Brownian survival and Lifshitz tail in perturbed lattice disorder

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Random Processes and Systems
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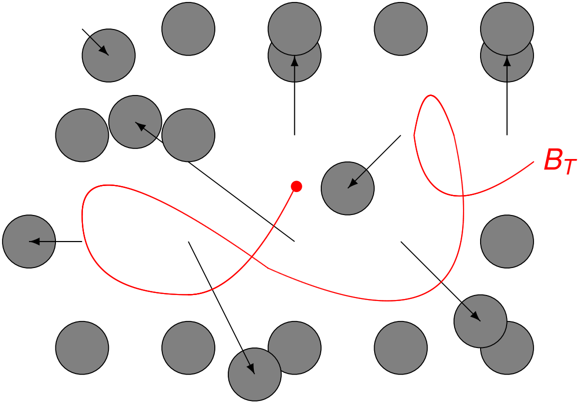
B_T

1. Model

- $\left(\{B_t\}_{t\geq 0}, P_x\right)$: standard Brownian motion on $\mathbb{R}^d$.
- $\left(\{\xi_q\}_{q\in\mathbb{Z}^d}, \mathbb{P}_\theta\right)$: i.i.d. with $\mathbb{P}_\theta(\xi_q \in dx) \asymp \exp\{-|x|^\theta\}dx$.

We call $\xi := \sum \delta_{q+\xi_q}$ the perturbed lattice.

Killing traps

We define killing traps by

$$V_\xi(x) := \sum_{q\in\mathbb{Z}^d} W(x - q - \xi_q).$$

Survival probability

The main object of this talk is

$$S_{T,\xi} = E_0\Big[\exp\Big\{-\int_0^T V_\xi(B_s)\,ds\Big\}\Big].$$

Example 1. When ξ is the lattice points,

$$\log S_{T,\xi} \sim -c_1 T, \qquad T \to \infty.$$

Example 2. (Donsker-Varadhan '75, Sznitman '90, '93)
When ξ is the Poisson points,

$$\log \mathbb{E}_\theta[S_{T,\xi}] \sim -c_2 T^{\frac{d}{d+2}}, \qquad T \to \infty,$$
$$\log S_{T,\xi} \sim -c_3 T/(\log T)^{2/d}, \quad T \to \infty, \text{ a.s.}$$

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2. Annealed asymptotics

2.1 Reduction step

We take $W = \infty \cdot 1_{\overline{B}(0,1)}$ for simplicity. Then,

$$\mathbb{E}_\theta[S_{T,\xi}] = \mathbb{E}_\theta\left[P_0(H_{\mathrm{supp}\,V_\xi} > T)\right]$$
$$\sim \sum_U \mathbb{P}_\theta(\xi(U) = 0) P_0(T_U > T) \quad (U = \mathbb{R}^d \setminus \mathrm{supp}\,V_\xi)$$
$$\sim \sup_U \mathbb{P}_\theta(\xi(U) = 0) P_0(T_U > T) \quad \text{(Laplace principle)},$$

where $\sum$ and sup are taken over $U \in \{\mathbb{R}^d \setminus \mathrm{supp}\,V_\xi\}$.

- $\log S_T \sim \sup_U \{\log \mathbb{P}_\theta(\xi(U) = 0) + \log P_0(T_U > T)\}$

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2.2 Hole probability

It is well known that

$$\log P_0(T_U > T) \sim -\lambda_1(U)T,$$

where $\lambda_1(U)$ is the smallest eigenvalue of $-1/2\Delta_D$ in U.

Lemma

Under some regularity assumptions on $U \subset \mathbb{R}^d$,

$$\log \mathbb{P}_\theta(\xi(U) = 0) \sim -\int_U \mathrm{dist}(x, \partial U)^\theta dx.$$

- $\log S_T \sim -\inf_U \left\{ \lambda_1(U)T + \int_U \mathrm{dist}(x, \partial U)^\theta dx \right\}$

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By the scaling $U = rU_r$,

$$\begin{aligned}\log S_T &\sim -\inf_U \left\{ \lambda_1(U)T + \int_U \mathrm{dist}(x, \partial U)^\theta dx \right\} \\ &= -\inf_{U_r} \left\{ \lambda_1(U_r)Tr^{-2} + r^{d+\theta} \int_{U_r} \mathrm{dist}(x, \partial U_r)^\theta dx \right\} \\ &= -Tr^{-2} \inf_{U_r} \left\{ \lambda_1(U_r) + \frac{r^{d+\theta}}{Tr^{-2}} \int_{U_r} \mathrm{dist}(x, \partial U_r)^\theta dx \right\}.\end{aligned}$$

'Natural' choice: $Tr^{-2} = r^{d+\theta}$? ($\Rightarrow r = T^{1/(d+\theta+2)}$)

An inconvenient truth

$$\inf_{U_r} \left\{ \lambda_1(U_r) + \int_{U_r} \mathrm{dist}(x, \partial U_r)^\theta dx \right\} \to 0 \quad (r \to \infty).$$

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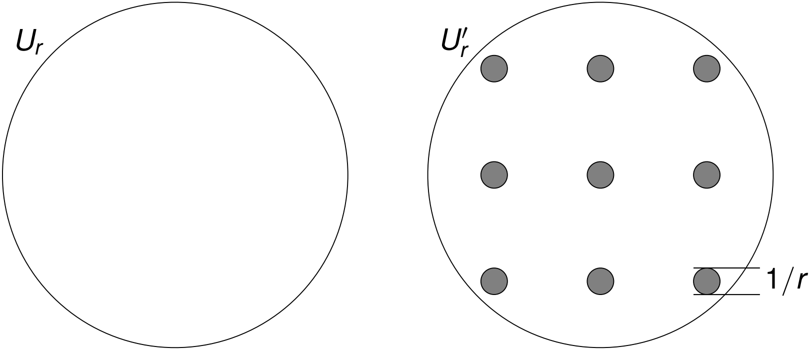

$$\int_{U_r} \mathrm{dist}(x, \partial U_r)^\theta \, dx \quad \gg \quad \int_{U_r} \mathrm{dist}(x, \partial U'_r)^\theta \, dx,$$

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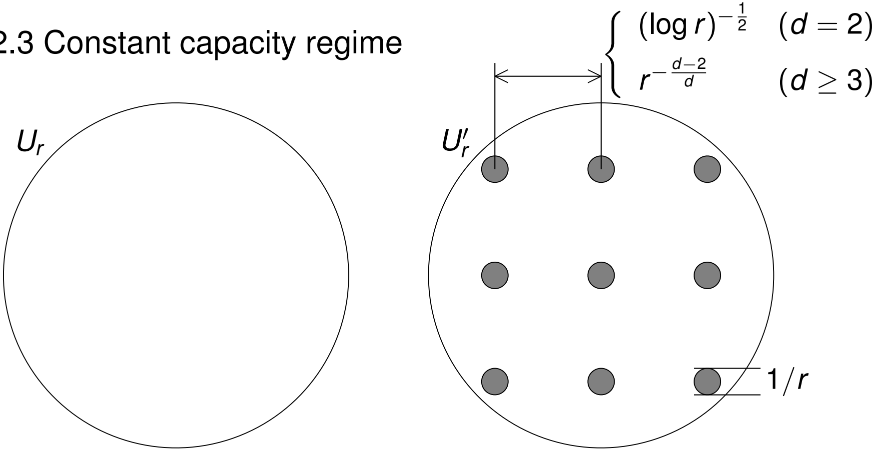
2.3 Constant capacity regime



$$\int_{U_r} \mathrm{dist}(x, \partial U_r)^\theta dx \qquad \gg \qquad \int_{U_r} \mathrm{dist}(x, \partial U'_r)^\theta dx,$$

$$\lambda_1(U_r) + \mathrm{const.} \qquad > \qquad \lambda_1(U'_r).$$

In this picture ($d \geq 3$),

$$\int_{U'_r} \mathrm{dist}(x, \partial U'_r)^\theta dx \asymp r^{-\frac{d-2}{d}\theta}.$$

This is optimal in the following sense:

Proposition

$$\inf_{r \geq 1} \inf_{U_r} \left\{ \lambda_1(U_r) + r^{\frac{d-2}{d}\theta} \int_{U_r} \mathrm{dist}(x, \partial U_r)^\theta dx \right\} > 0.$$

Therefore we should take $r^{d+\theta}/Tr^{-2} = r^{\frac{d-2}{d}\theta}$ in

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2.4 Results

Theorem 1

For any $\theta > 0$,

$$\log \mathbb{E}_\theta[S_{T,\xi}] \asymp \begin{cases} -T^{\frac{2+\theta}{4+\theta}}(\log T)^{-\frac{\theta}{4+\theta}} & (d=2), \\ -T^{\frac{d^2+2\theta}{d^2+2d+2\theta}} & (d \geq 3). \end{cases}$$

Remarks.

(1) Weak disorder limit: $\frac{d^2+2\theta}{d^2+2d+2\theta} \to 1 \quad \text{as} \quad \theta \to \infty.$

(2) Strong disorder limit: $\frac{d^2+2\theta}{d^2+2d+2\theta} \to \frac{d}{d+2} \quad \text{as} \quad \theta \to 0.$

3. Lifshitz tail

Define the density of states of $-1/2\Delta + V_\xi$ by

$$N(\lambda) := \lim_{N\to\infty} \frac{1}{(2N)^d} \mathbb{E}_\theta \Big[\#\{\lambda_k^\xi((-N,N)^d) \le \lambda\} \Big].$$

Corollary

$$\log N(\lambda) \asymp \begin{cases} -\lambda^{-1-\frac{\theta}{2}} (\log \lambda^{-1})^{-\frac{\theta}{2}} & (d=2), \\ -\lambda^{-\frac{d}{2}-\frac{\theta}{d}} & (d \ge 3). \end{cases}$$

complete lattice

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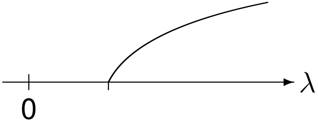

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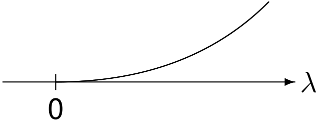

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4. Quenched asymptotics

4.1 The smallest eigenvalue and IDS

We start with a standard estimate:

$$S_{T,\xi} \leq E_0\Big[\exp\Big\{-\int_0^T V_\xi(B_s)\,ds\Big\};\ T_{(-t,t)^d} > T\Big] + e^{-cT}$$
$$\sim \exp\big\{-\lambda_1^\xi\big((-T,T)^d\big)\,T\big\}$$

Key estimate

$$N(\lambda) = \sup_{N\in\mathbb{N}} \frac{1}{(2N)^d}\mathbb{E}_\theta\Big[\#\{\lambda_k^\xi((-N,N)^d) \leq \lambda\}\Big]$$
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Using the Corollary, we have (when $d \geq 3$)

$$\mathbb{P}_\theta\big(\lambda_1^\xi((-N,N)^d) \leq \lambda\big) \leq (2N)^d \exp\{-c_1\lambda^{-L}\} \quad \Big(L = \frac{d}{2} + \frac{\theta}{d}\Big)$$

for any $N \in \mathbb{N}$ and small $\lambda > 0$.

If we take $N = T$ and $\lambda = (c_2 \log T)^{-1/L}$,

$$\mathbb{P}_\theta\big(\lambda_1^\xi((-T,T)^d) \leq (c_2 \log T)^{-1/L}\big) \leq 2^d T^{d-c_1/c_2}.$$

Thus for small $c_2 > 0$,

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4.2 Results

We can also prove (essentially) the same lower bound and obtain:

Theorem 2

For any $\theta > 0$, we have

$$\log S_{T,\xi} \asymp \begin{cases} -T(\log T)^{-\frac{2}{2+\theta}}(\log\log T)^{-\frac{\theta}{2+\theta}} & (d=2), \\ -T(\log T)^{-\frac{2d}{d^2+2\theta}} & (d \geq 3), \end{cases}$$

with $\mathbb{P}_\theta$-probability one.