Zero temperature limit for Brownian directed polymer in Poisonian disasters

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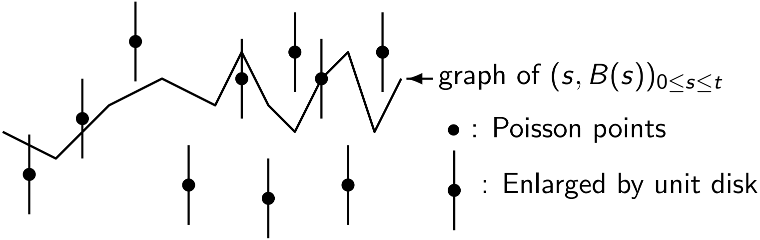
Brownian directed polymer in Poissonian environment

- $((B(t))_{t\geq 0}, P_x)$: standard Brownian motion on $\mathbb{R}^d$, $B(0) = x$.
- $(\omega = \sum_i \delta_{(t_i,x_i)}, \mathbb{P})$: Poisson point process on $(0,\infty) \times \mathbb{R}^d$ with unit intensity.



Directed polymer measure:

$$\mu_t^{\omega,\beta}(\mathrm{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta\#\{\text{hitting to } \phi \text{ up to } t\}} P_0(\mathrm{d}B).$$

$\beta < 0 \Rightarrow$ attractive, $\beta > 0 \Rightarrow$ repulsive.

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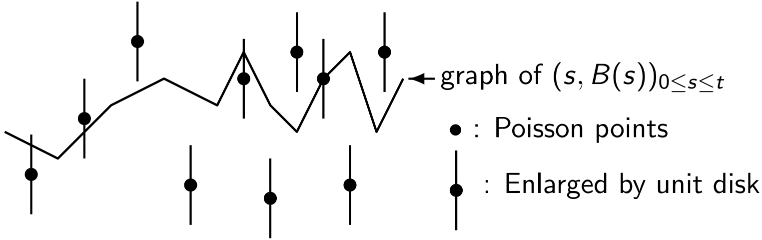

Directed polymer measure:

$$\mu_t^{\omega,\beta}(\mathrm{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta \# \{\text{hitting to } \phi \text{ up to } t\} - \int_0^t |\dot{B}(s)|^2 \mathrm{d}s}.$$

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Known results ($\beta \in \mathbb{R}$)

Partial list of known results:

1. Localization transition: Comets–Yoshida (2004, 2005, 2013).
2. Bounds on "fluctuation exponent": Comets–Yoshida (2005).
3. KPZ in "intermediate disorder regime": Cosco (2018+).
4. Survey article: Comets–Cosco (2018+).

Some results are better than in the discrete random walk model.
Stochastic analysis provides powerful tools.

Free energy at positive temperature

Reminder: $\mu_t^{\omega,\beta}(\mathrm{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta \#\{\text{hitting to } \phi \text{ up to } t\}} P_0(\mathrm{d}B)$.

An important quantity is the free energy:

$$\varphi(\beta) = \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\beta} = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\Big[\log Z_t^{\omega,\beta}\Big].$$

For example, criterion for the localization is

$$\varphi(\beta) \neq \varphi^{\mathrm{ann}}(\beta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[Z_t^{\omega,\beta}\Big].$$

Existence of $\varphi(\beta)$ for $\beta \in \mathbb{R}$ is standard:

- Either by sub-additive ergodic theorem or
- super-additivity of the mean & concentartion around mean.

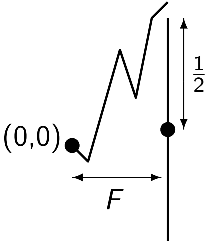
Free energy at zero temperature

At $\beta = -\infty$, the model does not make sense. Impurities are infinitely attractive.

At $\beta = \infty$, the model does make sense but $\mathbb{E}\left[\log Z_t^{\omega,\infty}\right] = -\infty$.
Let $\tau(\omega)$ be the hitting time to ϕ so that $Z_t^{\omega,\infty} = P_0(\tau(\omega) > t)$.

Proof.
Brownian motion has to avoid the first disaster in $[0,\infty] \times [-\frac{1}{4}, \frac{1}{4}]$. If it occurs at time F, then

$$\log P_0(\tau(\omega) > t) \lesssim \log \exp\left(-(\tfrac{1}{4})^2/F\right) = -\frac{1}{4F}.$$



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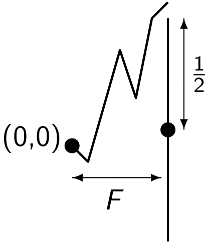

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$\longrightarrow$ Direct sub-additivity argument fails.

Main results

Theorem

There exists $p(\infty) \in (-\infty, 0)$ *such that the following hold:*

(i) $\mathbb{P}$*-almost surely,* $\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\infty} = p(\infty)$*;*

(ii) $\lim_{\beta\to\infty} p(\beta) = p(\infty)$*.*

Some elements of the proof

Modified death time

Lemma (non-integrability is due to the first disaster)

Let F_t be the first disaster in $[0, t] \times [-\frac{7}{2}, \frac{7}{2}]^d$. Then there exists $c > 0$ such that

$$\mathbb{E}\left[\log P_0(\tau(\omega) > t) \,\middle|\, F_t\right] \geq -c(t + F_t^{-1}).$$

Thus the following modification ensures the integrability:

$$\tau^1(\omega) := \inf\left\{s \geq 1 \colon (s, B_s) \text{ hits a disaster}\right\}.$$

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Problem 1: We need to revert $\tau^1 \to \tau$ in the end. This looks harmless but in fact requires a quite complicated argument. Due to the time limitation, we do not address this issue here.

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Problem 2: Standard argument for super-additivity yields

$$\begin{aligned}&\mathbb{E}\left[\log P(\tau^1(\omega) \geq s + t)\right] \\ &\quad \geq \mathbb{E}\left[\log P(\tau^1(\omega) \geq s)\right] + \mathbb{E}\left[\log P(\tau(\omega) \geq t)\right].\end{aligned}$$

Effect of changing disasters in a slab

We show an almost super-additivity by estimating

$$\log P\big(\tau^1(\omega) \geq s + t\big) - \log P\big(\tau^1(\omega_{[s,s+1]^c}) \geq s + t\big)$$
$$= \log P\left(\tau^1(\omega) \geq s + t \;\middle|\; \tau^1(\omega_{[s,s+1]^c}) \geq s + t\right).$$

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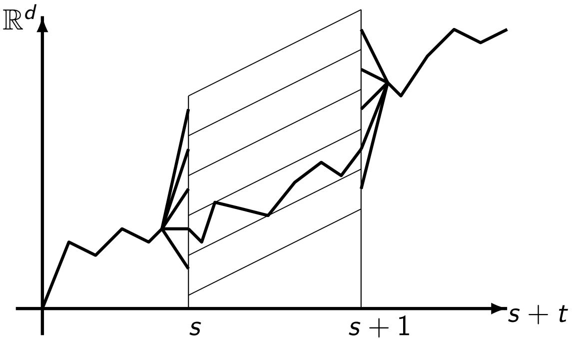
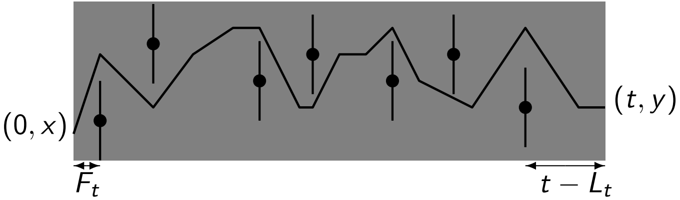


We need a control on the survival in tubes and that the polymer is "spread out" under $P(\cdot \mid \tau^1(\omega_{[s,s+1]^c}) \geq s+t)$.

Survival in tube

Lemma
Let F_t and L_t be the first and last disaster in $[0,t] \times [-\frac{7}{2}, \frac{7}{2}]$ respectively. Then

$$\inf_{x,y \in [-5/2, 5/2]^d} \mathbb{E}\left[\log P_{0,0}^{t,y}(\tau(\omega) \wedge \tau_{[-3,3]} > t) \,\middle|\, F_t, L_t \right] \geq -c(t + F_t^{-1} + (t - L_t)^{-1}).$$



Concentration bound

Previous Lemma and "spread-out" estimate for polymer measure (skipped) yield almost super-additivity

$\Rightarrow$ Existence of $\displaystyle\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log P(\tau^1(\omega) > t)]$.

Control on the effect of changing disasters in a slab

$\Rightarrow$ Concentration around the mean

$\Rightarrow$ Existence of $\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1(\omega) > t)$, $\mathbb{P}$-a.s.

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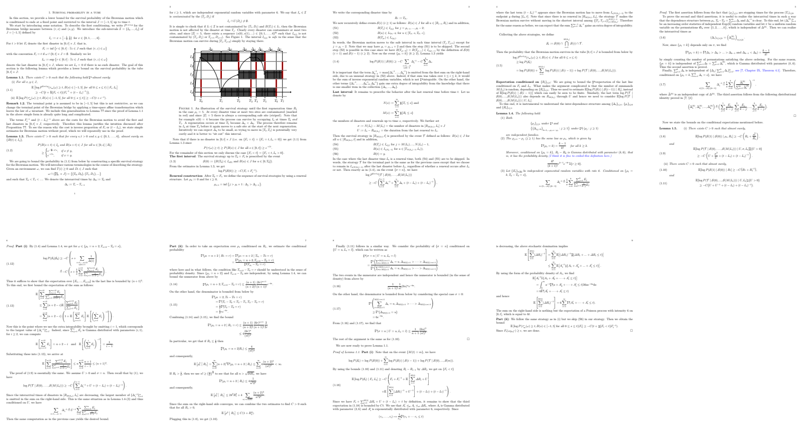
Moreover, once we get a concentration around the mean, there is a standard argument to derive a rate of convergence for

$$\left| \frac{1}{t}\mathbb{E}[\log P(\tau^1(\omega) > t)] - p(\infty) \right| \to 0.$$

The same holds for finite temperature **uniformly** in $\beta \in \mathbb{R}$. This yields the continuity of $p(\beta)$.

Proof of survival in tube Lemma



Related works

Remark

For the model based on simple random walk,

$$Z_n^{\eta,\beta} = E_0^{\mathrm{SRW}}\left[\exp\left(-\beta\sum_{k=1}^{n}\eta(k,X_k)\right)\right]$$
$$\approx \exp\left(-\beta\inf_{X:\ \mathrm{path}}\sum_{k=1}^{n}\eta(k,X_k)\right)$$

if $\operatorname{ess\,inf}\eta < 0$*. Thus as* $\beta\to\infty$*,*

$$\frac{1}{\beta n}\log Z_n^{\eta,\beta} \sim -\frac{1}{n}\inf_{X:\ \mathrm{path}}\sum_{k=1}^{n}\eta(k,X_k).$$

Related works

- Comets-F.–Nakajima–Yoshida (2015): Continuity of the free energy for a long range random walk model with Bernoulli disasters.
- Nakajima (2018): Getting rid of a parameter restriction.
- Bakhtin–Li (2018+): Convergence of the polymer measure defined by

$$\mu^{\eta,\beta}(\mathrm{d}X) := \frac{1}{Z_n^{\eta,\beta}} \exp\left(-\beta \sum_{k=1}^{n} \left(\eta(k, X_k) + |X_{k-1} - X_k|^2\right)\right) \mathrm{d}X.$$

  The limit is a kind of first passage percolation. (Similar models have been studied by Berger–Torri recently.)

Thank you!