On exponential growth for a certain class of linear systems

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Universality and Scaling Limits in
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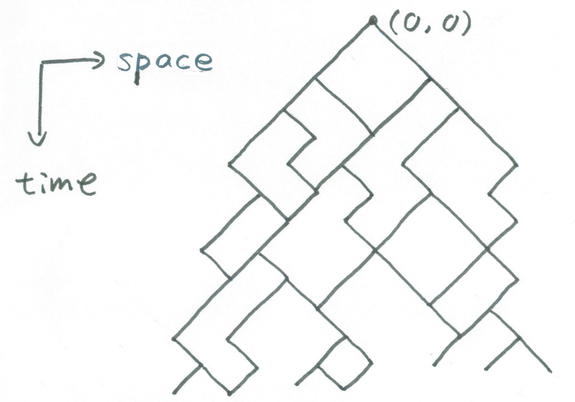
Simplest Example

(1+1)-dimensional oriented percolation.



Q. Asymptotics of the number of open paths of length n?
(We regard it as the "population".)

Related models

- Homogeneous branching random walks: each particle splits and moves independently.
- Branching random walks in space time random environment: the branching law depends on sites.
- Path counting on the oriented percolation: every particle at the same site splits and moves in the same manner.

Trivial half

$$p \leq p_c \Longrightarrow \text{extinct a.s.}$$

($p = p_c$: Grimmett-Hiemer, *Progr. Probab.*, 2002.)

Let $p > p_c$. How fast the population grows on the event of survival?

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Branching process analogy

For the supercritical Galton-Watson process,

$$\text{survival} \overset{a.s.}{\Longleftrightarrow} \text{exponential growth}$$

Remark

Let $|N_n|$ denote the total population at time n.

- For the supercritical Galton-Watson process with the Kesten-Stigum condition,

$$P\left(\lim_{n\to\infty}\frac{|N_n|}{E[|N_n|]}>0\right)>0.$$

Remark

Let $|N_n|$ denote the total population at time n.

- For the supercritical Galton-Watson process with the Kesten-Stigum condition,

$$P\left(\lim_{n\to\infty}\frac{|N_n|}{E[|N_n|]} > 0\right) > 0.$$

- Without the Kesten-Stigum condition, we still have

$$P\left(\lim_{n\to\infty}\frac{|N_n|}{C_n} > 0\right) > 0.$$

 for some $C_n \approx E[|N_n|]$.

- N. Yoshida (*J. Stat. Phys.*, 2008) proved that

$$P\left(\lim_{n\to\infty}\frac{|N_n|}{E[|N_n|]} > 0\right) > 0.$$

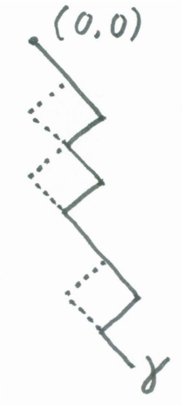
for the $(1+d)$-dimensional oriented percolation if $d \geq 3$ and p is sufficiently large. (sufficient: $p >$return probability of SRW)

However, he also proved that it fails for $d = 1, 2$.

Simple proof for the (1+1)-dimensional oriented percolation

Consider the rightmost infinite open path γ.

1. It changes the direction "many times".
2.  are open with probability p^2 independently.



Simple proof for the (1+1)-dimensional oriented percolation

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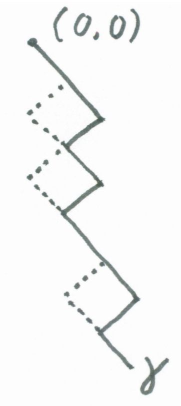
The law of large numbers shows that γ has $O(n)$-bypasses until n.

Since the population doubles whenever it has a bypass,...

$\Longrightarrow$ exponential growth along γ $\square$



Remark

Kesten-Nazarov-Peres-Sidoravicius have recently studied

$$\text{“maximal paths”} \stackrel{\text{def}}{=} \text{paths going through maximal number of open sites.}$$

Their reselt says:

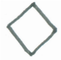
For any $p \in (0, 1)$ and any space dimension, the number of maximal paths of length n grows exponentially.

We are interested in another generalization.

(1) General space dimension
(2) Non nearest neighbor immigration
(3) Spatial correlation

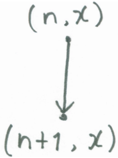
The "simple proof" breaks down since

(1) or (2) $\Longrightarrow$ no rightmost path,

(3) in addition $\Longrightarrow$ (possibly) no -bypass

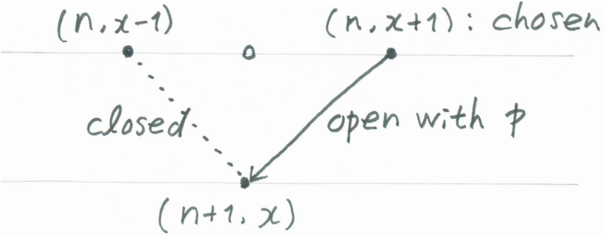
but -bypass.

Example (a discrete version of contact process)

(1) 

is open with probability q.

(2) For non-vertical bonds, choose one nearest neighbor site at the previous level uniformly and

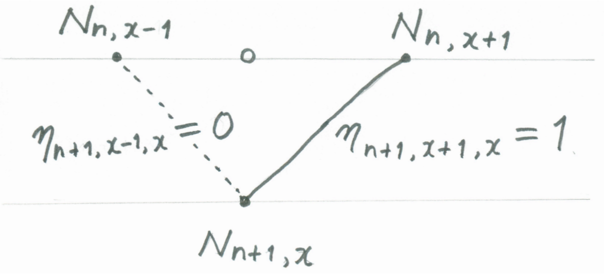


General model

$$\begin{cases} N_{n,x} : \text{the population at } (n,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d, \\ N_n = (N_{n,x})_{x \in \mathbb{Z}^d} \end{cases}$$

The evolution rule of the oriented percolation is

$$N_{n+1,x} = \sum_y N_{n,y} \cdot \eta_{n+1,y,x} \cdot 1_{\{|x-y|=1\}}$$



We can rewrite the equation as

$$N_{n+1} = N_n A_{n+1},$$

where $A_{n,x,y} = \eta_{n,x,y} \cdot 1_{\{|x-y|=1\}}$.

Key properties of $\{A_n\}_{n\in\mathbb{N}}$

(1) $\mathbb{Z}_+$-valued elements,

(2) I.I.D. in $n \in \mathbb{N}$,

(3) $(A_{n,x,y})_{x,y\in\mathbb{Z}^d} \overset{\text{law}}{=} (A_{n,x+z,y+z})_{x,y\in\mathbb{Z}^d}$ for any $z \in \mathbb{Z}^d$,

(4) For some $m \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$P\left((A_1 \cdots A_m)_{0,x} \geq 2\right) > 0.$$

The last condition (4) ensures the existence of a bypass. Indeed,

$$(A_1 \cdots A_m)_{0,x} = \sum_{x_1,\ldots,x_{m-1}} A_{1,0,x_1} \cdots A_{m,x_{m-1},x}.$$

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If $\{A_n\}_{n\in\mathbb{N}}$ are of finite range, then (4) is necessary for exponential growth since

$$\text{(4) fails} \Longrightarrow N_{n,x} \in \{0,1\},$$
$$\text{finite range} \Longrightarrow \#\{\text{occupied sites}\} = O(n^d).$$

Viewing $A_1 \cdots A_m$ as one step, we replace (4) by

(5) $P(A_{1,0,x} \geq 2) > 0$ for some $x \in \mathbb{Z}^d$.

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Definition

$$N_n = (\delta_{0,x})_{x \in \mathbb{Z}^d} A_1 \cdots A_n.$$

Theorem (F. and Yoshida)

Suppose (1)–(3) and (5) hold and the process survives with positive probability. Then,

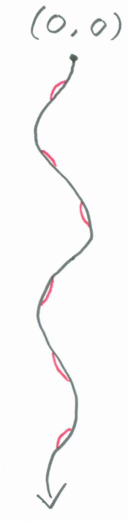
$$\text{survival} \overset{\text{a.s.}}{\Longleftrightarrow} \text{exponential growth}$$

Idea of the proof

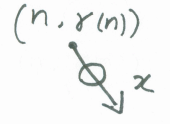
We are going to find an open path on which $A_{n,\gamma(n-1),\gamma(n)} \geq 2$ many times.

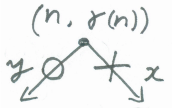
However, for an infinite open path, it is difficult to establish "independence" required to use the law of large numbers.

Thus we first construct a path which is not open in general.

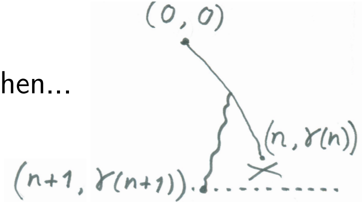


Let $x \in \mathbb{Z}^d$ be such that $P(A_{1,0,x} \geq 2) > 0$.
Set $\gamma(0) = 0$ and given $\gamma(n)$,

(1) $\gamma(n+1) = \gamma(n) + x$ if 

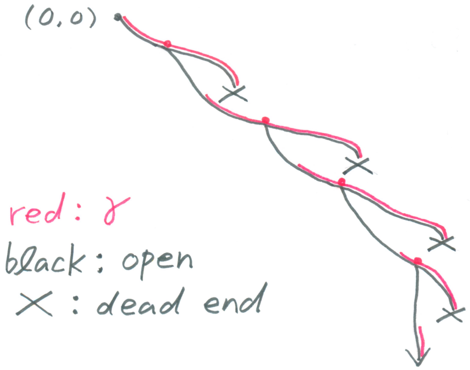
(2) $\gamma(n+1) = \gamma(n) + y$ $(y \neq x)$ if 

(3) If $(n, \gamma(n))$ is a dead end, then...



Properties of γ

- prefers to choose x-direction,
- does not refer to the future,
- does not refer to the past beyond a percolation point,
- $\exists$ open path going through all percolation points on γ.



Define the good events by

$$G_n = \{A_{n,\gamma(n-1),\gamma(n-1)+x} \geq 2 \text{ and } (n, \gamma(n)) \rightsquigarrow \infty\}$$

(1) $\exists$ open path going through all sites where G_n occurs,

(2) $\{G_n\}_{n\in\mathbb{N}}$ is stationary,

(3) $$\begin{aligned} P(G_n) &= P(A_{n,\gamma(n-1),\gamma(n-1)+x} \geq 2)P((n,\gamma(n)) \rightsquigarrow \infty) \\ &= P(A_{1,0,x} \geq 2)P(\text{survival}) \\ &> 0, \end{aligned}$$

(4) $\{G_n\}_{n\in\mathbb{N}}$ is mixing.

$$
\begin{aligned}
&P(G_m \cap G_n) \\
&\quad = P(A_{m,\gamma(m-1),\gamma(m-1)+x} \geq 2, (m, \gamma(m)) \rightsquigarrow \infty, \\
&\qquad\qquad A_{n,\gamma(n-1),\gamma(n-1)+x} \geq 2, (n, \gamma(n)) \rightsquigarrow \infty) \\
&\quad = P(A_{m,\_,\_} \geq 2, (m, \gamma(m)) \rightsquigarrow (n-1, \gamma(n-1))) \\
&\qquad \times P(A_{n,\_,\_} \geq 2, (n, \gamma(n)) \rightsquigarrow \infty) \\
&\quad \sim P(G_m)P(G_n) \quad \text{as} \quad n - m \to \infty.
\end{aligned}
$$

The law of large numbers shows that

$$\frac{1}{n}\sum_{k=1}^{n} 1_{G_k} \longrightarrow P(A_{1,0,x} \geq 2)P(\text{survival}).$$

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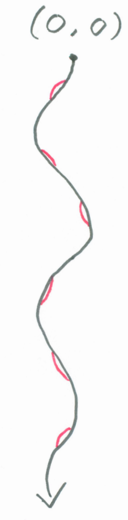
By (1), there exists an open path going through all sites where G_n occurs. Therefore we have an open path with many bypasses.

$\Longrightarrow$ exponential growth.



Thank you!