Number of paths in oriented percolation as zero temperature limit of directed polymer

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Joint work with Stefan Junk (AIMR Tohoku University).

Oriented percolation

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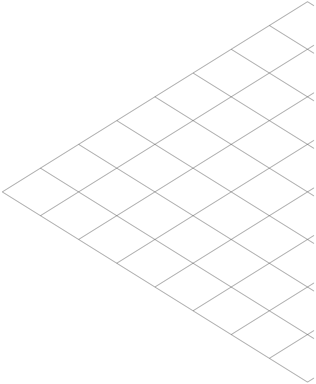
The result at $p = \vec{p_c}$ is due to Bezuidenhout–Grimmett (1990).



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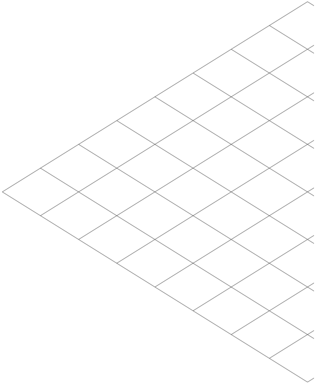

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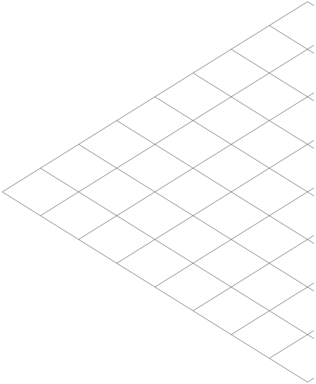

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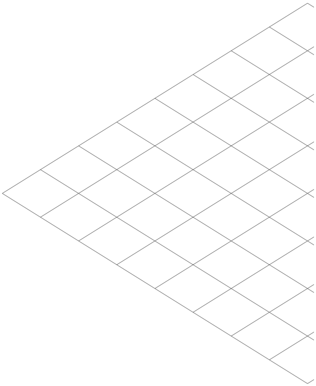

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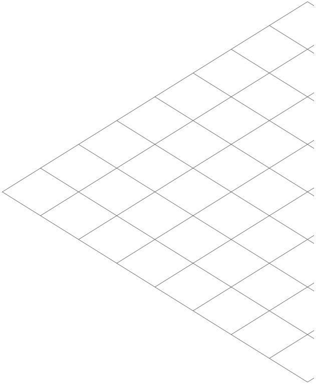

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Disclaimer: We focus on $(1+1)$-dimension for simplicity.

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Let $N_n = \#\{\text{open path from } (0,0) \text{ to } (n, \mathbb{Z}^d)\}$.

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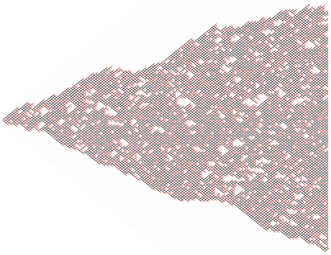
On $\{(0,0) \leftrightarrow \infty\}$, $\exists \alpha_p = \lim_{n\to\infty} \frac{1}{n} \log N_n > 0$. *In fact, the "directional growth rate"* $\alpha_p(v)$ *exists and* $\alpha_p = \sup_v \alpha_p(v)$.



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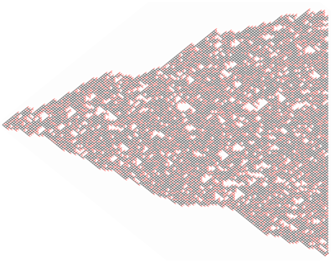

Question 1: Is α_p continuous in p?
Question 2: Is $\alpha_p(v)$ strictly concave in v?

We will answer the first question.
The second question seems to be hard.

First set of results

Theorem (F.–Junk (2021))

The growth rate α_p is continuous in $p \in (\vec{p_c}, 1]$.

We deduce this from the following results, which establish a "good finite volume approximation".

Proposition (F.–Junk (2021))

For any $\delta, \varepsilon > 0$ and $r > 0$, there exists $c > 0$ such that for all $p \geq \vec{p_c} + \varepsilon$,

$$\mathbb{P}_p\Big(\big|\log N_n - \mathbb{E}_p[\log N_n \mid (0,0) \leftrightarrow \infty]\big| \geq n^{\frac{1}{2}+\delta} \;\Big|\; (0,0) \leftrightarrow \infty\Big) \leq cn^{-r}.$$

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Directed polymer in random environment

There is a positive temperature version of our model:

$$Z_n^\beta(\omega) := \sum_{\pi:\ \text{path of length } n} e^{-\beta \sum_{t=1}^{n} \omega(t,\pi(t))}.$$

Indeed, we have $\lim_{\beta\to\infty} Z_n^\beta(\omega) = N_n(\omega)$ as long as $n \in \mathbb{N}$ is fixed.

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For $0 \le \beta < \infty$, the existence of the growth rate is relatively easy.

Theorem (Comets–Shiga–Yoshida (2003))

For every $\beta \in [0,\infty)$, there exists $\mathfrak{f}(\beta,p) \in (0, \log(2d)]$ such that, $\mathbb{P}_p$-almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n^\beta(\omega) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_p[\log Z_n^\beta(\omega)] = \mathfrak{f}(\beta,p).$$

Question: Is $\alpha_p = \lim_{\beta\to\infty} \mathfrak{f}(\beta,p)$?

Second set of results

Theorem (F.–Junk (2021))

For any $p \in (\vec{p_c}, 1]$, $\alpha_p = \lim_{\beta\to\infty} \mathfrak{f}(\beta, p)$.

This follows from the following results.

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How it goes when $\beta < \infty$: super-additivity

For $\beta < \infty$, we have a simple structure:

$$\begin{aligned}
\log Z_{m+n}^{\beta} &= \log \sum_x Z^{\beta}_{(0,0)\to(m,x)} Z^{\beta}_{(m,x)\to(m+n,\mathbb{Z}^d)} \\
&= \log Z_m^{\beta} + \log \sum_x \frac{Z^{\beta}_{(0,0)\to(m,x)}}{Z_m^{\beta}} Z^{\beta}_{(m,x)\to(m+n,\mathbb{Z}^d)} \\
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Taking expectation, we get

$$\begin{aligned}
&\mathbb{E}_p[\log Z^\beta_{m+n}] \geq \mathbb{E}_p[\log Z^\beta_m] + \mathbb{E}_p[\log Z^\beta_n] \\
&\Rightarrow \exists \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_p[\log Z^\beta_n].
\end{aligned}$$

How it goes when $\beta < \infty$: concentration

Basic principle in measure concentration

For a function of many independent random variables:

Stable under re-sampling coordinates $\Rightarrow$ well-concentrated.

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Consider the environment changed at one time:

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Then we have

$$\begin{aligned} D_k &:= |\log Z_n^\beta(\hat{\omega}_k) - \log Z_n^\beta(\omega)| \\ &= \left| \log \frac{\sum_\pi e^{-\beta \sum_{t=1}^n \hat{\omega}_k(t,\pi(t))}}{\sum_\pi e^{-\beta \sum_{t=1}^n \omega(t,\pi(t))}} \right| \\ &\le \sup_\pi \left| \log \frac{e^{-\beta \hat{\omega}(k,\pi(k))}}{e^{-\beta \omega(k,\pi(k))}} \right|, \end{aligned}$$

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which is bounded. By the bounded difference inequality,

$$\mathbb{P}_p\Big(\big|\log Z_n^\beta - \mathbb{E}_p[\log Z_n^\beta]\big| \geq n^{\frac{1}{2}+\delta}\Big) \leq \exp\left\{-c\frac{n^{1+2\delta}}{\sum_{k=1}^n \|D_k\|_\infty^2}\right\}.$$

Troubles around $\beta = \infty$

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Even for $\beta < \infty$, the bound on D_k depends on β and hence we don't get a uniform concentration around the mean.

At $\beta = \infty$, it gets worse. For the super-additivity, we can write

$$\log N_{m+n} \geq \log N_m + \sum_x \frac{N_{(0,0)\to(m,x)}}{N_m} \log N_{(m,x)\to(m+n,\mathbb{Z}^d)}.$$

But $(0,0) \leftrightarrow \infty$ does not imply $(m,x) \leftrightarrow (m+n,\mathbb{Z}^d)$, and hence the right-hand side is typically $-\infty$. As for the influence, it is possible that $N_n(\omega) > 0$ and $N_n(\hat{\omega}_k) = 0$, which implies $D_k = \infty$.

<u>Common Problem</u>: An open path can be discontinued.

Plan of the remaining talk

In the remaining part, I will explain how to prove the concentration around the mean, that is,

$$\left|\frac{1}{n}\log N_n - \frac{1}{n}\mathbb{E}_p[\log N_n \mid (0,0) \leftrightarrow \infty]\right| \leq cn^{-\frac{1}{2}+\delta}$$

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I will not discuss the other matters since

- the same repairing procedure can be used to prove *almost* super-additivity for $\frac{1}{n}\mathbb{E}_p[\log N_n \mid (0,0) \leftrightarrow \infty]$,
- the rate of convergence for $\frac{1}{n}\mathbb{E}_p[\log N_n \mid (0,0) \leftrightarrow \infty] - \alpha_p$ follows from the concentration thanks to Yu Zhang's argument in his 2010 paper.

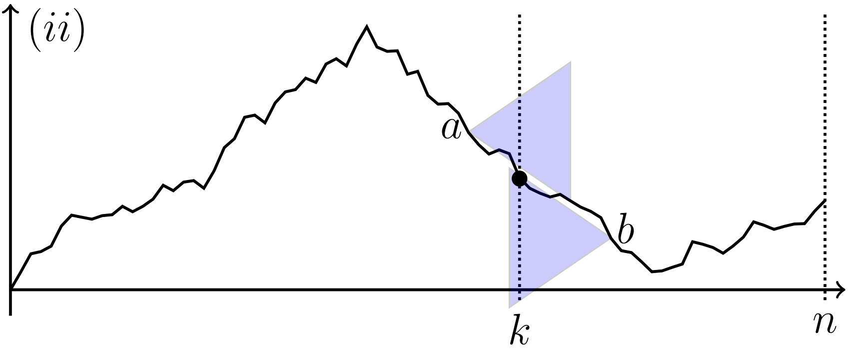
Basic principle and naive expectation

Basic principle in measure concentration

For a function of many independent random variables:

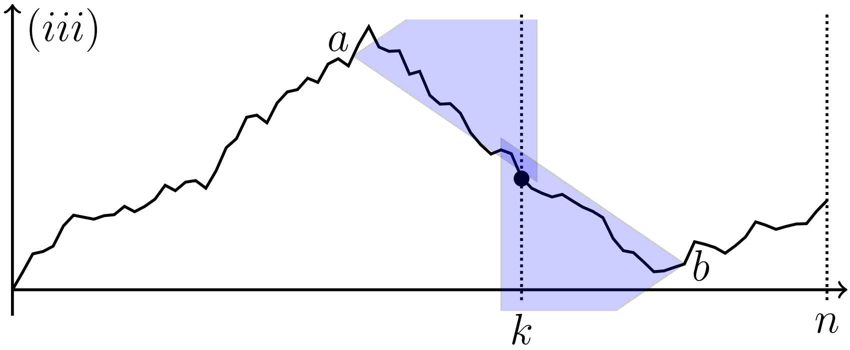
Stable under re-sampling coordinates $\Rightarrow$ well-concentrated.

Repairing: the issue



For a path having an extremal slope, the percolation cones starting from a and b might miss each other. $\longrightarrow$ Impossible to repair the connection.

Repairing: the issue — continued.



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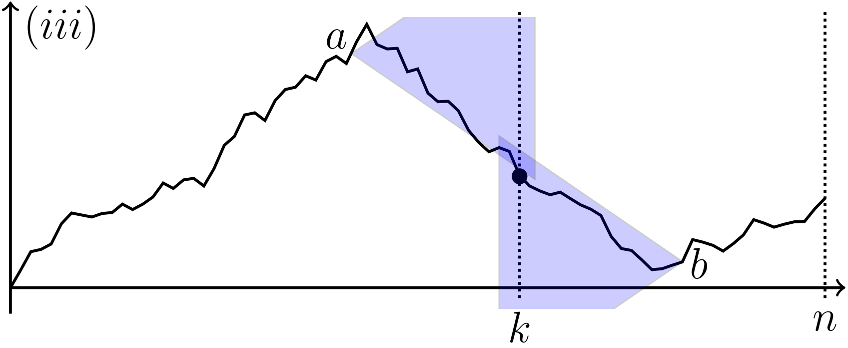

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This means that between the paths before and after re-sampling, there is only many-to-one correspondence, which is useless.

Burkholder's inequality

Theorem (Burkholder 1966)

For every $q \in \mathbb{N}$, there exists $C > 0$ such that for every martingale $((M_n)_{n\in\mathbb{N}}, \mathbb{P})$,

$$\mathbb{E}\left[(M_n - M_0)^{2q}\right] \le C\mathbb{E}\left[\left(\sum_{k=1}^{n}(M_k - M_{k-1})^2\right)^q\right].$$

We apply this to the martingale $M_k := \mathbb{E}_p[\log N_n \mid \mathcal{F}_k]$ under the measure "$\mathbb{P}_p(\cdot \mid (0,0) \leftrightarrow \infty)$".

Then, we only need a moment bound for the martingale difference

$$M_k - M_{k-1} = \mathbb{E}_p[\log N_n \mid \mathcal{F}_k] - \mathbb{E}_p[\log N_n \mid \mathcal{F}_{k-1}].$$

Burkholder's inequality

Theorem

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Once we get a good bound on this martingale difference, we are done!

We … e

"$\mathbb{P}_p$ …

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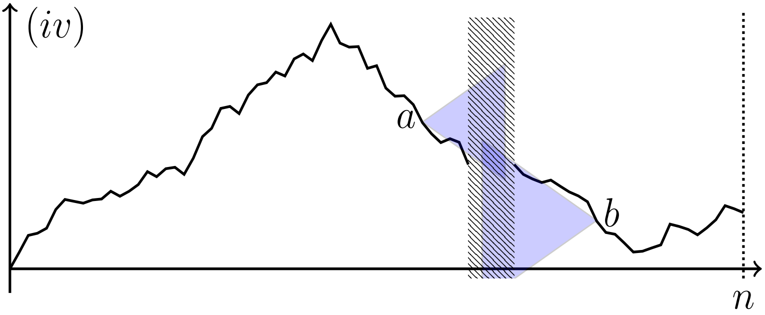
The bounded difference inequality suggests to bound this by using $(\omega, \hat{\omega}_k)$. We use a different coupling.

Repairing: the key idea

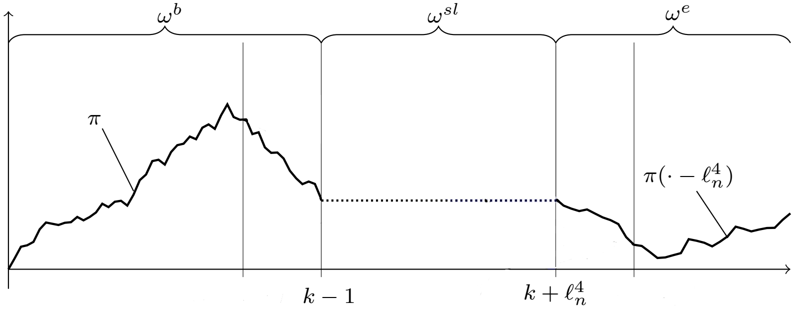
The key idea is to write the martingale difference as

$$\begin{aligned}&\mathbb{E}_p[\log N_n \mid \mathcal{F}_k] - \mathbb{E}_p[\log N_n \mid \mathcal{F}_{k-1}] \\ &\quad = \mathbb{E}_p^{\mathsf{sl,e}}\Big[\log N_n([\omega, \omega^{\mathsf{e}}]_k) - \log N_n([\omega, \omega^{\mathsf{sl}}, \omega^{\mathsf{e}}]_{k-1,k+(\log n)^8})\Big].\end{aligned}$$



The only remaining problem is how to extend the cones into the slab.

Repairing: the procedure



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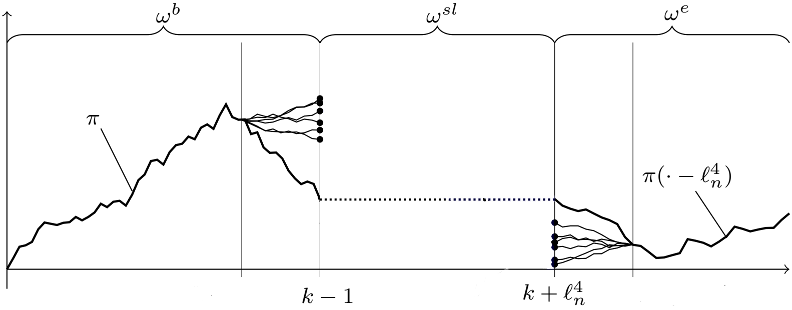

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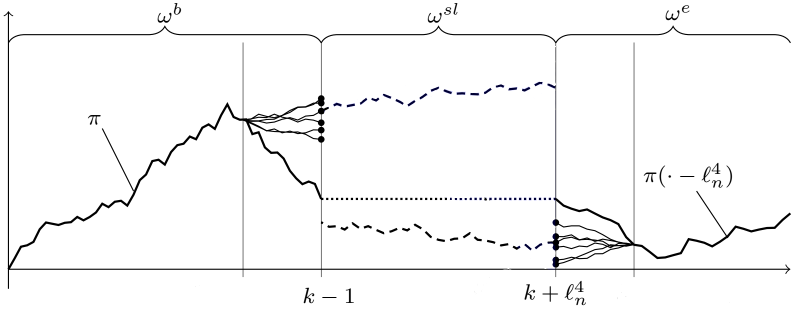

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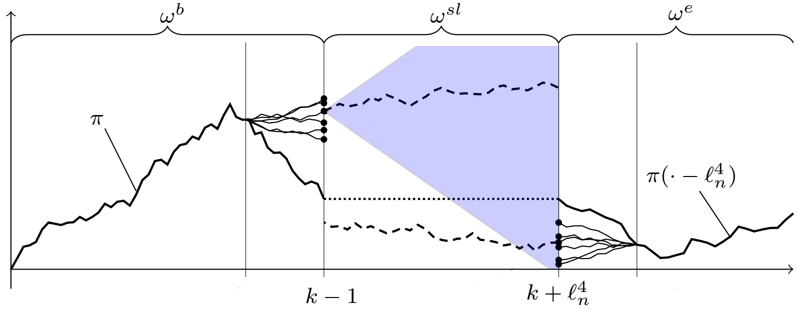

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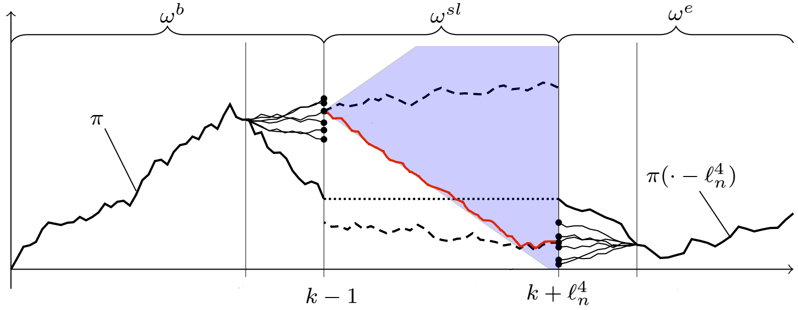

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4. From a percolation point, the cluster grows like a cone.
5. The backward connection must be captured by the cone.

Conclusion

The path constructed in this way can be used to repair at most $(2d)^{2(\log n)^4}$ open paths.

Thus with very high probability,

$$\mathbb{E}_p[\log N_n \mid \mathcal{F}_k] - \mathbb{E}_p[\log N_n \mid \mathcal{F}_{k-1}] \leq 2(\log n)^4 \log(2d).$$

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The other bound

$$\mathbb{E}_p[\log N_n \mid \mathcal{F}_k] - \mathbb{E}_p[\log N_n \mid \mathcal{F}_{k-1}] \geq -2(\log n)^4 \log(2d).$$

can be proved similarly. These bounds are good enough to get the desired concentration inequality. □

Final remarks

1. Duminil-Copin, Kesten, Nazarov, Peres and Sidoravicius (2020) studied maximal paths in the directed last passage percolation. The number of paths question makes sense even in the sub-critical phase.
   - Exponential growth is proved.
   - Growth rate does NOT go to zero as $p \searrow \vec{p}_c$.
   - Existence of the growth rate is left open.

Final remarks

1. Duminil-Copin, Kesten, Nazarov, Peres and Sidoravicius (2020) studied maximal paths in the directed last passage percolation. The number of paths question makes sense even in the sub-critical phase.
   - Exponential growth is proved.
   - Growth rate does NOT go to zero as $p \searrow \vec{p}_{\mathsf{c}}$.
   - Existence of the growth rate is left open.
2. The number of self-avoiding paths on non-directed percolation cluster is also an interesting object. From mathematical side, Lacoin (2014, two papers) proved the non-coincidence of quenched vs. annealed growth rate. Existence of the growth rate is left open.

Thank you!