Zero temperature limit for Brownian directed polymer in Poisonian disasters

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Joint work with Stefan Junk (Technical University of Munich)

Brownian directed polymer in Poissonian environment

- ((B(t))_{t≥0}, P_x): standard Brownian motion on ℝ^d, B(0) = x.
 (ω = ∑_i δ_(t_i,x_i), ℙ): Poisson point process on (0,∞) × ℝ^d
 - with unit intensity.



Directed polymer measure:

$$\mu_t^{\omega,\beta}(\mathsf{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta \# \{\text{hitting to } \neq \text{ up to } t\}} P_0(\mathsf{d}B).$$

 $\beta < \mathbf{0} \Rightarrow$ attractive, $\beta > \mathbf{0} \Rightarrow$ repulsive.

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Directed polymer measure:

$$\mu_t^{\omega,\beta}(\mathsf{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta \#\{\text{hitting to } \neq \text{ up to } t\} - \int_0^t |\dot{B}(s)|^2 \mathsf{d}s}$$

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Partial list of known results:

- 1. Localization transition: Comets-Yoshida (2004, 2005, 2013).
- 2. Bounds on "fluctuation exponent": Comets-Yoshida (2005).
- 3. KPZ in "intermediate disorder regime": Cosco (2018+).
- 4. Survey article: Comets-Cosco (2018+).

Some results are better than in the discrete random walk model. Stochastic analysis provides powerful tools.

Free energy at positive temperature

Reminder:
$$\mu_t^{\omega,\beta}(dB) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta \# \{\text{hitting to } \neq \text{ up to } t\}} P_0(dB).$$

An important quantity is the free energy:

$$\varphi(\beta) = \lim_{t \to \infty} \frac{1}{t} \log Z_t^{\omega,\beta} = \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \Big[\log Z_t^{\omega,\beta} \Big].$$

For example, criterion for the localization is

$$arphi(eta)
eq arphi^{\mathsf{ann}}(eta) := \lim_{t o \infty} rac{1}{t} \log \mathbb{E}\Big[Z^{\omega,eta}_t \Big].$$

Existence of $\varphi(\beta)$ for $\beta \in \mathbb{R}$ is standard:

- Either by sub-additive ergodic theorem or
- super-additivity of the mean & concentartion around mean.

Free energy at zero temperature

At $\beta=-\infty,$ the model does not make sense. Impurities are infinitely attractive.

At $\beta = \infty$, the model does make sense but $\mathbb{E}\left[\log Z_t^{\omega,\infty}\right] = -\infty$. Let $\tau(\omega)$ be the hitting time to ϕ so that $Z_t^{\omega,\infty} = P_0(\tau(\omega) > t)$. Proof.

Brownian motion has to avoid the first disaster in $[0, \infty] \times [-\frac{1}{4}, \frac{1}{4}]$. If it occurs at time *F*, then

$$\log P_0(\tau(\omega) > t) \lesssim \log \exp\left(-(rac{1}{4})^2/F
ight) \ = -rac{1}{4F}.$$



Since $F \stackrel{d}{=} Exp(1/2)$, 1/F is not integrable.

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Since $F \stackrel{d}{=} Exp(1/2)$, 1/F is not integrable. \longrightarrow Direct sub-additivity argument fails.

Main results

Theorem

There exists $p(\infty) \in (-\infty, 0)$ such that the following hold:

(i) \mathbb{P} -almost surely, $\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\infty} = p(\infty);$

(ii) $\lim_{\beta\to\infty} p(\beta) = p(\infty)$.

Some elements of the proof

Modified death time

Lemma (non-integrability is due to the first disaster) Let F_t be the first disaster in $[0, t] \times [-\frac{7}{2}, \frac{7}{2}]^d$. Then there exists c > 0 such that

$$\mathbb{E}\left[\log P_0(\tau(\omega) > t) \,\Big|\, \mathsf{F}_t\right] \geq -c(t + \mathsf{F}_t^{-1}).$$

Thus the following modification ensures the integrability:

 $au^1(\omega) := \inf \left\{ s \ge 1 \colon (s, B_s) \text{ hits a disaster}
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Problem 1: We need to revert $\tau^1 \rightarrow \tau$ in the end. This looks harmless but in fact requires a quite complicated argument. Due to the time limitation, we do not address this issue here.

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Problem 2: Standard argument for super-additivity yields

$$egin{aligned} \mathbb{E}\left[\log P(au^1(\omega)\geq s+t)
ight]\ &\geq \mathbb{E}\left[\log P(au^1(\omega)\geq s)
ight]+\mathbb{E}\left[\log P(au(\omega)\geq t)
ight]. \end{aligned}$$

Effect of changing disasters in a slab

We show an almost super-additivity by estimating

$$\begin{split} \log P\big(\tau^1(\omega) \geq s+t\big) - \log P\big(\tau^1(\omega_{[s,s+1]^c}) \geq s+t\big) \\ = \log P\left(\tau^1(\omega) \geq s+t \mid \tau^1(\omega_{[s,s+1]^c}) \geq s+t\right). \end{split}$$

Effect of changing disasters in a slab

We show an almost super-additivity by estimating

$$egin{aligned} \log egin{aligned} & \log egin{aligned} & \log egin{aligned} & \left(au^1(\omega) \geq s + t
ight) - \log egin{aligned} & P\left(au^1(\omega) \geq s + t
ight| \ au^1(\omega_{[s,s+1]^c}) \geq s + t
ight). \end{aligned}$$



We need a control on the survival in tubes and that the polymer is "spread out" under $P(\cdot \mid \tau^1(\omega_{[s,s+1]^c}) \ge s+t)$.

Survival in tube

Lemma

Let F_t and L_t be the first and last disaster in $[0, t] \times [-\frac{7}{2}, \frac{7}{2}]$ respectively. Then

$$\inf_{\substack{x,y \in [-5/2,5/2]^d}} \mathbb{E} \left[\log P_{0,0}^{t,y}(\tau(\omega) \wedge \tau_{[-3,3]} > t) \, \middle| \, F_t, L_t \right] \\ \geq -c(t + F_t^{-1} + (t - L_t)^{-1}).$$



Concentration bound

Previous Lemma and "spread-out" estimate for polymer measure (skipped) yield almost super-additivity

$$\Rightarrow \text{Existence of } \lim_{t \to \infty} \frac{1}{t} \mathbb{E}[\log P(\tau^1(\omega) > t)].$$

Control on the effect of changing disasters in a slab

- \Rightarrow Concentration around the mean
- \Rightarrow Existence of $\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1(\omega) > t)$, \mathbb{P} -a.s.

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Control on the effect of changing disasters in a slab

- \Rightarrow Concentration around the mean
- \Rightarrow Existence of $\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1(\omega) > t)$, \mathbb{P} -a.s.

Moreover, once we get a concentration around the mean, there is a standard argument to derive a rate of convergence for

$$\left|rac{1}{t}\mathbb{E}[\log P(au^1(\omega)>t)]-p(\infty)
ight|
ightarrow 0.$$

The same holds for finite temperature **uniformly** in $\beta \in \mathbb{R}$. This yields the continuity of $p(\beta)$.

Proof of survival in tube Lemma





$$\begin{split} \mathbf{p}_{(\mathbf{x}_i)}^{\prime} = & \mathbf{s} + \mathbf{s} \left(\left| \hat{\mathbf{x}}_{\mathbf{x}_i} = \mathbf{s} \right) + \mathbf{p}_{(\mathbf{x}_i)}^{\prime} = \mathbf{s} + \mathbf{s} \left(\left| \hat{\mathbf{x}}_{\mathbf{x}_i} - \hat{\mathbf{x}}_{\mathbf{s}} + \mathbf{s} \right) \right| \\ & = \frac{\mathbf{p}_{(\mathbf{x}_i)}^{\prime}}{\mathbf{p}_{(\mathbf{x}_i)}^{\prime} = \mathbf{s} + \mathbf{s} \left(\hat{\mathbf{x}}_{\mathbf{x}_i} - \hat{\mathbf{x}}_{\mathbf{s}} + \mathbf{s} \right) } \end{split}$$

 $p_{(n_1+n_2+1)} = \tilde{c}_1 + c_1 + \frac{(n_1+n_2)(n_1)}{(n_1+1)(n_2+1)} =$

1.44

Part 4 + 4(6) 1 + 1

Fig. 1 (Sec. 1) (Sec. 2) (S

 $(\frac{1}{2})^{2}(1 - \hat{t}_{1} + \hat{t}_{2} + \hat{t}_{1})$ $(\frac{1}{2})^{2}(1 - \hat{t}_{2} + \hat{t}_{2})$

i , we assume induce more to the sole model in out time should (I_i, I_{i+1}) sample for $i \in I$. Since there is a proper strategy of the solution of the

$$\label{eq:product} h_{0}(P) \phi(\eta)(||\theta||\eta) \equiv -i \sum_{\alpha,\beta_{\alpha},\beta_{\alpha}} \mathbf{A}_{\alpha}^{\alpha} - i \sum_{\alpha,\beta_{\alpha},\beta_{\alpha}} \mathbf{A}_{\alpha}^{\alpha},$$

$$|\hat{x}|_{[n]} := \sum_{i=1}^n \mathbf{1}_i |\hat{x}_{-i}|_{i=1} \text{ and }$$

 $A(x) = \sum_{i=1}^{n} \frac{1}{2} (A_i + x)$

```
\equiv -i^{2}\left(\sum_{i}^{2}A_{i}^{-1}+\sum_{i}A_{i}^{-1}+y-\bar{a}_{i}^{-1}+y-\bar{a}_{i}^{-1}\right)
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Barrie and Barrier

former, contrast at (a + i), b - b is former detting a for the relation density of the i and b is the set of the b

Participation - and a second second

by analy contrary for contrary of protocolour with (eq. the down values), for the own near $\{p_{1}^{(i)}, i\}$ is adapted at $P_{1}^{(i)}, A_{1}^{(i)}, Y_{2}^{(i)}, A_{2}^{(i)}, Y_{2}^{(i)}, Y_{2}^{(i$

 $\sum_{i=1}^{n} a_i \cdot \left(\frac{1}{2} \sum_{i=1}^{n} \left(\frac{a_i^{(0)}}{\sum_{i=1}^{n} a_i^{(0)}}\right)\right)$

 $\left(\hat{\omega}^{+}, \hat{\omega}^{+}, \dots, \hat{\omega}^{+} \right) \in \left(\sum_{i=1}^{r} \frac{d_{i}}{1+i} \sum_{i=1}^{r} \frac{d_{i}}{1+i} \cdots \sum_{i=r-1}^{r} \frac{d_{i}}{1+i} \right)$

 $\widehat{A}[h_{0},P(\theta)] \mid A(\theta) \mid (a,b) \mid = -1 \left(b_{0} + \frac{b_{0}}{2}\right)$

 $\lambda_{1}(P(\boldsymbol{\beta}_{i}|\boldsymbol{\beta}_{i})) = - \mathcal{O}\left(- \sum_{m \in \mathcal{M}_{i}} \frac{1}{2m} \right)$ $\delta = \left(- \frac{1}{2} \sum_{i=1}^{n} \frac{y_{i+1}^{-i} \delta_{i}}{y_{i+1}^{-i} \frac{y_{i+1}^{-i} \delta_{i}}}{y_{i+1}^{-i} \frac{y_{i+1}^{-i} \delta_{i}}{y_{i+1}^{-i} \frac{y_{i+1}^{-i} \delta_{i}}}{y_{i+1}^{-i} \frac{y_{i+1}^{-$ * 200 <u>2000</u> North Street $\mathbb{E}\left[\sum_{i=1}^{n} d_i\right] = \dots = \mathbb{E}\left[\left(\sum_{i=1}^{n} d_i\right)^{-1}\right] = \frac{1}{n+1}$

 $= \left\{ \frac{1}{2} \frac{1}{1 + 1} \frac{1}{1 + 1} \frac{1}{1 + 1} \right\} = \left\{ \frac{1}{2} \frac{1}{1 + 1} \frac{1}{1 + 1} + \left\{ \frac{1}{1 + 1} \frac{1}{1 + 1} + \left\{ \frac{1}{1 + 1} + \frac{1$ (100 g and all (11) is mentiody the same. We assume if its final of its a Taka condition by (1), we

 $\log |\mathcal{D}(f')| d(p), \dots, d(|\mathcal{U}(h_i)|) \geq - \mathcal{O}\left(\sum_{i=1}^{i-1} h_i^{-1} + V + |V-h_i| + |V-h_i|^{-1}\right).$

 $\sum A_i = \int dx - \int \frac{\sum A_i}{\sum \sum A_i}$

-----Personal strends of the line $\mathbb{E}\left[\log P(\mathbf{k} \mid (d_1, \mathbf{k}) \mid z - t) \left(d_1 + P_1^{-1} + \mathbb{E}\left[\sum_{i=1}^{n-1} di \mathbf{k} + t \right] \right]$

 $\mathbb{E}\left[g\left(|h_{1}\right)+\sum_{i=1}^{n}|h_{i}|+g\left(\nabla_{g_{1}}+u_{1}+v\right)|h_{i}|\right)+\sum_{i=1}^{n}\frac{|h_{1}+\beta|^{2}}{|h_{1}|^{2}}+u_{i}|$ Received in the second

 $\frac{P\left[\frac{1}{1+\alpha},\frac{1}{\alpha}+\alpha,A_{12},\ldots, +A_{12},\ldots, +A_{12}$

 $-\lambda \mathbb{E}\left[\sum_{j=1}^{n+1} (ab_j^{-1} + b^{-1}] + (r - b_j^{-1} + (r - b_j^{-1}))\right]$

 $= \int_{0}^{1} e^{-i\theta} H_{\theta} + d + \dots + d \leq (|\theta|_{\theta} + \frac{1}{2} d_{\theta})$ $+ i H_{\theta} H_{\theta} + \dots + H_{\theta} \leq 0$

Related works

Remark

For the model based on simple random walk,

$$Z_n^{\eta,\beta} = E_0^{\text{SRW}} \left[\exp\left(-\beta \sum_{k=1}^n \eta(k, X_k)\right) \right]$$
$$\approx \exp\left(-\beta \inf_{X: \text{ path}} \sum_{k=1}^n \eta(k, X_k)\right)$$

if essinf $\eta < 0$. Thus as $\beta \to \infty$,

$$rac{1}{eta n}\log Z_n^{\eta,eta}\sim -rac{1}{n}\inf_{X: ext{ path }}\sum_{k=1}^n\eta(k,X_k).$$

Related works

- Comets-F.–Nakajima–Yoshida (2015): Continuity of the free energy for a long range random walk model with Bernoulli disasters.
- ▶ Nakajima (2018): Getting rid of a parameter restriction.
- Bakhtin–Li (2018+): Convergence of the polymer measure defined by

$$\mu^{\eta,\beta}(\mathsf{d} X) := \frac{1}{Z_n^{\eta,\beta}} \exp\left(-\beta \sum_{k=1}^n \left(\eta(k,X_k) + |X_{k-1} - X_k|^2\right)\right) \mathsf{d} X.$$

The limit is a kind of first passage percolation. (Similar models have been studied by Berger–Torri recently.)

Thank you!