A directed polymer in random environment with unbounded jumps

Ryoki Fukushima (Kyoto University)

16th Stochastic Analysis on Large Scale Interacting Systems
November 9, 2017

Partly based on a joint work with Francis Comets (University of Paris 7), Shuta Nakajima (Kyoto university) and Nobuo Yoshida (Nagoya university)

Model

- $(\{X_n\}_{n\in\mathbb{N}}, P)$: Random walk on $\mathbb{Z}^d$ with

$$P(X_{n+1} = x | X_n = y) = c_1 \exp\{-|x-y|_1^\alpha\};$$

- $(\{\eta(j,x)\}_{(j,x)\in\mathbb{N}\times\mathbb{Z}^d}, Q)$: IID, Ber($p$).

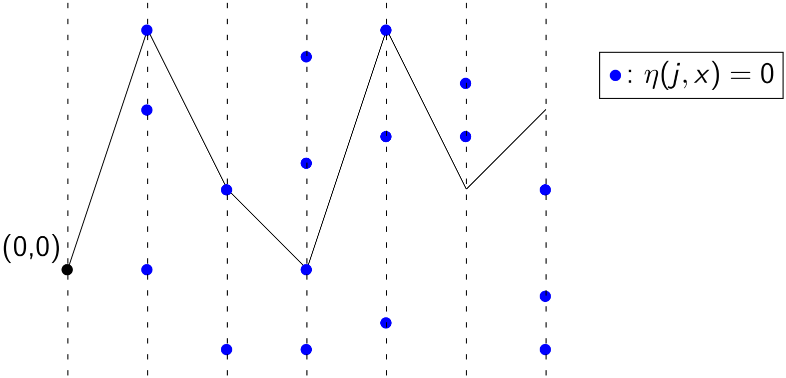
Directed polymer measure:

$$\mu_n^{\eta,\beta}(\mathrm{d}X) = \frac{1}{Z_n^{\eta,\beta}} \exp\left\{\beta \sum_{j=1}^n \eta(j, X_j)\right\} P_0(\mathrm{d}X),$$

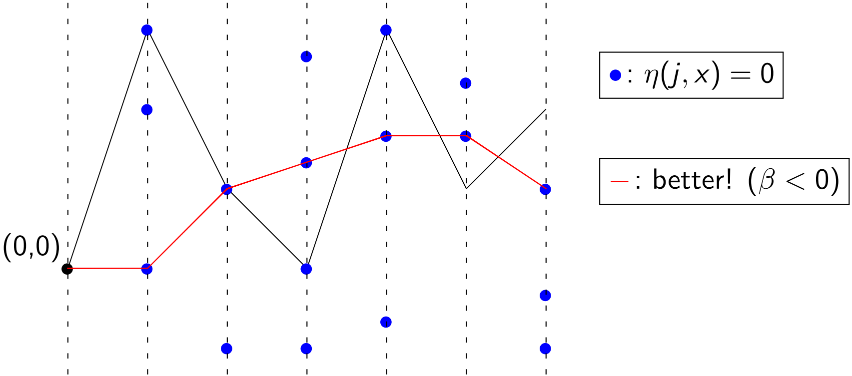
$$Z_n^{\eta,\beta} = P\left[\exp\left\{\beta \sum_{j=1}^n \eta(j, X_j)\right\}\right].$$

$\beta > 0 \Rightarrow$ attractive, $\beta < 0 \Rightarrow$ repulsive.

$$Z_n^{\eta,\beta} = \sum_{X:\ \text{path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[\beta\eta(j, X_j) - |X_{j-1} - X_j|_1^{\alpha} \right] \right\}.$$



$$Z_n^{\eta,\beta} = \sum_{X:\ \text{path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[\beta\eta(j, X_j) - |X_{j-1} - X_j|_1^{\alpha} \right] \right\}.$$



Free energy

It is standard to show the existence of the free energy:

$$\varphi(p,\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\eta,\beta} = \lim_{n\to\infty} \frac{1}{n} Q[\log Z_n^{\eta,\beta}].$$

If we naturally define $Z_n^{\eta,-\infty} = P(\sum_{j=1}^n \eta(j, X_j) = 0)$, this holds even at $\beta = -\infty$.

The key ingredient is that $Q[\log Z_n^{\eta,-\infty}] < \infty$,

Free energy

It is standard to show the existence of the free energy:

$$\varphi(p,\beta)= \lim_{n\to\infty} \frac{1}{n}\log Z_n^{\eta,\beta}= \lim_{n\to\infty} \frac{1}{n} Q[\log Z_n^{\eta,\beta}].$$

If we naturally define $Z_n^{\eta,-\infty} = P(\sum_{j=1}^n \eta(j,X_j)=0)$, this holds even at $\beta=-\infty$.

The key ingredient is that $Q[\log Z_n^{\eta,-\infty}] < \infty$, which FAILS to hold for SRW model and Brownian polymer in Poissonian environment.

Digression to nearest neighbor model

Let N_n be the number of open paths of length n in the oriented percolation. Suppose $p > p_c$ and always assume percolation.

- F.-Yoshida (2012): N_n grows exponentially.
- Garet-Gouéré-Marchand (2017): $\frac{1}{n} \log N_n$ converges.

Difficulty: $\log N_n$ is nearly super-additive but is $-\infty$ on the event of no percolation.

Digression to nearest neighbor model

Let N_n be the number of open paths of length n in the oriented percolation. Suppose $p > p_c$ and always assume percolation.

- F.-Yoshida (2012): N_n grows exponentially.
- Garet-Gouéré-Marchand (2017): $\frac{1}{n} \log N_n$ converges.

Difficulty: $\log N_n$ is nearly super-additive but is $-\infty$ on the event of no percolation.

This N_n is the (negative) zero-temperature version of the directed SRW in Bernoulli environment.

Question: Continuity of the free energy at $\beta = -\infty$?

Part 1: Free energy asymptotics

Zero temperature limit

In our model, we know $\varphi(p, -\infty)$ exists.

Theorem (Comets–F.–Nakajima–Yoshida 2015)

For any $\alpha < d$,

$$\varphi(p, \beta) \xrightarrow{\beta \searrow -\infty} \varphi(p, -\infty).$$

Remark

1. *The joint continuity in (p, β) is easy on $|\beta| < \infty$ region.*
2. *The proof shows that for any $\epsilon > 0$, we can choose large negative $\beta < 0$ such that*

   $$Z_n^{\eta, -\infty} \leq Z_n^{\eta, \beta} \leq e^{\epsilon n} Z_n^{\eta, -\infty}.$$

   This gives an alternative proof of the existence of $\varphi(p, -\infty)$.

A word on the proof: $\alpha < d$

The proof roughly goes as follows:

$$Z_n^{\eta,\beta} = \sum_{X:\text{ path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[\beta\eta(j, X_j) - |X_{j-1} - X_j|_1^{\alpha} \right] \right\}$$

$$= \left(\sum_{\text{no traps}} + \sum_{\text{few traps}} + \sum_{\text{many traps}} \right) c_1^n \exp\{\cdots\}.$$

$\sum_{\text{no traps}} = Z_n^{\eta,-\infty}$ and $\sum_{\text{many traps}}$ is negligible when $\beta \sim -\infty$.

A word on the proof: $\alpha < d$

The proof roughly goes as follows:

$$Z_n^{\eta,\beta} = \sum_{X:\text{ path}} c_1^n \exp\left\{ \sum_{j=1}^n \left[\beta\eta(j, X_j) - |X_{j-1} - X_j|_1^\alpha \right] \right\}$$

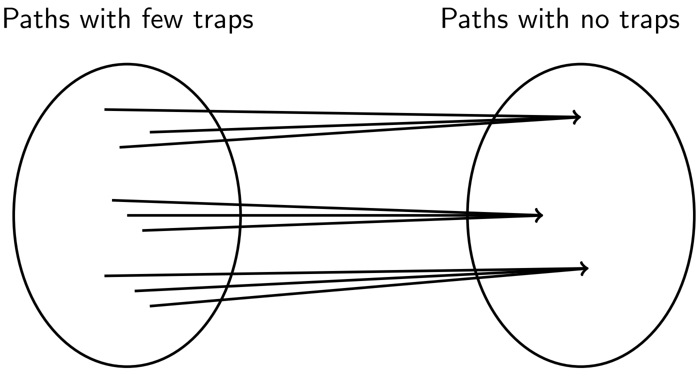
$$= \left(\sum_{\text{no traps}} + \sum_{\text{few traps}} + \sum_{\text{many traps}} \right) c_1^n \exp\{\cdots\}.$$

$\sum_{\text{no traps}} = Z_n^{\eta,-\infty}$ and $\sum_{\text{many traps}}$ is negligible when $\beta \sim -\infty$.

If a path X go through only few traps, we can deform (or map) it to a trap free path:

- it can be done without too much extra cost;
- not too many paths are mapped to the same trap free path.

Bounding $\sum_{\text{few traps}}$ by $\sum_{\text{no traps}}$



$\#\{\text{Paths with few traps}\} \leq \#\{\text{Paths with no traps}\} \times$ "Multiplicity"

The case $\alpha \geq d$

The other case $\alpha \geq d$ require a different technique and has been done by

S. Nakajima: Concentration results for directed polymer with unbounded jumps, `arXiv:1603.05032`.

In order to apply the method of bounded differences, a good control on the jumps is important.

Lemma

For any $\alpha > 1$, "typical" polymers of length n jumps at most $n^{o(1)}$.

High density limit

When $\beta = -\infty$ and $p = 1$, it is natural to set $\varphi(1, -\infty) = -\infty$.

Theorem (Comets *et al.* 2015)

For any $\alpha > 0$*, there exists* $\mu_1 > 0$ *such that*

$$\varphi(p, -\infty) \overset{p \nearrow 1}{\sim} -\mu_1 (1 - p)^{-\alpha/d}.$$

Remark

When $p \sim 1$*, RW has to jump* $(1 - p)^{-1/d}$ *to find an "open" site.*
The constant μ_1 *is a time constant of a certain FPP model.*

A directed first passage percolation

Denote the scaled open sites by

$$\omega_p = \{(k, (1-p)^{1/d}x) \colon \eta(k,x) = 0\}.$$

This converges to a homogeneous Poisson point process $(\omega_1, \mathbb{P})$ on $\mathbb{N} \times \mathbb{R}^d$ as $p \nearrow 1$.

A directed first passage percolation

Denote the scaled open sites by

$$\omega_p = \{(k, (1-p)^{1/d} x) \colon \eta(k, x) = 0\}.$$

This converges to a homogeneous Poisson point process $(\omega_1, \mathbb{P})$ on $\mathbb{N} \times \mathbb{R}^d$ as $p \nearrow 1$.

For each $p \in (0, 1]$, consider a first passage percolation:

$$T_n(\omega_p) = \min \left\{ \sum_{k=1}^{n} |x_{k-1} - x_k|^\alpha : x_0 = 0 \text{ and } \{(k, x_k)\}_{k=1}^n \subset \omega_p \right\}.$$

(cf. Howard–Newmann (1996) studied non-directed model.)

$$\exists \mu_p = \lim_{n \to \infty} \frac{1}{n} T_n(\omega_p), \quad Q\text{-a.s.}$$

Continuity of time constant

As $T_n(\omega_p)$ is the least cost for a path to avoid traps,

$$P\left(\sum_{j=1}^{n}\eta(j,X_j)=0\right)\geq c_1^n\exp\{-(1-p)^{-\alpha/d}T_n(\omega_p)\}$$
$$\gtrsim c_1^n\exp\{-(1-p)^{-\alpha/d}\mu_p n\}.$$

the lower bound follows once we show

Proposition

$\lim_{p\nearrow 1}\mu_p=\mu_1.$

Continuity of time constant

As $T_n(\omega_p)$ is the least cost for a path to avoid traps,

$$P\left(\sum_{j=1}^{n}\eta(j,X_j)=0\right) \geq c_1^n \exp\{-(1-p)^{-\alpha/d}\,T_n(\omega_p)\}$$
$$\gtrsim c_1^n \exp\{-(1-p)^{-\alpha/d}\mu_p n\}.$$

the lower bound follows once we show

Proposition

$\lim_{p\nearrow 1}\mu_p = \mu_1.$

To prove the upper bound, we need to show that there are not too many paths which are "nearly optimal". This is done by a standard block argument.

Part 2: Geometry of optimal path

Reminder

Lemma (Nakajima, 2017+)

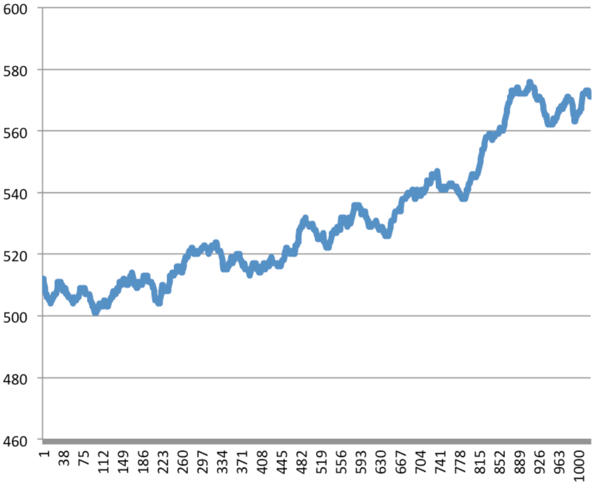
For any $\alpha > 1$, "typical" polymers of length n jumps at most $n^{o(1)}$.

This remains true (in fact easier) in the FPP setting. Let us call paths attaining $T_n(\omega_p)$ *optimal paths*.

Lemma (Nakajima, 2017+)

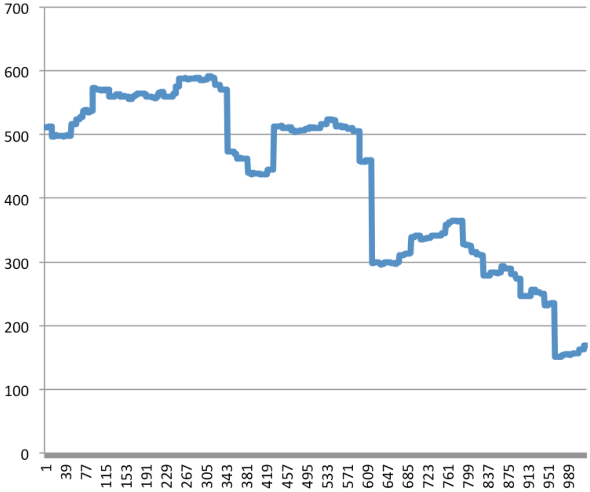
For any $\alpha > 1$, any optimal path jumps at most $n^{o(1)}$.

$(1+1)$-dim, $p = 1/2$, 1024 steps, $\alpha = 1.2$



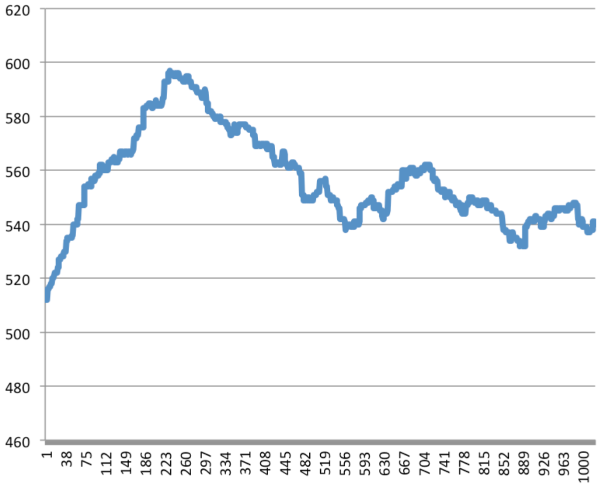
Maximal jump=3

$(1+1)$-dim, $p = 1/2$, 1024 steps, $\alpha = 0.4$



Maximal jump=162

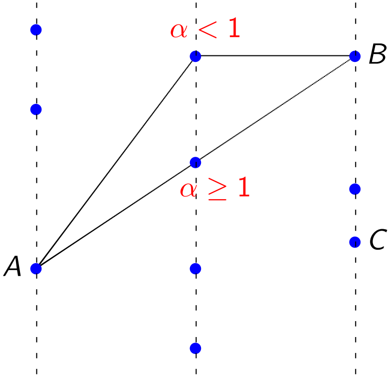
$(1+1)$-dim, $p = 1/2$, 1024 steps, $\alpha = 0.8$



Maximal jump=8

Why large jump for $\alpha < 1$?

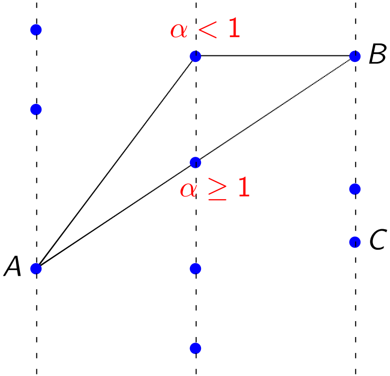
The optimal way to go from A to B is



because of convexity/concavity of the cost $|x - y|^\alpha$.

Why large jump for $\alpha < 1$?

The optimal way to go from A to B is

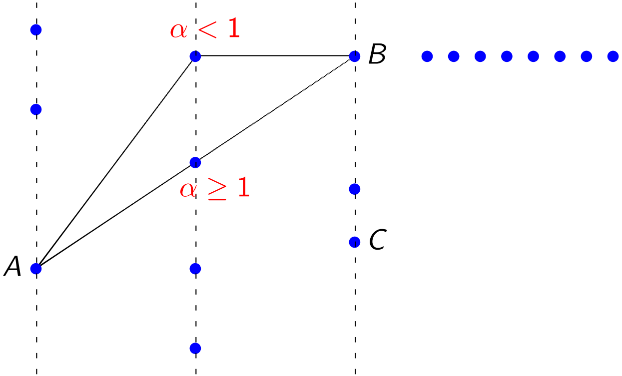


because of convexity/concavity of the cost $|x - y|^{\alpha}$.

Why go to B instead of C?

Why large jump for $\alpha < 1$?

The optimal way to go from A to B is



because of convexity/concavity of the cost $|x - y|^\alpha$.

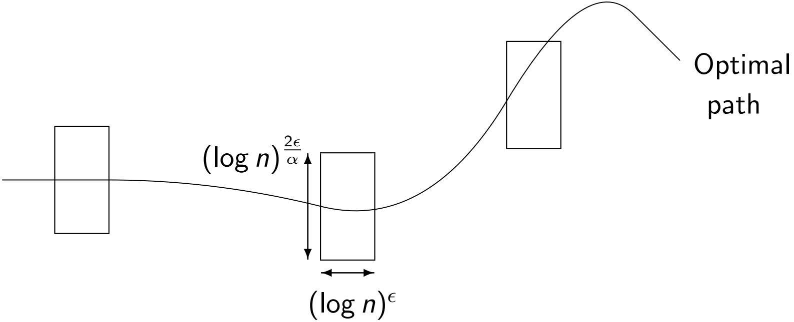
Why go to B instead of C? To find a "good" environment!

A result

Proposition

For any $\alpha \in (0, \infty)$, there exists $\epsilon > 0$ such that the maximal jump of any optimal path is larger than $(\log n)^{\epsilon}$.

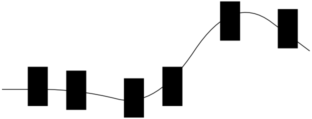
Schematic argument



We call a box *black* if its passage time is close to $\mu_p(\log t)^{\epsilon}$ and the path exits from right side. This is a typical situation and most of the boxes are black.

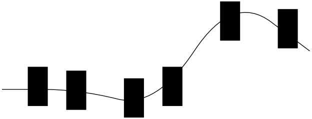
Schematic argument

There are $n^{1-o(1)}$ many black boxes.
On each black box, we *re-sample*
the configuration.



Schematic argument

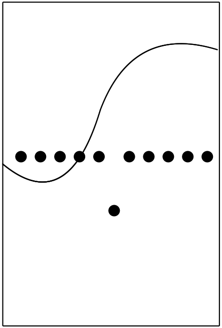
There are $n^{1-o(1)}$ many black boxes.
On each black box, we *re-sample* the configuration.



We find a configuration that has

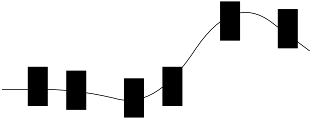
- almost straightly aligned points;
- the middle point $(\log n)^{\epsilon}$ away;
- otherwise no points inside the box;

with probability at least $\exp\{-(\log n)^{\frac{10\epsilon}{\alpha}}\}$,



Schematic argument

There are $n^{1-o(1)}$ many black boxes. On each black box, we *re-sample* the configuration.

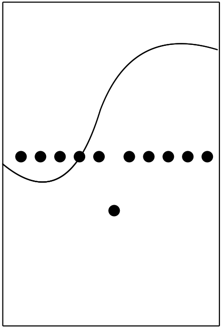


We find a configuration that has

- almost straightly aligned points;
- the middle point $(\log n)^{\epsilon}$ away;
- otherwise no points inside the box;

with probability at least $\exp\{-(\log n)^{\frac{10\epsilon}{\alpha}}\}$, that is much larger than $n^{-1+o(1)}$ for small ϵ.

If there is such a box, the optimal path does make a $(\log n)^{\epsilon}$ jump.



Schematic argument

There are two ways to make sense/use of "re-sampling".

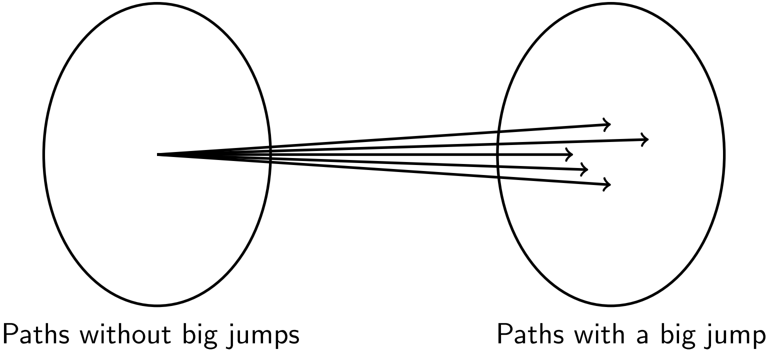
- Van den Berg–Kesten: By looking at mean values;
- Duminil-Copin *et al.*: Multi-valued map principle.

Schematic argument

There are two ways to make sense/use of "re-sampling".

- Van den Berg–Kesten: By looking at mean values;
- Duminil-Copin *et al.*: Multi-valued map principle.

We use the latter that relies on a combinatorial interpretation of the re-sampling:

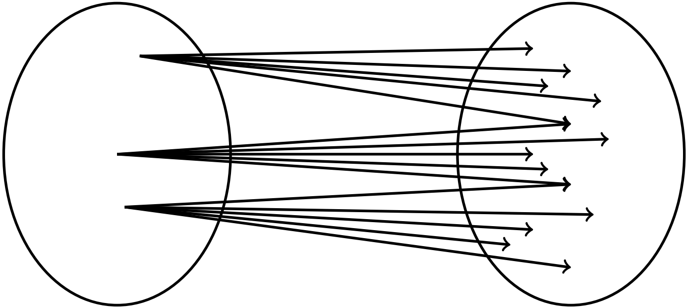


Schematic argument

There are two ways to make sense/use of "re-sampling".

- Van den Berg–Kesten: By looking at mean values;
- Duminil-Copin *et al.*: Multi-valued map principle.

We use the latter that relies on a combinatorial interpretation of the re-sampling:



$\#\{\text{Paths without big jumps}\} \quad \ll \quad \#\{\text{Paths with a big jump}\}$

Thank you!