Quenched tail estimate for the random walk in random scenery and in random layered conductance

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The 7th Pacific Rim Conference on Mathematics
July 1, 2016

Joint work with J.-D. Deuschel (TU Berlin).
Slides will be avairable at my webpage.

Random walk in random scenery (RWRS)

- $(\{z(x)\}_{x\in\mathbb{Z}^d}, \mathbb{P})$: IID random variables,
- $(S_n)_{n\in\mathbb{Z}_+}$: Random walk on $\mathbb{Z}^d$.

Random walk in random scenery:

$$W_n := \sum_{k=1}^{n} z(S_k).$$

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- Introduced by Borodin and Kesten-Spitzer in 1979.
- Scaling limit (under $\mathbb{P}\otimes P_0$) yields a self-similar process.

  $d=1$, z: α-stable, S: $\beta_{>1}$-stable $\Rightarrow$ index $= 1-\frac{1}{\beta}+\frac{1}{\alpha\beta}$.
- CLT holds in transient case.
- $d=2$ too (!) but for $\frac{1}{\sqrt{n\log n}}W_n$ (Bolthausen 1989).

RWRS: continuous time

In this talk,

- $(\{z(x)\}_{x\in\mathbb{Z}^d},\mathbb{P})$: IID, ≥ 0 with $\mathbb{P}(z(x)\geq r)=r^{-\alpha+o(1)}$,
- $((S_t)_{t\geq 0},(P_x)_{x\in\mathbb{Z}^d})$: continuous time simple random walk.

Continuous time version of RWRS:

$$A_t := \int_0^t z(S_u)\mathrm{d}u.$$

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Naturally appears in random media:

- $E_x\left[f(X_t)e^{A_t}\right]$ is a solution of $\partial_t u = \Delta u + zu$, $u(0,x) = f(x)$.
- $(S_t^1, S_t^2 + A_t^1)_{t\geq 0}$: diffusion in random shear flow.
- $(S^1_{A_t^2}, S_t^2)_{t\geq 0}$: random walk in layered conductance (later).

Tail estimates for RWRS

Many *annealed* results: $\mathbb{P} \otimes P_0(A_t \geq t^\rho)$.

- Natural tail assumption is $\mathbb{P}(z(x) \geq r) \approx \exp(-r^\alpha)$.

  "z has high exceedance" & "RW use it": both exponential.
  (To be explained more in the next slide.)
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  - Google search "Large and moderate deviations for random walks in random scenery: a review" by F. Castell.

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Not so many *quenched* results: $P_0(A_t \geq t^\rho)$ for typical z.

- Brownian motion in Gaussian scenery,
  - Large deviation for $\frac{1}{t\sqrt{\log t}} A_t$: Asselah-Castell (2003),
  - Moderate deviations: Castell (2004),
- Brownian motion in bounded scenery,
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No results in Borodin and Kesten-Spitzer setting.

Light tail vs Heavy tail

Let

$$\ell_t := \int_0^t \delta_{S_u} \mathrm{d}u \Rightarrow A_t = \langle \ell_t, z \rangle.$$

The strategy for $\{A_t \geq t^\rho\}$ under $\mathbb{P} \otimes P_0$ is

$$\{z(\cdot) \approx a_t \psi(\cdot)\} \text{ and } \{\langle \ell_t, \psi \rangle \geq t^\rho / a_t\}.$$

Assume $\mathbb{P}(z(x) \geq r) \approx \exp(-r^\alpha)$.

- $\alpha < 1 \Rightarrow$ optimal $\psi = \delta_0$.
- $\alpha > 1 \Rightarrow$ optimal ψ has a non-trivial profile.

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The quenched results by Asselah-Castell corresponds to the second regime. Search for a high exceedance in the *physical space* instead of the probability space.

Main result I: $\mathbb{P}(z(x) \geq r) = r^{-\alpha+o(1)}$.

Let $\rho > 0$. Then $\mathbb{P}$-almost surely,

$$P_0\left(A_t \geq t^{\rho}\right) = \exp\left\{-t^{p(\alpha,\rho)+o(1)}\right\}$$

as $t \to \infty$, where

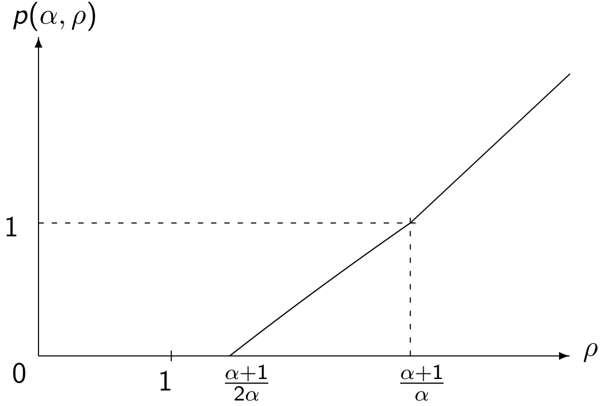
$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho}{\alpha+1} - 1, & \rho \in \left(\frac{\alpha+1}{2\alpha} \vee 1, \frac{\alpha+1}{\alpha}\right], \\ \alpha(\rho-1), & \rho > \frac{\alpha+1}{\alpha} \end{cases}$$

for $d = 1$ and

$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho-d}{2\alpha+d}, & \rho \in \left(\frac{d}{2\alpha} \vee 1, \frac{\alpha+d}{\alpha}\right], \\ \frac{\alpha(\rho-1)}{d}, & \rho > \frac{\alpha+d}{\alpha} \end{cases}$$

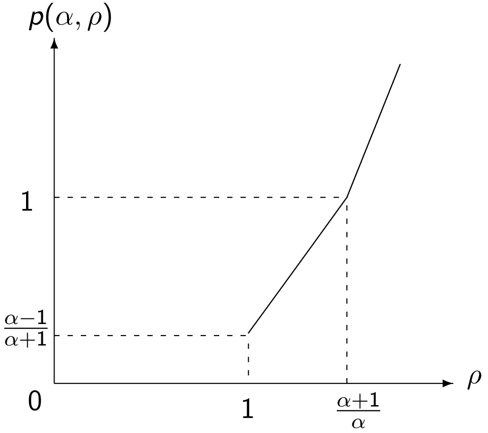
for $d \geq 2$.

Illustration: $d = 1, \alpha \leq 1$ ($d \geq 2, \alpha \leq \frac{d}{2}$ is similar)



For $\rho < \frac{\alpha+1}{2\alpha}$, we in fact have a polynomial decay. The threshold $\frac{\alpha+1}{2\alpha}$ is the self-similar index found by Borodin and Kesten-Spitzer.

Illustration: $d = 1, \alpha > 1$ ($d \geq 2, \alpha > \frac{d}{2}$ is similar)



When $\alpha > 1$, $\mathbb{E}[z(x)] < \infty$ and $P_0(A_t \geq ct) \to 1$ for $c < \mathbb{E}[z(x)]$.
For $c \geq \mathbb{E}[z(x)]$, this is the standard large deviation regime.

Main result II: $\mathbb{P}(z(x) \geq r) = r^{-\alpha+o(1)}$.

Let $d = 1$ and $\alpha > 1$ or $d \geq 2$ and $\alpha > \frac{d}{2}$. Then for any $c > \mathbb{E}[z(x)]$, $\mathbb{P}$-almost surely,

$$P_0\left(A_t \geq ct\right) = \begin{cases} \exp\left\{-t^{\frac{\alpha-1}{\alpha+1}+o(1)}\right\}, & d = 1, \\ \exp\left\{-t^{\frac{2\alpha-d}{2\alpha+d}+o(1)}\right\}, & d \geq 2 \end{cases}$$

as $t \to \infty$. (I.e., the extrapolation gives the correct exponent.)

Outline of the argument

Let us see how to get for $d = 1$ and $\alpha > 1$

$$P_0(A_t \geq ct) = \exp\left\{-t^{\frac{\alpha-1}{\alpha+1}+o(1)}\right\}.$$

Lower bound: Let the random walk

- explore $[-t^{\frac{\alpha}{\alpha+1}}, t^{\frac{\alpha}{\alpha+1}}]$,
- there is $z(x) \sim t^{\frac{1}{\alpha+1}}$ almost surely,
- leave the local time $\ell_t(x) \gtrsim t^{\frac{\alpha}{\alpha+1}}$.

The first and the third event have probability

$$\approx \exp\{-t^{\frac{\alpha-1}{\alpha+1}}\}.$$

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Upper bound:

- Consider level sets $\mathcal{H}_k = \left\{|x| \leq t^{\frac{\alpha}{\alpha+1}} : \, t^{(k-1)\epsilon} \leq z(x) < t^{k\epsilon}\right\}$,
- fine control on the "geometry" of $\mathcal{H}_k$,
- a tail estimate for additive functional by Xia Chen (2001),

$\Rightarrow$ Too difficult to get contribution from lower level sets:

$$P_0\left(\ell_t(\mathcal{H}_k) \geq t^{1-k\epsilon}\right) \ll \exp\left\{-t^{\frac{\alpha-1}{\alpha+1}}\right\}.$$

Xia Chen's theorem, Stoch. Proc. & Appl. 2001

Suppose $f \geq 0$ and

$$E_x \left[\int_0^t f(S_u) \mathrm{d}u \right] \lesssim a(t)$$

uniformly $x \in \mathrm{supp} f$. Let $0 \ll b(t) \ll t$. Then for $\lambda > 4$,

$$P_0 \left(\int_0^t f(S_u) \mathrm{d}u \geq \lambda a \left(\frac{t}{b(t)} \right) b(t) \right) \leq \exp \left\{ -c(\lambda) b(t) \right\}.$$

Remark

- *There is a corresponding lower bound.*
- *If $a(\cdot)$ varies regularly, sharper bound available.*
- *Simpler case dates back to Khas'minskii (1959).*

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Apply this to $f = 1_{\mathcal{H}_k}$.

- $E_x[\ell_t(\mathcal{H}_k)] = E_x[\int_0^t f(S_u)\mathrm{d}u] = \sum_y 1_{\mathcal{H}_k}(y) \int_0^t p_u(x,y)\mathrm{d}u$,
- Concentration inequality & Borel-Cantelli $\Rightarrow$ uniform estimate.
- Q: $E_x[\int_0^t z(S_u)\mathrm{d}u] \sim \sum_y G(x,y)z(y)$: extreme values?

Application to random layered conductance model

Let $((X_t)_{t\geq 0}, (P_x^\omega)_{x\in\mathbb{Z}^{1+d}})$ be a continuous time Markov chain on $\mathbb{Z}^2$ with jump rates

$$\omega(x, x \pm \mathbf{e}_i) = \begin{cases} z(x_2), & i = 1, \\ 1, & i = 2. \end{cases}$$

By using CTSRW (S^1, S^2) on $\mathbb{Z}^2$,

$$(X_t^1, X_t^2)_{t\geq 0} = (S^1_{A_t^2}, S_t^2)_{t\geq 0} \text{ with } A_t^2 = \int_0^t z(S_u^2)\mathrm{d}u.$$

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Anomalous behavior expected (Andres-Deuschel-Slowik 2016) to

$$P_0^\omega\left(X_t = t^\delta \mathbf{e}_1\right) \approx P_0\left(S^1_{A_t^2} = t^\delta \mathbf{e}_1\right) = E_0\left[p_{A_t^2}(0, t^\delta \mathbf{e}_1)\right].$$

Tail estimate for layered conductance model

For $\mathbb{P}$-almost every ω,

$$P_0^\omega\left(X_t = t^\delta \mathbf{e}_1\right) = \exp\left\{-t^{q(\alpha,\delta)+o(1)}\right\}$$

as $t \to \infty$, where

$$q(\alpha,\delta) = \begin{cases} 0, & \delta < \frac{1}{2} \vee \frac{\alpha+1}{4\alpha}, \\ 2\delta - 1, & \delta \in \left[\frac{1}{2}, \frac{\alpha}{\alpha+1}\right), \\ \frac{4\alpha\delta-\alpha-1}{3\alpha+1}, & \delta \in [\frac{\alpha}{\alpha+1} \vee \frac{\alpha+1}{4\alpha}, \frac{2\alpha+1}{2\alpha}], \\ \frac{\alpha(2\delta-1)}{\alpha+1}, & \delta \in (\frac{2\alpha+1}{2\alpha}, \frac{\alpha}{(\alpha-1)_+}), \\ \delta, & \delta \geq \frac{\alpha}{(\alpha-1)_+}. \end{cases}$$

$\exists$ Extensions to higher dimensions & non-horizontal displacement.

Thank you!