Fluctuation of spectra in random media revisited

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Based on joint work with M. Biskup (UCLA) and W. König (WIAS)

Anderson Hamiltonian

Anderson Hamiltonian is the random Schrödinger operator of the form

$$H_{\xi} = -\Delta + \xi$$
 on $\ell^2(\mathbb{Z}^d)$,

where $\xi = \{\xi(x)\}_{x \in \mathbb{Z}^d}$ is random, stationary and ergodic potential.

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Main streams

- 1. Localization of low energy eigenfunctions.
- 2. Localization of the wave function $e^{itH_{\xi}}\psi$.
- 3. Localization of the diffusion $e^{-tH_{\xi}}\psi$.

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$$e^{-tH_{\xi}}\psi = E_x\left[\psi(X_t)\exp\left\{-\int_0^t\xi(X_s)\mathrm{d}s\right\}\right]$$

Setting of the problem

We are interested in the so-called "homogenization" problem.

- $D \subset \mathbb{R}^d$: a bounded domain with smooth boundary;
- $D_{\epsilon} = D \cap \epsilon \mathbb{Z}^d$: a natural discretization;
- $\Delta_{\epsilon}f(x) = \epsilon^{-2} \sum_{|y-x|=\epsilon} (f(y) f(x));$
- $\xi = \{\xi(x) \colon x \in D_{\epsilon}\}$: a random potential.

Let $\{\lambda_{D_{\epsilon},\xi}^{(k)}\}_{k\geq 1}$ be the eigenvalues of the operator (matrix)

$$-\Delta_{\epsilon} + \xi$$

with the Dirichlet (zero) boundary condition outside D_{ϵ} .

Related works 1

Crushed ice problem

 Kac (1974) and Rauch-Taylor (1975): homogenization of eigenvalues of -Δ in a randomly perforated domain;



Surface area does not control the cooling efficiency.

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Mark Kac, in his 1974 paper:

"Here the probabilistic treatment is extremely useful, because from the analytic point of view the problem looks impossible, unless you do it by the perturbation method, which few of us are willing to buy." Figari-Orlandi-Teta (1985) and Ozawa (1990): When d = 3,

$$\sqrt{N} \left[\lambda_{D \setminus \text{balls}}^{(k)} - (\lambda_D^{(k)} + \alpha)
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with $\sigma^2 = \int_D (\varphi_k(x)^2 - |D|^{-1})^2 dx$ by a heavy perturbation analysis.

Figari-Orlandi-Teta (1985) and Ozawa (1990): When d = 3,

$$\sqrt{N} \left[\lambda_{D \setminus \text{balls}}^{(k)} - (\lambda_D^{(k)} + \alpha) \right] \to \mathcal{N}(0, \sigma) \quad \text{as } N\epsilon \to 1$$

with $\sigma^2 = \int_D (\varphi_k(x)^2 - |D|^{-1})^2 dx$ by a heavy perturbation analysis.

Shin Ozawa, in Japanese article in 1992:

"Perturbation methods is hard but it yields a result which has not been achieved by the Winer sausage method. To probabilists: Give a probabilistic proof for the CLT."

Related works 2

▶ Bal (2008): Consider

$$-\Delta + \xi(\cdot/\epsilon)$$
 on $D \subset \mathbb{R}^d$ $(d \leq 3)$,

where ξ is stationary, centered and assume either

- $1. \ \mbox{boundedness}$ and a certain mixing condition or
- 2. $\mathbb{E}[\xi^6(0)] < \infty$ and a stronger mixing condition.

Then

$$\lambda_{D_{\epsilon},\xi}^{(k)}
ightarrow \lambda_D^{(k)} = k$$
'th eigenvalue of $-\Delta$ on D

as $\epsilon \downarrow 0$ in probability for each $k \ge 1$.

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as $\epsilon \downarrow 0$ in probability for each $k \ge 1$. Moreover, for distinct simple eigenvalues $\lambda_D^{(k_1)}, \ldots, \lambda_D^{(k_n)}$,

$$\epsilon^{-d/2} \big(\lambda_{D_{\epsilon},\xi}^{(k_1)} - \lambda_D^{(k_1)}, \dots, \lambda_D^{(k_n)} - \lambda_D^{(k_n)} \big) \xrightarrow{\epsilon \downarrow 0} \mathscr{N}(0,\sigma)$$

in law, where $\sigma_{ij}^2 := \operatorname{var}(\xi) \int_D \varphi_D^{(k_j)}(x)^2 \varphi_D^{(k_j)}(x)^2 dx$.

Remark

- 1. The Green's function $(-\Delta)^{-1}(x, \cdot) \in L^{2+}_{loc}$ is essential in his argument. ($\Leftrightarrow d \leq 3$.)
- 2. $\mathbb{E}[\xi^4(0)] < \infty$ suffices for IID.

Outline of Bal's argument

Let $G = (-\Delta)^{-1}$ and $G_{\xi} = (-\Delta + \xi)^{-1}$. The starting point of the argument is the following perturbative representation of the eigenvalue difference:

$$\begin{split} \lambda_{D_{\epsilon},\xi}^{-1} - \lambda_{D}^{-1} &= \langle \varphi_{D}, (G_{\xi} - G)\varphi_{D} \rangle \\ &+ \left\langle \varphi_{D_{\epsilon},\xi} - \varphi_{D}, [(G_{\xi} - \lambda_{D_{\epsilon},\xi}^{-1}) - (G - \lambda_{D}^{-1})]\varphi_{D} \right\rangle. \end{split}$$

By the formal expansion

$$G_{\xi} = (-\Delta_{\epsilon} + \xi)^{-1} = G - G\xi G + G\xi G\xi G - \cdots,$$

we get $\lambda_{D_{\epsilon},\xi}^{-1} - \lambda_D^{-1} \sim \langle \varphi_D, -G\xi G\varphi_D \rangle = -\lambda_D^{-2} \langle \varphi_D, -\xi \varphi_D \rangle.$

Lemma

$$\max\{\|G\xi G\|_{2\to 2}, \|G\xi G\xi\|_{2\to 2}, \|G_{\xi} - G\|_{2\to 2}\} = O(\epsilon^{d/2})$$
 and

$$\|G\xi G\xi G\|_{2\to 2} = o(\epsilon^{d/2}).$$

in probability.

Proof. $\|G\xi G\xi f\|_{2}^{2} = \sum_{x \in D_{\epsilon}} \epsilon^{d} \left| \sum_{y \in D_{\epsilon}} \sum_{z \in D_{\epsilon}} \epsilon^{2d} g(x, y) \xi(y) g(y, z) \xi(z) f(z) \right|^{2}$ $\leq \|f\|_{2}^{2} \sum_{x \in D_{\epsilon}} \sum_{z \in D_{\epsilon}} \epsilon^{2d} \left| \sum_{y \in D_{\epsilon}} \epsilon^{d} g(x, y) \xi(y) g(y, z) \xi(z) \right|^{2}$ $= \|f\|_{2}^{2} \sum_{x, y_{1}, y_{2}, z \in D_{\epsilon}} \epsilon^{4d} g(x, y_{1}) \xi(y_{1}) g(y_{1}, z) g(x, y_{2}) \xi(y_{2}) g(y_{2}, z) \xi(z)^{2}.$

Noting that $\mathbb{E}[\xi(y_1)\xi(y_2)\xi(z)^2] \leq \delta_{y_1,y_2}\mathbb{E}[\xi(z)^4]$, we find

$$\mathbb{E}[\|G\xi G\xi\|_{2\to 2}^{2}] \leq \operatorname{const.} \epsilon^{d} \sum_{x, y, z \in D_{\epsilon}} \epsilon^{3d} g(x, y)^{2} g(y, z)^{2}.$$

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Homogenization of eigenvalues

• $\lambda_D^{(k)}$: k-th smallest eigenvalue of $-\Delta$ on D.

Theorem (homogenization, Biskup-F.-König) If ξ is a centered IID with $\mathbb{E}[|\xi|^K] < \infty$ for some $K > 1 \lor d/2$,

$$\lambda_{D_{\epsilon},\xi}^{(k)}
ightarrow \lambda_{D}^{(k)}$$
 as $\epsilon \downarrow 0$

in probability for each $k \ge 1$.

Remark

The moment condition is optimal in the sense that if $\mathbb{E}[\xi(x)_{-}^{K}] = \infty$ for some K < d/2, then $\underline{\lim}_{\epsilon \downarrow 0} \lambda_{D_{\epsilon},\xi} = -\infty$.

Fluctuation around the mean

•
$$\lambda_D^{(k)}$$
: k-th smallest eigenvalue of $-\Delta$ on D.

• $\varphi_D^{(k)}$: corresponding eigenfunction, $\|\varphi_D^{(k)}\|_2 = 1$.

Theorem (fluctuation, BFK)

If ξ is IID with $\mathbb{E}[|\xi|^K] < \infty$ for some $K > 2 \lor d/2$ and $\lambda_D^{(k_1)}, \ldots, \lambda_D^{(k_n)}$ are distinct simple eigenvalues. Then,

$$\epsilon^{-d/2} \big(\lambda_{D_{\epsilon},\xi}^{(k_1)} - \mathbb{E} \lambda_{D_{\epsilon},\xi}^{(k_1)}, \dots, \lambda_{D_{\epsilon},\xi}^{(k_n)} - \mathbb{E} \lambda_{D_{\epsilon},\xi}^{(k_n)} \big) \xrightarrow{\epsilon \downarrow 0} \mathcal{N}(0,\sigma)$$

in law, where

$$\sigma_{ij}^2 := \operatorname{var}(\xi) \int_D \varphi_D^{(k_i)}(x)^2 \varphi_D^{(k_j)}(x)^2 \, \mathrm{d}x.$$

We focus on the first eigenvalue and drop the superscript $^{(1)}$.

Rayleigh-Ritz formula

$$\begin{split} \lambda_{D_{\epsilon},\xi} &= \inf_{\substack{g \in \ell_0^2(D_{\epsilon}), \|g\|_2 = 1}} \left\{ \|\nabla_{\epsilon} g\|_2^2 + \langle \xi, g^2 \rangle \right\}, \\ \lambda_D &= \inf_{\substack{\psi \in H_0^1(D), \|\psi\|_2 = 1}} \|\nabla \psi\|_2^2. \end{split}$$

 $\rightarrow \varphi_{D_{\epsilon},\xi}$ and φ_{D} are minimizers.

λ_{D_ε,ξ} ≲ λ_D by substituting φ_D to the first formula;
 λ_{D_ε,ξ} ≳ λ_D by substituting φ_{D_ε,ξ} to the second formula.

The first step

$$\begin{split} \lambda_{D_{\epsilon},\xi} &\leq \|\nabla_{\epsilon} \varphi_{D}\|_{2}^{2} + \langle \xi, \varphi_{D}^{2} \rangle \\ & \xrightarrow{\epsilon \downarrow \mathbf{0}} \|\nabla \varphi_{D}\|_{2}^{2} = \lambda_{D} \end{split}$$

is nothing but the weak law of large numbers.

The first step

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The second step

$$\lambda_D \leq \underbrace{\|\nabla \varphi_{D_{\epsilon},\xi}\|_2^2}_{\text{need an interpolation}} \sim \|\nabla_{\epsilon} \varphi_{D_{\epsilon},\xi}\|_2^2 + \underbrace{\langle \xi, \varphi_{D_{\epsilon},\xi}^2 \rangle}_{\text{randomly weighted sum}} = \lambda_{D_{\epsilon},\xi}$$

is more problematic.

We use the following two tools:

Finite element method

 $\exists \text{ piecewise affine interpolation } \widetilde{\varphi_{D_{\epsilon},\xi}} \text{ such that } \\ \|\nabla_{\!\epsilon} \, \varphi_{D_{\epsilon},\xi}\|_2 = \|\nabla \widetilde{\varphi_{D_{\epsilon},\xi}}\|_2.$

Elliptic regularity

 $\|\nabla_{\epsilon} \varphi_{D_{\epsilon},\xi}\|_2^2$ is bounded (with high probability).

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 $\|\nabla_{\epsilon} \varphi_{D_{\epsilon},\xi}\|_2^2$ is bounded (with high probability).

For a step function, we can use weak LLN with a tail bound step-wise. (Independence is essential here.)

Proof of the fluctuation (martingale decomposition)

We use a martingale CLT. Assume $\mathbb{E}[\xi] = 0$ and $Var(\xi) = 1$. Let $D_{\epsilon} = \{x_1, \ldots, x_n\}$ and $\mathcal{F}_m = \sigma[\xi(x_1), \ldots, \xi(x_m)]$.

$$\lambda_{D_{\epsilon},\xi} - \mathbb{E}[\lambda_{D_{\epsilon},\xi}] = \sum_{m=1}^{n} \mathbb{E}[\lambda_{D_{\epsilon},\xi} | \mathcal{F}_{m}] - \mathbb{E}[\lambda_{D_{\epsilon},\xi} | \mathcal{F}_{m-1}]$$
$$=: \sum_{m=1}^{n} Z_{m}.$$

Need to check:

(1)
$$\epsilon^{-d} \sum_{m} \mathbb{E}[Z_{m}^{2} | \mathcal{F}_{m-1}] \xrightarrow{\epsilon \downarrow 0} \int_{D} \varphi_{D}(x)^{4} dx$$
 in prob.;
(2) $\epsilon^{-d} \sum_{m} \mathbb{E}[Z_{m}^{2} \mathbb{1}_{\{|Z_{m}| > \delta \epsilon^{d/2}\}} | \mathcal{F}_{m-1}] \xrightarrow{\epsilon \downarrow 0} 0$ in prob. (easy)

Proof of the fluctuation (Hadamard's formula)

By independence,

$$\begin{split} Z_m &= \mathbb{E}[\lambda_{D_{\epsilon},\xi}|\mathcal{F}_m] - \mathbb{E}[\lambda_{D_{\epsilon},\xi}|\mathcal{F}_{m-1}] \\ &= \hat{\mathbb{E}}\left[\lambda_{D_{\epsilon},\xi_{\leq m},\widehat{\xi}_{>m}} - \lambda_{D_{\epsilon},\xi_{< m},\widehat{\xi}_{\geq m}}\right] \\ &= \hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \partial_m \lambda_{D_{\epsilon},\xi_{< m},\widetilde{\xi}_{m},\widehat{\xi}_{>m}} \mathsf{d}\widetilde{\xi}_m\right] \\ &= \hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \epsilon^d \varphi_{D_{\epsilon},\xi_{< m},\widetilde{\xi}_m,\widehat{\xi}_{>m}}^2(\mathbf{x}_m) \mathsf{d}\widetilde{\xi}_m\right]. \end{split}$$

The last = is Hadamard's first variation formula:

$$\partial_m \lambda_{D_{\epsilon},\xi} = \epsilon^d \varphi_{D_{\epsilon},\xi}(x_m)^2.$$

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Proof of the fluctuation (completion: technicality aside)

We expect

$$\mathbb{E}[Z_m^2|\mathcal{F}_{m-1}] = \epsilon^{2d} \int \mathbb{P}(\mathsf{d}\xi_m) \hat{\mathbb{E}} \left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi_{D_{\epsilon},\xi_{< m},\widetilde{\xi}_m,\widehat{\xi}_{>m}}^2(x_m) \mathsf{d}\widetilde{\xi}_m \right]^2$$
$$\stackrel{?}{\sim} \epsilon^{2d} \int \mathbb{P}(\mathsf{d}\xi_m) \hat{\mathbb{E}} \left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi_D^2(x_m) \mathsf{d}\widetilde{\xi}_m \right]^2$$
$$= \epsilon^{2d} \varphi_D(x_m)^4$$

$$\Rightarrow \epsilon^{-d} \sum_m \mathbb{E}[Z_m^2 | \mathcal{F}_{m-1}] \sim \sum_m \epsilon^d \varphi_D(x_m)^4 \sim \int_D \varphi_D(x)^4 dx.$$

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$$\stackrel{?}{\sim} \epsilon^{2d} \int \mathbb{P}(\mathsf{d}\xi_m) \hat{\mathbb{E}} \left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi_D^2(x_m) \mathsf{d}\widetilde{\xi}_m \right]^2$$
$$= \epsilon^{2d} \varphi_D(x_m)^4$$

$$\Rightarrow \epsilon^{-\mathfrak{a}} \sum_{m} \mathbb{E}[\mathcal{Z}_{m}^{*}|\mathcal{F}_{m-1}] \sim \sum_{m} \epsilon^{\mathfrak{a}} \varphi_{D}(\mathbf{x}_{m})^{*} \sim \int_{D} \varphi_{D}(\mathbf{x})^{*} \mathrm{d}\mathbf{x}.$$

But the dummy variable ξ_m prevent us from using ANY probability estimates to establish $\stackrel{?}{\sim}$.

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$$\stackrel{?}{\sim} \epsilon^{2d} \int \mathbb{P}(\mathsf{d}\xi_m) \hat{\mathbb{E}} \left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi_D^2(x_m) \mathsf{d}\widetilde{\xi}_m \right]^2$$
$$= \epsilon^{2d} \varphi_D(x_m)^4$$

$$\Rightarrow \epsilon^{-d} \sum_m \mathbb{E}[Z_m^2 | \mathcal{F}_{m-1}] \sim \sum_m \epsilon^d \varphi_D(x_m)^4 \sim \int_D \varphi_D(x)^4 \mathrm{d}x.$$

But the dummy variable $\tilde{\xi}_m$ prevent us from using ANY probability estimates to establish $\stackrel{?}{\sim}$.

 \longrightarrow concentration of measure, Kash'minskii's lemma,...

Random vs. non-random

- ▶ Perturbation methods → CLT around the homogenized eqigenvalues for d ≤ 3, (under mixing condition)
- ▶ Probabilistic method → CLT around the mean for any dimension. (under independence)

Random vs. non-random

- ▶ Perturbation methods \rightarrow CLT around the homogenized eqigenvalues for $d \leq 3$, (under mixing condition)
- Probabilistic method CLT around the mean for any dimension. (under independence)

We can always write

$$\lambda_{D_{\epsilon},\xi} - \lambda_{D} = \underbrace{\lambda_{D_{\epsilon},\xi} - \mathbb{E}[\lambda_{D_{\epsilon},\xi}]}_{\{D_{\epsilon},\xi\}} + \underbrace{\mathbb{E}[\lambda_{D_{\epsilon},\xi}] - \lambda_{D}}_{\{D_{\epsilon},\xi\}}.$$

random shift non-random shift

Q: Can we prove that the non-random part is

$$\begin{cases} = o(\epsilon^{d/2}), & \text{when } d \leq 3, \\ \gg \epsilon^{d/2}, & \text{when } d \geq 4? \end{cases}$$

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Random vs. non-random: local time heuristics

Let ξ be IID standard Gaussian and $D = (\mathbb{R}/\mathbb{Z})^d$. $(\lambda_D = 0.)$

$$\mathbb{E}\left[\exp\left\{-\epsilon^{-d/2}\lambda_{D_{\epsilon},\xi}\right\}\right] \sim \mathbb{E}\left[e^{-\epsilon^{-d/2}H_{\xi}}\mathbf{1}(0)\right]$$
$$= \mathbb{E}\left[E_{0}\left[\exp\left\{-\int_{0}^{\epsilon^{-d/2}}\xi(X_{\epsilon^{-2}s})ds\right\}\right]\right]$$
$$= \mathbb{E}\left[E_{0}\left[\exp\left\{-\epsilon^{2}\sum_{x}\xi(x)\ell_{\epsilon^{-2-d/2}}(x)\right\}\right]\right]$$
$$= E_{0}\left[\exp\left\{\frac{\epsilon^{4}}{2}\sum_{x}\ell_{\epsilon^{-2-d/2}}(x)^{2}\right\}\right],$$

where X is SRW on $(\mathbb{R}/\epsilon^{-1}\mathbb{Z})^d$ and $\ell_t(x) = \int_0^t \mathbb{1}_{\{X_s=x\}} ds$.

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Random vs. non-random: local time heuristics

$$\mathbb{E}\left[\exp\left\{-\epsilon^{-d/2}\lambda_{D_{\epsilon},\xi}\right\}\right] \sim E_0\left[\exp\left\{\frac{\epsilon^4}{2}\sum_{x}\ell_{\epsilon^{-2-d/2}}(x)^2\right\}\right].$$

Easy to check:

$$E_0\left[\|\ell_{\epsilon^{-2-d/2}}\|_2^2\right] \approx \begin{cases} \epsilon^{-4}, & d \leq 3, \\ \epsilon^{-2-d/2} \gg \epsilon^{-4}, & d \geq 5. \end{cases}$$

This infers (but does not prove) that we need a different scaling in higher dimensions.

Thank you!

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Proof of the replacement

Essential part of the proof is

$$\int \mathbb{P}(\mathsf{d}\xi_m) \hat{\mathbb{E}} \left[\int_{\widehat{\xi}_m}^{\xi_m} g_{D_{\epsilon},\,\xi_{\leq m},\,\widetilde{\xi}_m,\,\widehat{\xi}_{>m}}^2(x_m) \mathsf{d}\widetilde{\xi}_m \right]^2$$
$$\stackrel{?}{\sim} \int \mathbb{P}(\mathsf{d}\xi_m) \hat{\mathbb{E}} \left[\int_{\widehat{\xi}_m}^{\xi_m} g_{D_{\epsilon},\,\xi_{< m},\,\overline{\xi}_m,\,\widehat{\xi}_{>m}}^2(x_m) \mathsf{d}\widetilde{\xi}_m \right]^2.$$

Lemma

$$\partial_{m} \log \varphi_{D_{\epsilon},\xi}(x_{m}) = \left\langle \delta_{x_{m}}, (H_{D_{\epsilon},\xi} - \lambda_{D_{\epsilon},\xi})^{-1} P_{1}^{\perp} \delta_{x_{m}} \right\rangle$$
$$=: \epsilon^{d} G_{D_{\epsilon}}(x_{m}, x_{m}; \xi)$$

with P_1^{\perp} the orthogonal projection onto $\langle \varphi_{D_{\epsilon},\xi} \rangle^{\perp}$.

Proof of the replacement (comparison)

For some large $\lambda > 0$,

$$egin{aligned} \mathcal{G}_{D_\epsilon}(x_m,x_m;\xi) &= \sum_{k\geq 2} rac{1}{\lambda_{D_\epsilon,\xi}^{(k)} - \lambda_{D_\epsilon,\xi}} arphi_{D_\epsilon,\xi}^{(k)}(x_m)^2 \ &\lesssim \sum_{k\geq 1} rac{1}{\lambda_{D_\epsilon,\xi}^{(k)} + \lambda} arphi_{D_\epsilon,\xi}^{(k)}(x_m)^2 \ &= (H_{D_\epsilon,\xi} + \lambda)^{-1}(x_m,x_m). \end{aligned}$$

If we can replace $H_{D_{\epsilon},\xi}$ by $H_{D_{\epsilon},0}$, we are done:

$$(H_{D_{\epsilon},0}+\lambda)^{-1}(x_m,x_m)\lesssim egin{cases} 1, & d=1,\ \lograc{1}{\epsilon}, & d=2,\ \epsilon^{2-d}, & d\geq 3. \end{cases}$$

 Proof of the replacement (Khas'minskii's lemma)

We write

$$(H_{D_{\epsilon},\xi}+\lambda)^{-1}(x_m,x_m)=\int_0^{\infty}e^{-t(H_{D_{\epsilon},\xi}+\lambda)}(x_m,x_m)\mathrm{d}t.$$

Khas'minskii's lemma

$$\exists \tau > 0, \sup_{z \in D_{\epsilon}} I_{\tau,z}(\xi) := \sup_{z \in D_{\epsilon}} \int_{0}^{\tau} e^{-sH_{D_{\epsilon},0}} \xi_{-}(z) \mathrm{d}s < 1/2$$

$$\Rightarrow e^{-tH_{D_{\epsilon},\xi}}(x_{m}, x_{m}) \leq e^{t\zeta(\tau)} e^{-tH_{D_{\epsilon},0}}(x_{m}, x_{m}).$$

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Remark

This is "incredible" at the first sight since it deduces a bound on $E^{z}[e^{-\int_{0}^{\tau} \xi(X_{s})ds}]$ from that of $E^{z}[\int_{0}^{\tau} \xi_{-}(X_{s})ds]$.

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If we can find the above au,

$$egin{aligned} (H_{D_\epsilon,\xi}+\lambda)^{-1}(x_m,x_m)&\leq \int_0^\infty e^{-t(H_{D_\epsilon,0}+\lambda-\zeta(au))}(x_m,x_m)\mathrm{d}t\ &=(H_{D_\epsilon,0}+\lambda-\zeta(au))^{-1}(x_m,x_m). \end{aligned}$$

Proof of the replacement (finding τ)

Note that $\mathbb{E}[I_{\tau,z}] = \mathbb{E}[\int_0^{\tau} e^{-sH_{D_{\epsilon},0}}\xi_{-}(z)ds] \leq \tau \max_y \mathbb{E}[\xi_{-}(y)].$ Moreover, since

$$egin{aligned} |I_{ au,z}(\xi)-I_{ au,z}(\eta)| &\leq \int_0^ au \|e^{-s\Delta_\epsilon}(z,\cdot)\|_2\|\xi-\eta\|_2\mathrm{d}s\ &= \|\xi-\eta\|_2\int_0^ au e^{-2s\Delta_\epsilon}(z,z)^{1/2}\mathrm{d}s\ &\lesssim \|\xi-\eta\|_2 egin{cases} au^{1-d/4}\epsilon^{d/2}, & d\leq 3,\ \epsilon^2\log(au\epsilon^{-2}), & d=4,\ \epsilon^2, & d\geq 5, \end{aligned}$$

Talagrand's inequality implies concentration around the mean.