \in \mathbb{Z}\}, P_\mu^{\mathrm{HT}})$ denotes the holding times.

- $\mathcal{G}_t^1 = \left\{\mu\colon \max_{-\epsilon t < x < vt} \mu(x) \leq t^{\frac{1}{\alpha+1}}\right\}$,
- $\mathcal{G}_t^3 = \left\{Z\colon \inf_{0 \leq n < H(vt)} Z_n > -\epsilon t\right\}$.
- $\mathcal{G}_t^4 = \left\{Z\colon \sup_{-\epsilon t < x < vt} \ell_{vt}(x) \leq t^{\frac{\alpha}{\alpha+1}}\right\}$.
- $\mathcal{G}_t^5 = \left\{Z\colon \sum_{-\epsilon t < x < vt} \ell_{vt}(x) \leq (v + 2\epsilon)t\right\}$.

On $\mathcal{G}_t^1 \cap \mathcal{G}_t^3 \cap \mathcal{G}_t^4$,

$$
\begin{aligned}
&E_\mu^{\mathrm{HT}}\left[\exp\left\{\epsilon t^{-\frac{1}{\alpha}} H^X(vt)\right\}\right] \\
&= E_\mu^{\mathrm{HT}}\left[\exp\left\{\epsilon t^{-\frac{1}{\alpha+1}} \sum_{-\epsilon t<x<vt} \sum_{k=1}^{\ell_{vt}(x)} \tau_k(x)\right\}\right] \\
&= \prod_{-\epsilon t<x<vt} \left(1-\epsilon t^{-\frac{1}{\alpha+1}}\mu(x)\right)^{-\ell_{vt}(x)}
\end{aligned}
$$

and we can use $(1-x)^{-1} \le \exp\{x/(1-\epsilon)\}$ $(x \le \epsilon)$ to get

$$
\begin{aligned}
&\mathbb{E}\left[(1-\epsilon t^{-\frac{1}{\alpha+1}}\mu(x))^{-\ell} \colon \mu(x) \le t^{\frac{1}{\alpha+1}}\right] \\
&\le \alpha C \int_0^{t^{\frac{1}{\alpha+1}}} r^{\alpha-1} \exp\left\{\frac{\epsilon}{1-\epsilon} t^{-\frac{1}{\alpha+1}} \ell r - r^\alpha\right\} \mathrm{d}r.
\end{aligned}
$$

$\alpha \le 1$ and $\ell \le t^{\frac{\alpha}{\alpha+1}} \Rightarrow \frac{\epsilon}{1-\epsilon} t^{-\frac{1}{\alpha+1}} \ell$ is “small” $\Rightarrow$ “linearized”.

$$
\begin{aligned}
&\mathbb{E}\left[(1-\epsilon t^{-\frac{1}{\alpha+1}}\mu(x))^{-\ell}\colon \mu(x)\le t^{\frac{1}{\alpha+1}}\right]\\
&\le \alpha C\int_0^{t^{\frac{1}{\alpha+1}}} r^{\alpha-1}\exp\Big\{\frac{\epsilon}{1-\epsilon}t^{-\frac{1}{\alpha+1}}\ell r - r^{\alpha}\Big\}\mathrm{d}r\\
&\sim 1+\frac{\epsilon}{1-\epsilon}t^{-\frac{1}{\alpha+1}}\ell.
\end{aligned}
$$

Using $1+x\le e^x$, we find that on $\mathcal{G}_t^3\cap\mathcal{G}_t^4\cap\mathcal{G}_t^5$,

$$
\begin{aligned}
&\mathbb{E}E_\mu^{\mathrm{HT}}\big[\exp\big\{\epsilon t^{-\frac{1}{\alpha+1}}H^X(vt)\big\}\colon \max_{-\epsilon t<x<vt}\le t^{\frac{1}{\alpha+1}}\big]\\
&\lesssim \exp\Big\{\frac{\epsilon}{1-\epsilon}t^{-\frac{1}{\alpha+1}}\sum_{-\epsilon t<x<vt}\ell_{vt}(x)\Big\}\\
&\le \exp\Big\{\frac{\epsilon}{(1-\epsilon)}\frac{v+2\epsilon}{v_\alpha}t^{\frac{\alpha}{\alpha+1}}\Big\}
\end{aligned}
$$

and we are done. □

Happy 60th birthday Francis!

祝 還暦