Geometry of the random walk range conditioned on survival among Bernoulli obstacles

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Setting

- $(S := (S_n)_{n\geq 0}, \mathbf{P}_x)$: SRW on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$;
- $(\omega = (\omega_x)_{x\in\mathbb{Z}^d}, \mathbb{P})$: IID, Bernoulli(p).

Let $\mathcal{O} = \{x \in \mathbb{Z}^d : \omega_x = 0\}$. The random walk is killed upon hitting $\mathcal{O}$:

$$\tau_{\mathcal{O}} := \inf\{n \geq 0 : S_n \in \mathcal{O}\}.$$

The question is how S (and $\mathcal{O}$) behaves conditioned on $\{\tau_{\mathcal{O}} > N\}$, i.e., under the measure

$$\mu_N((S, \mathcal{O}) \in \cdot) := \mathbb{P} \otimes \mathbf{P}((S, \mathcal{O}) \in \cdot \mid \tau_{\mathcal{O}} > N).$$

This is called the *annealed law* since the average is taken over the environment.

Setting

In particular, we are interested in the law of the random walk range

$$S_{[0,N]} := \{S_i : 0 \leq i \leq N\}$$

under the conditioned measure $\mu_N = \mathbb{P} \otimes \mathbf{P}(\cdot \mid \tau_{\mathcal{O}} > N)$.

The range is "intrinsic" to μ_N. Since

$$\mathbb{P}(\tau_{\mathcal{O}} > N) = \mathbb{P}(S_{[0,N]} \cap \mathcal{O} = \emptyset) = p^{|S_{[0,N]}|},$$

one can integrate out the $\mathcal{O}$-marginal to find

$$\mu_N(S \in \cdot) = \frac{\mathbf{E}\left[p^{|S_{[0,N]}|} : S \in \cdot\right]}{\mathbf{E}\left[p^{|S_{[0,N]}|}\right]}.$$

This can be viewed as a *self-attractive polymer* model.

Earlier works 1: partition function

The first result I mention is due to Donsker–Varadhan (1979).

Theorem

For $d \geq 2$,

$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) = \exp\Big\{-c(d,p)N^{\frac{d}{d+2}}(1+o(1))\Big\},$$

$$\text{with } c(d,p) = \inf_U \{|U| \log(1/p) + \lambda(U)\},$$

where $\lambda(U)$ is the principal Dirichlet eigenvalue of $-\frac{1}{2d}\Delta$ in U.

Remark

Due to the Faber–Krahn isoperimetric inequality, the infimum is achieved by a ball $B(0; \varrho_1)$.

Earlier works 1: partition function

The proof roughly goes as follows:

$$\begin{aligned}\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) &= \sum_{U} \mathbb{P}(\mathcal{O} \cap U = \emptyset)\mathbf{P}(S_{[0,N]} = U) \\ &\approx \max_{U} p^{|U|} \exp\{-N\lambda(U)\} \\ &= \exp\left\{-N^{\frac{d}{d+2}} \inf_{U}\{|U| \log(1/p) + \lambda(U)\}\right\}.\end{aligned}$$

The above approximation is a kind of Laplace principle.

- Donsker–Varadhan proved it by the large deviation principle,
- Antal (1995) gave another proof by Sznitman's "method of enlargement of obstacles".

Anyway, this "indicates" that the best strategy —to stay in a ball of radius $\varrho_N = \varrho_1 N^{\frac{1}{d+2}}$— dominates others.

Earlier works 2: confinement property

This "indication" has been made rigorous by Sznitman (1991), Bolthausen (1994) and Povel (1999) in the following stronger form:

Theorem (Confinement property)

For any $d \geq 2$*, there exist* $\epsilon_1 \in (0,1)$ *and* $x_N = x_N(\mathcal{O}) \in B(0; \varrho_N)$ *such that*

$$\lim_{N\to\infty} \mu_N\big(S_{[0,N]} \subset B(x_N; \varrho_N + \varrho_N^{\epsilon_1})\big) = 1.$$

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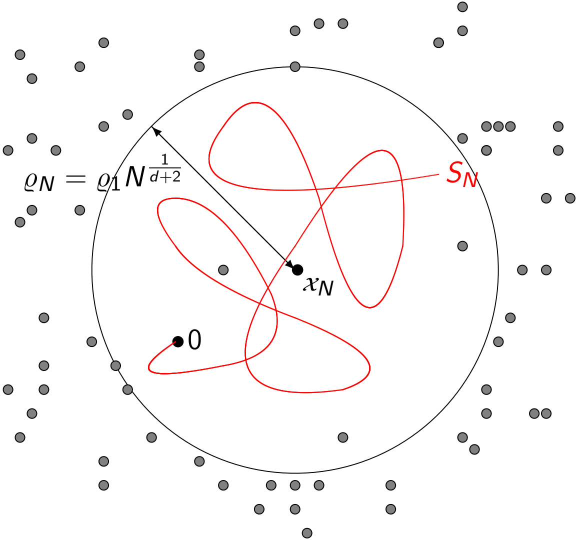
Remark

Why "stronger"? Because the large deviation principle only tells us that the random walk spends most of the time in a ball $B(x;\varrho_N)$.

Earlier works 2: confinement property



This picture is a bit misleading since almost all the sites should be visited $N/N^{\frac{d}{d+2}} = N^{\frac{2}{d+2}}$ times.

Earlier works 2.5: clearing/covering ball

In dimension two, we know more.

Proposition (Ball clearing: Sznitman (1991))

Let $d = 2$. Then for any $\epsilon \in (0,1)$,

$$\lim_{N\to\infty} \mu_N(\mathcal{O} \cap B(x_N; (1-\epsilon)\varrho_N) = \emptyset) = 1.$$

Proposition (Ball covering: Bolthausen (1994))

Let $d = 2$. Then for any $\epsilon \in (0,1)$,

$$\lim_{N\to\infty} \mu_N\big(B(x_N; (1-\epsilon)\varrho_N) \subset S_{[0,N]}\big) = 1.$$

Bolthausen used this in his proof of the confinement property and he conjectured that this remains true for $d \geq 3$.

Main result 1: ball covering in $d \geq 3$

Theorem (Ball covering: Ding, F., Sun, Xu (2018))

Let $d \geq 2$, and let ϱ_N and x_N be as in the confinement property. Then there exists $\epsilon_2 \in (0,1)$ such that

$$\lim_{N \to \infty} \mu_N \big(B(x_N; \varrho_N - \varrho_N^{\epsilon_2}) \subset S_{[0,N]} \big) = 1.$$

Remark

This confirms Bolthausen's conjecture in 1994. However, our proof relies on the confinement property and hence does not give a way to extend Bolthausen's proof of confinement to $d \geq 3$. Recently, Berestycki and Cerf announced a proof of the ball covering without assuming the confinement (arXiv:1811.04700).

Main result 2: boundary size

The confinement property and the ball covering theorem implies

$$\partial S_{[0,N]} \subset B(x_N; \varrho_N + \varrho_N^{\epsilon_1}) \setminus B(x_N; \varrho_N - \varrho_N^{\epsilon_2}).$$

The following theorem is a step toward understanding the surface fluctuation:

Theorem (Boundary size: Ding, F., Sun, Xu (2018))

Let $d \geq 2$, and let ϱ_N be as in the confinement property. Then there exists $\epsilon_3 > 0$ such that

$$\lim_{N\to\infty} \mu_N\big(|\partial S_{[0,N]}| \leq \varrho_N^{d-1}(\log \varrho_N)^{\epsilon_3}\big) = 1.$$

A consequence: partition function asymptotics

Lubensky (1984) deduced from a field theoretic computation that

$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) = \exp\Big\{-c(d,p)N^{\frac{d}{d+2}} - a_1 N^{\frac{d-1}{d+2}} + o(N^{\frac{d-1}{d+2}})\Big\}.$$

Mathematically: $-c_1 N^{\frac{d-1}{d+2}} \leq \text{2nd term} \leq c_2 N^{\frac{d-\kappa}{d+2}}$ for $\exists \kappa \in (0,1)$.

Our control on the size of the boundary allows us to substantially reduce the summands in

$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) = \sum_U \mathbb{P}(\mathcal{O} \cap U = \emptyset)\mathbf{P}(S_{[0,N]} = U)$$

so that we can deduce the following modest improvement:

$$\mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N) \leq \exp\Big\{-c(d,p)N^{\frac{d}{d+2}} + cN^{\frac{d-1}{d+2}}(\log N)^{\epsilon_3+1}\Big\}.$$

Related model

There is a general framework containing our setting called the parabolic Anderson model. For IID $\{\omega(x)\}_{x\in\mathbb{Z}^d}$,

$$\mu_N(\cdot) \propto \mathbb{E}\otimes\mathbf{E}\left[\exp\left\{\sum_{k=1}^{N}\omega(S_k)\right\} : (S,\omega)\in\cdot\right],$$

$$\mu_N^{\omega}(\cdot) \propto \mathbf{E}\left[\exp\left\{\sum_{k=1}^{N}\omega(S_k)\right\} : S\in\cdot\right].$$

- $\omega\in\{-\infty,0\}\longrightarrow$ Bernoulli obstacles;
- more generally, $\omega\le 0\longrightarrow$ Repulsive impurities;
- $\omega\ge 0\longrightarrow$ Attractive impurities.

Various localization results exist depending on the distribution of ω. But often the random walk range tends to be "smeared" and does not converge to a rigid shape.

Proof Idea for Ball Covering

Proof idea for weak version of ball covering

Our proof heavily relies on comparison arguments. The following lemma gives an illustrative example:

Lemma (clearing implies covering)

Suppose $\mu_N(\mathcal{O} \cap B(x_N; (1-\epsilon)\varrho_N) = \emptyset) = 1 - o(\varrho_N^{-d})$.

Then, $\lim_{N\to\infty} \mu_N\big(B(x_N; (1-\epsilon)\varrho_N) \subset S_{[0,N]}\big) = 1.$

Proof.

Suppose $\mu_N(\exists x \in B(x_N; (1-\epsilon)\varrho_N) \setminus S_{[0,N]}) \geq c > 0$. Then there is a point x such that

$$\mu_N(x \in B(x_N; (1-\epsilon)\varrho_N) \setminus S_{[0,N]}) \geq c\varrho_N^{-d}.$$

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$$\mu_N(x \in B(x_N; (1-\epsilon)\varrho_N) \setminus S_{[0,N]}) \geq c\varrho_N^{-d}.$$

But the left-hand side is bounded by

$$\frac{1}{1-p}\mu_N(x \in B(x_N; (1-\epsilon)\varrho_N) \setminus S_{[0,N]} \text{ and } x \in \mathcal{O})$$

and this contradicts the assumption. □

Proof idea for ball clearing

<u>To show</u>: $\lim_{N\to\infty} \mu_N(\mathcal{O} \cap B(x_N; (1-\epsilon)\varrho_N) = \emptyset) = 1.$

Suppose $x \in \mathcal{O} \cap B(x_N; (1-\epsilon)\varrho_N)$. Then, either

1. $B(x; \epsilon\varrho_N/2)$ contains a large density of obstacles or
2. $B(x; \epsilon\varrho_N/2)$ contains a small density of obstacles.

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1. $B(x; \epsilon\varrho_N/2)$ contains a large density of obstacles or
2. $B(x; \epsilon\varrho_N/2)$ contains a small density of obstacles.

- Case 1 is easy to exclude since it makes too hard for the random walk to survive.
- Case 2 is more complicated and split into two sub-cases...
  - 2.1 random walk comes close to x many times;
  - 2.2 random walk comes close to x few times.

We deal with them by using comparison arguments.

Proof idea for ball clearing

Case 2.1: $B(x; \epsilon\varrho_N/2)$ contains a small density of obstacles and random walk comes close to x many times.

We remove all the obstacles in $B(x; \epsilon\varrho_N/2)$. This operation

- imposes a cost in the environment probability;
- brings a gain in the random walk probability.

It turns out that the gain beats the cost:

$$\mathbb{P} \otimes \mathbf{P}(\text{Case } 2.1) \ll \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O}} > N, \mathcal{O} \cap B(x; \epsilon\varrho_N/2) = \emptyset).$$

Proof idea for ball clearing

Case 2.1: $B(x;\epsilon\varrho_N/2)$ contains a small density of obstacles and random walk comes close to x many times.

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However, it is not straightforward because

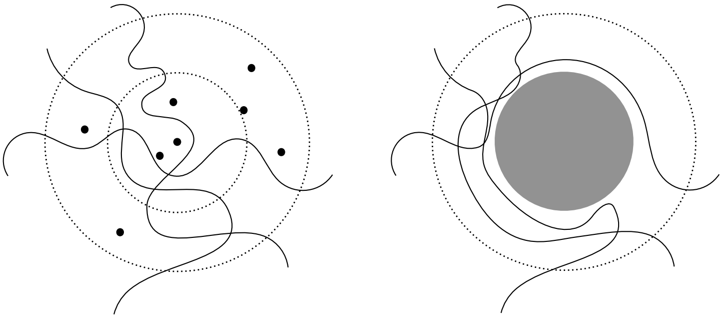
- the cost increases linearly in the number of obstacles in $B(x;\epsilon\varrho_N/2)$, while
- the gain DOES NOT increases linearly in the number of obstacles in $B(x;\epsilon\varrho_N/2)$.

Proof idea for ball clearing

Case 2.2: $B(x; \epsilon\varrho_N/2)$ contains a small density of obstacles and random walk comes close to x few times.

We remove all the obstacles in $B(x; \epsilon\varrho_N/2) \setminus B(x; \epsilon\varrho_N/4)$, let the random walk avoid $B(x; \epsilon\varrho_N/4)$, and then change the obstacles configuration in $B(x; \epsilon\varrho_N/4)$ to typical ones. This operation

- imposes a cost in the random walk probability;
- brings a gain in the environment probability.



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- imposes a cost in the random walk probability;
- brings a gain in the environment probability.

It turns out that the gain beats the cost:

$$\mathbb{P} \otimes \mathbf{P}(\text{Case } 2.2) \\ \ll \mathbb{P} \otimes \mathbf{P}(\tau_{\mathcal{O} \cup B(x;\epsilon\varrho_N/4)} > N, \mathcal{O} \cap B(x; \epsilon\varrho_N/4) \text{ is typical}).$$

Remark

This argument looks wasteful since we are comparing the LHS to a tiny part of the partition function. But it might be less wasteful than comparing with $\exp\{-c(d,p)N^{\frac{d}{d+2}} + o(N^{\frac{d}{d+2}})\}$.

Proof Idea for Boundary Size

"Truly"-open site

The key idea is to approximate the range of the random walk, $S_{[0,N]}$, by a set of *"truly"-open sites* $\mathcal{T}$.

Definition ("Truly"-open sites)

A site $x \in \mathbb{Z}^d$ is called "truly"-open if

$$\mathbf{P}_x\big(\tau_{\mathcal{O}} > (\log N)^5\big) \geq \exp\big\{-(\log N)^2\big\}.$$

$\mathcal{T}$: the cluster of "truly"-open sites inside the confinement ball $B(x_N; \varrho_N + \varrho_N^{\epsilon_1})$ containing the origin.

Remark

1. A "truly"-open site is atypically safe. For a typical site, the above probability decays like $\exp\{-(\log N)^{5+o(1)}\}$.
2. Whether x is "truly"-open or not depends only on the obstacles configuration inside $B(x; (\log N)^5)$.

"Truly"-open site approximates $S_{[0,N]}$

The following two facts justifies the approximation of $\partial S_{[0,N]}$ by the boundary of "truly"-open sites $\partial\mathcal{T}$.

- $\mu_N\big(S_{[0,N]} \subset \mathcal{T}\big) \xrightarrow{N\to\infty} 1,$
- $\mu_N\big(S_{[0,N]} \supset \{x \in \mathcal{T} \colon \mathrm{dist}(x, \partial\mathcal{T}) \geq (\log N)^3\}\big) \xrightarrow{N\to\infty} 1.$

It follows that $\mu_N\left(\partial S_{[0,N]} \subset \bigcup_{x\in\partial\mathcal{T}} B(x; (\log N)^3)\right) \xrightarrow{N\to\infty} 1.$

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- $\mu_N\big(S_{[0,N]} \subset \mathcal{T}\big) \xrightarrow{N\to\infty} 1$,
  - It is not good to visit a non-"truly"-open site.
- $\mu_N\big(S_{[0,N]} \supset \{x \in \mathcal{T} : \mathrm{dist}(x, \partial\mathcal{T}) \geq (\log N)^3\}\big) \xrightarrow{N\to\infty} 1$.
  - Just like the ball covering theorem.

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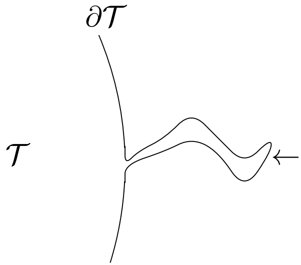
$\partial\mathcal{T}$ is smooth: heuristics

It suffices to prove

$$\mu_N\Big(|\partial\mathcal{T}| \le \varrho_N^{d-1}(\log N)^c\Big) \xrightarrow{N\to\infty} 1.$$

The $\partial\mathcal{T}$ should be rather smooth roughly because...



← The random walk does not go into such a "finger" going outward. Then there is no point in paying the cost to keep it "truly"-open.

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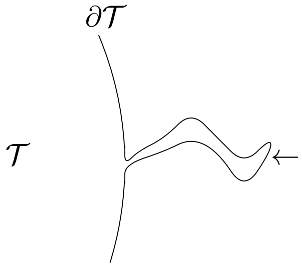


The random walk does not go into such a “finger” going outward. Then there is no point in paying the cost to keep it “truly”-open.

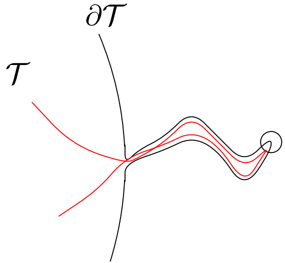
This argument does not exclude the inward “fingers”.
But in the actual proof, we do not distinguish outward and inward.

$\partial\mathcal{T}$ is smooth: proof

The crux of the proof is how to define/quantify "finger", or more generally "bad points". Our first definition is

$$x \in \partial\mathcal{T} \text{ and } P_0(\tau_{B(x;(\log N)^5)} < \tau_{\mathcal{O}}) < \varrho_N^{1-d-\epsilon},$$

i.e., a point is bad if it is difficult for the random walk to visit.



If the random walk visits a bad point, then we "switch" it to inside. Then we can "close" a "truly"-open site to gain a lot.

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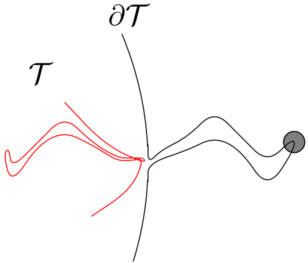

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$\Longrightarrow$ There is no bad point.

$\partial\mathcal{T}$ is smooth: proof

We have proved that

$$\forall x \in \partial\mathcal{T}, P_0(\tau_{B(x;(\log N)^5)} < \tau_{\mathcal{O}}) \geq \varrho_N^{1-d-\epsilon}.$$

On the other hand, it is simple to show

$$\sum_{x\in\partial\mathcal{T}} P_0(\tau_{B(x;(\log N)^5)} < \tau_{\mathcal{O}}) \leq (\log N)^C,$$

since the random walk can easily get trapped after hitting $\partial\mathcal{T}$.

Thus we get $\mu_N\Big(|\partial\mathcal{T}| \leq \varrho_N^{d-1+o(1)}\Big) \xrightarrow{N\to\infty} 1.$

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Thus we get $\mu_N\Big(|\partial\mathcal{T}| \leq \varrho_N^{d-1+o(1)}\Big) \xrightarrow{N\to\infty} 1.$

Finally, we use this to reduce the entropy ($\sum_U$) and bootstrap to get the final result ($\varrho_N^{o(1)}$ replaced by $(\log N)^{\epsilon_3}$).

Thank you for the attention.