Annealed Brownian motion in a heavy tailed Poissonian potential

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1. Setting

- $\left(\{B_t\}_{t\geq 0}, P_x\right)$: $\kappa\Delta$-Brownian motion on $\mathbb{R}^d$
- $\left(\omega = \sum_i \delta_{\omega_i}, \mathbb{P}\right)$: Poisson point process on $\mathbb{R}^d$ with unit intensity

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Potential

For a non-negative and integrable function v,

$$V_\omega(x) := \sum_i v(x - \omega_i).$$

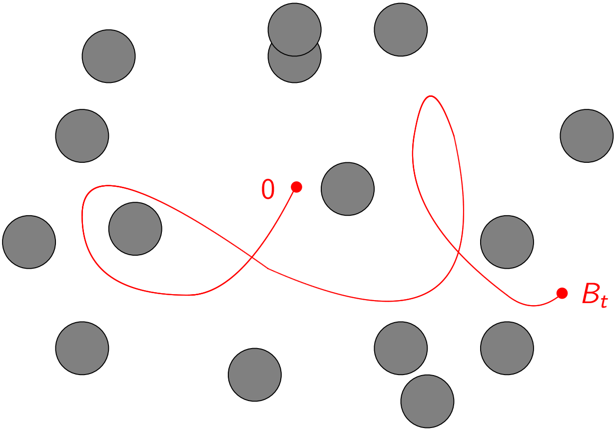
(Typically $v(x) = 1_{B(0,1)}(x)$ or $|x|^{-\alpha} \wedge 1$ with $\alpha > d$.)

Annealed measure

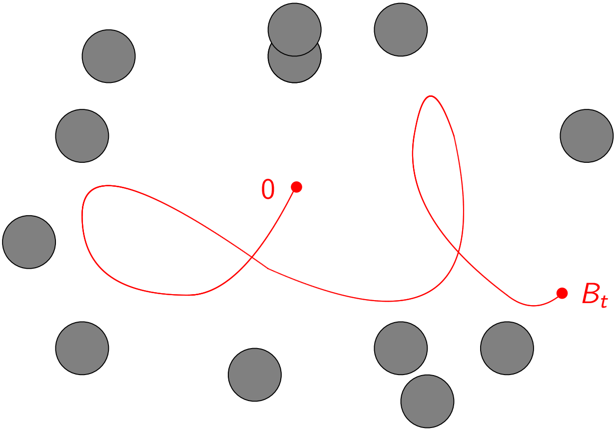
We are interested in the behavior of Brownian motion under the measure

$$Q_t(\,\cdot\,) = \frac{\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\mathbb{P}\otimes P_0(\,\cdot\,)}{\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\right]}.$$

The configuration is not fixed and hence Brownian motion and ω_i's tend to avoid each other.



$\exp\{-\int_0^t V_\omega(B_s)\mathrm{d}s\}$: large, $\quad \mathbb{P}$: large, $\quad P_0$: small



$\exp\{-\int_0^t V_\omega(B_s)\mathrm{d}s\}$: large, $\quad \mathbb{P}$: small, $\quad P_0$: large

2. Light tailed case

Donsker and Varadhan (1975)

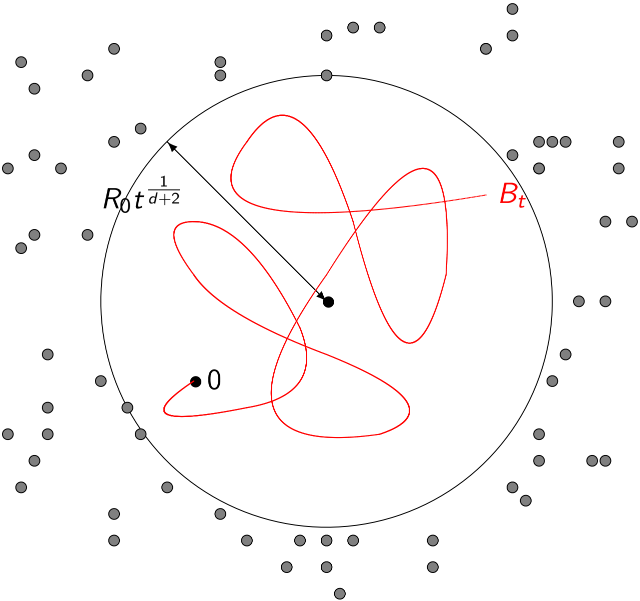
When $v(x) = o(|x|^{-d-2})$ as $|x| \to \infty$,

$$
\begin{aligned}
&\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(B_s)\, ds \right\} \right] \\
&= \exp \left\{ -c(d,\kappa) t^{\frac{d}{d+2}} (1 + o(1)) \right\} \\
&= P_0 \left(B_{[0,t]} \subset B(x, t^{\frac{1}{d+2}} R_0) \right) \mathbb{P} \left(\omega(B(x, t^{\frac{1}{d+2}} R_0)) = 0 \right),
\end{aligned}
$$

as $t \to \infty$.

Remark

$$c(d,\kappa) = \inf_U \{ \kappa \lambda^D(U) + |U| \}.$$


$R_0 t^{\frac{1}{d+2}}$
B_t
0

One specific strategy gives dominant contribution to the partition function.

$$\Downarrow$$

It occurs with high probability under the annealed path measure.

Sznitman (1991, $d = 2$) and Povel (1999, $d \geq 3$)

When v has a compact support, there exists

$$D_t(\omega) \in B\big(0, t^{\frac{1}{d+2}}(R_0 + o(1))\big)$$

such that

$$Q_t\left(B_{[0,t]} \subset B\big(D_t(\omega), t^{\frac{1}{d+2}}(R_0 + o(1))\big)\right) \xrightarrow{t\to\infty} 1.$$

Remark

Bolthausen (1994) proved the corresponding result for two-dimensional random walk model.

3. Heavy tailed case

Pastur (1977)

When $v(x) \sim |x|^{-\alpha}$ ($\alpha \in (d, d+2)$) as $|x| \to \infty$,

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(B_s) \mathrm{d}s \right\} \right] = \exp \left\{ -a_1 t^{\frac{d}{\alpha}} (1 + o(1)) \right\},$$

where $a_1 = |B(0,1)| \Gamma\left(\frac{\alpha - d}{\alpha} \right)$.

In fact, Pastur's proof goes as follows:

$$\begin{aligned}
&\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(B_s) \mathrm{d}s \right\} \right] \\
&\quad \approx \mathbb{E}[\exp\{-tV_\omega(0)\}] \\
&\quad \sim \exp\{-a_1 t^{\frac{d}{\alpha}}\}.
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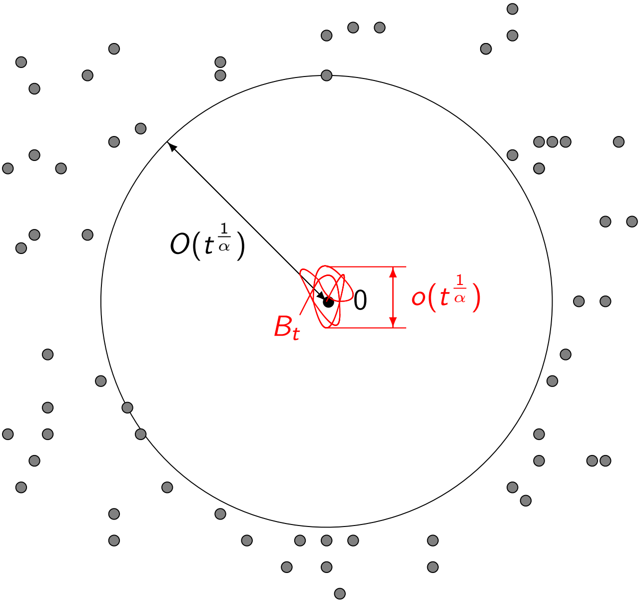
$$

The effort of the Brownian motion is hidden in the lower order terms.


$O(t^{\frac{1}{\alpha}})$
0
$o(t^{\frac{1}{\alpha}})$
B_t

F. (2011)

When $v(x) = |x|^{-\alpha} \wedge 1$ $(d < \alpha < d+2)$,

$$\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(B_s)\mathrm{d}s \right\} \right] = \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (a_2 + o(1)) t^{\frac{\alpha+d-2}{2\alpha}} \right\},$$

where

$$a_2 := \inf_{\|\phi\|_2 = 1} \left\{ \int \kappa |\nabla \phi(x)|^2 + C(d,\alpha)|x|^2 \phi(x)^2 \, \mathrm{d}x \right\}.$$

Remark

The proof is an application of the general machinery developed by Gärtner-König 2000.

Recalling the Donsker-Varadhan LDP

$$P_0\left(\frac{1}{t}\int_0^t \delta_{B_s}\mathrm{d}s \sim \phi^2(x)\mathrm{d}x\right) \approx \exp\left\{-t\int \kappa|\nabla\phi(x)|^2\mathrm{d}x\right\},$$

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In particular, since $P_0\left(B_{[0,t]} \subset B(x,R)\right) \approx \exp\{-tR^{-2}\}$, the localization scale should be

$$tR^{-2} = t^{\frac{\alpha+d-2}{2\alpha}} \Leftrightarrow R = t^{\frac{\alpha-d+2}{4\alpha}}.$$

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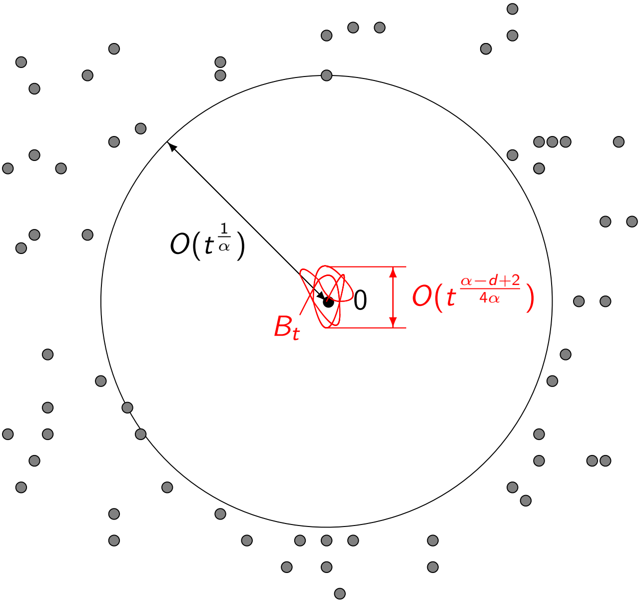
In addition, the term $\int C(d,\alpha)|x|^2\phi(x)^2\,\mathrm{d}x$ says that V_ω (locally) looks like a quadratic function.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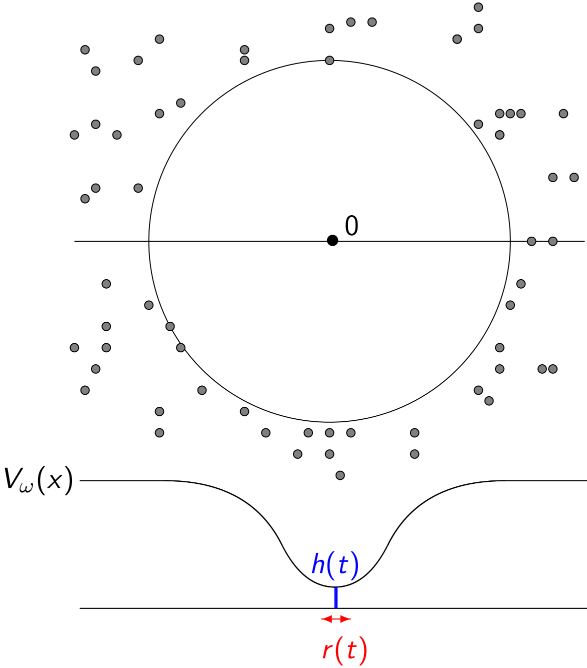

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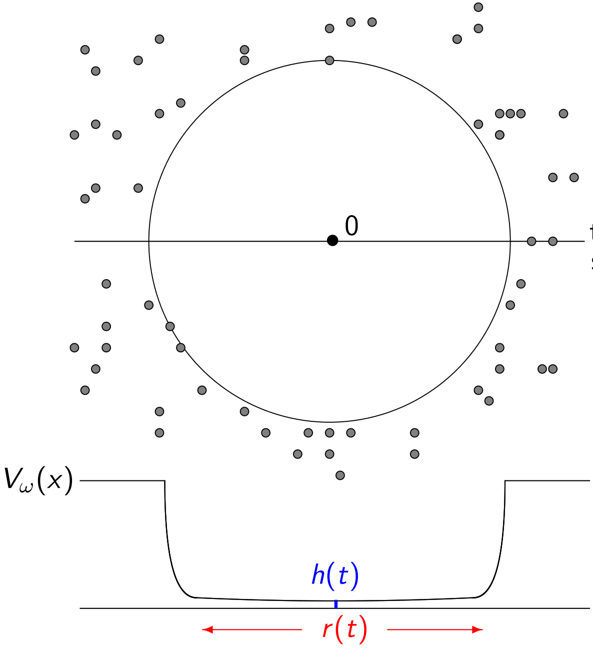
Heavy tailed case

$$t h(t) \gg t/r(t)^2$$



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Light tailed case

$$th(t) \ll t/r(t)^2$$

Main Theorem (F. 2012)

$$Q_t\left(B_{[0,t]} \subset B\left(0, t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\epsilon}\right)\right) \xrightarrow{t\to\infty} 1,$$

$$Q_t\Big(V_\omega(x) - V_\omega(m_t(\omega)) \sim t^{-\frac{\alpha-d+2}{\alpha}} C(d,\alpha)|x - m_t(\omega)|^2$$
$$\text{in } B(0, t^{\frac{\alpha-d+2}{4\alpha}+\epsilon})\Big) \xrightarrow{t\to\infty} 1,$$

$$\left\{t^{-\frac{\alpha-d+2}{4\alpha}} B_{t^{\frac{\alpha-d+2}{2\alpha}}s}\right\}_{s\geq 0} \xrightarrow{\text{in law}} \text{OU-process with}$$
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where $m_t(\omega)$ is the minimizer of V_ω in $B(0, t^{\frac{\alpha-d+2}{4\alpha}} \log t)$.

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where $m_t(\omega)$ is the minimizer of V_ω in $B(0, t^{\frac{\alpha-d+2}{4\alpha}} \log t)$.

Remark: $X_s := t^{-\frac{\alpha-d+2}{4\alpha}} B_{t^{\frac{\alpha-d+2}{2\alpha}}s} \Rightarrow B_t = t^{\frac{\alpha-d+2}{4\alpha}} X_{t^{\frac{\alpha+d-2}{2\alpha}}}$

4. Outline of the proof (of localization)

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To see 1st $\Rightarrow$ 2nd, let $L_t := \frac{1}{t}\int_0^t \delta_{B_s}\mathrm{d}s$ and rewrite

$$Q_t\left(\mathrm{d}\omega\right) = \frac{1}{Z_t} E_0\left[\mathbb{E}\left[e^{-t\langle L_t, V_\omega\rangle}\right]\mathbb{P}_t^{L_t}(\mathrm{d}\omega)\right],$$

where Z_t is the normalizing constant and

$$\mathbb{P}_t^{L_t}(\mathrm{d}\omega) := \frac{\exp\{-t\langle L_t, V_\omega\rangle\}\mathbb{P}(\mathrm{d}\omega)}{\mathbb{E}\left[\exp\{-t\langle L_t, V_\omega\rangle\}\right]}.$$

Assuming the 1st statement, we may replace L_t by $\delta_{m_{L_t}}$ with $m_{L_t} = \int x L_t(\mathrm{d}x)$ in the following:

$$\mathbb{P}_t^{L_t}\left(V_\omega(x) - V_\omega(m_t(\omega)) \sim C(d,\alpha) t^{-\frac{\alpha-d+2}{\alpha}} |x|^2\right) \to 1?$$

Assuming the 1st statement, we may replace L_t by $\delta_{m_{L_t}}$ with $m_{L_t} = \int x L_t(\mathrm{d}x)$ in the following: Let $m_{L_t} = 0$ for simplicity.

$$\mathbb{P}_t^{\delta_0}\left(V_\omega(x) - V_\omega(0) \sim C(d,\alpha) t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \right) \to 1?$$

This can be easily proved since $(\omega, \mathbb{P}_t^{\delta_0})$ is nothing but the Poisson point process with intensity $e^{-t(|x|^{-\alpha}\wedge 1)}\mathrm{d}x$.

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Remark

In fact, a slightly weaker localization bound is enough to do the above replacement.

This observation is useless (circular argument) as it is. But due to the last remark, there is a chance to go as follows:

$$\begin{aligned} &\text{crude control on the potential,} \\ \Rightarrow\ &\text{crude control on the trajectory,} \\ \Rightarrow\ &\text{fine control on the potential,} \\ \Rightarrow\ &\text{fine control on the trajectory.} \end{aligned}$$

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Assume V_ω attains its local minimum at 0 for simplicity.

4.1 Crude control on the potential

Lemma 1

$$Q_t\left(V_\omega(0) \in \frac{d}{\alpha} a_1 t^{-\frac{\alpha-d}{\alpha}} + t^{-\frac{3\alpha-3d+2}{4\alpha}}(-M_1, M_1)\right) \to 1.$$

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Idea

$$Z_t \le \mathbb{E}[\exp\{-tV_\omega(0)\}] \begin{cases} = \exp\left\{-a_1 t^{\frac{d}{\alpha}}\right\}, \\ \approx \sup_{h>0}\left[e^{-th}\mathbb{P}(V_\omega(0) \approx h)\right]. \end{cases}$$

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$\frac{d}{\alpha}a_1t^{-\frac{\alpha-d}{\alpha}}=h(t)$ is the maximizer.

$$\begin{aligned}\Rightarrow&\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}:V_\omega(0)\text{ is far from }h(t)\right]\\ &\le\mathbb{E}\left[\exp\{-tV_\omega(0)\}:V_\omega(0)\text{ is far from }h(t)\right]=o(Z_t).\end{aligned}$$

Lemma 2

$$Q_t\Big(V_\omega(0) + V_\omega(x) \ge 2h(t) + c_1 t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \\ \text{for } t^{\frac{\alpha-d+6}{8\alpha}} < |x| < M_2 t^{\frac{\alpha-d+6}{8\alpha}} \Big) \to 1.$$

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$$Q_t\Big(V_\omega(0)+V_\omega(x)\ge 2h(t)+c_1t^{-\frac{\alpha-d+2}{\alpha}}|x|^2 \\ \text{for } t^{\frac{\alpha-d+6}{8\alpha}}<|x|<M_2t^{\frac{\alpha-d+6}{8\alpha}}\Big)\to 1.$$

Idea

By Lemma 1,

$$\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\lesssim \exp\{-th(t)\}=\exp\left\{-\frac{d}{\alpha}a_1t^{\frac{d}{\alpha}}\right\}$$

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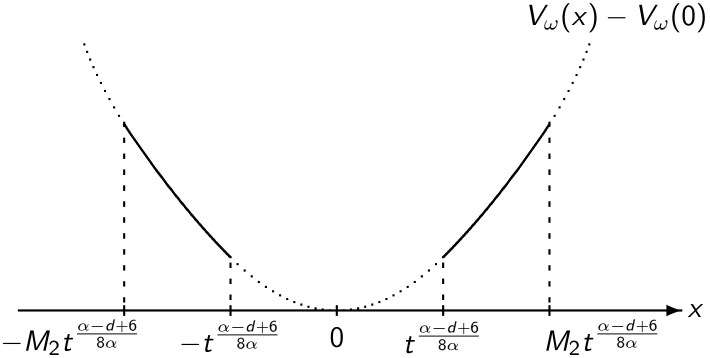
Then, use

$$\mathbb{E}\left[\exp\left\{-\frac{t}{2}(V_\omega(0)+V_\omega(x))\right\}\right]\approx \exp\left\{-a_1 t^{\frac{d}{\alpha}}-c_2 t^{\frac{d-2}{\alpha}}|x|^2\right\}$$

and Chebyshev's inequality.

4.2 Crude control on the trajectory



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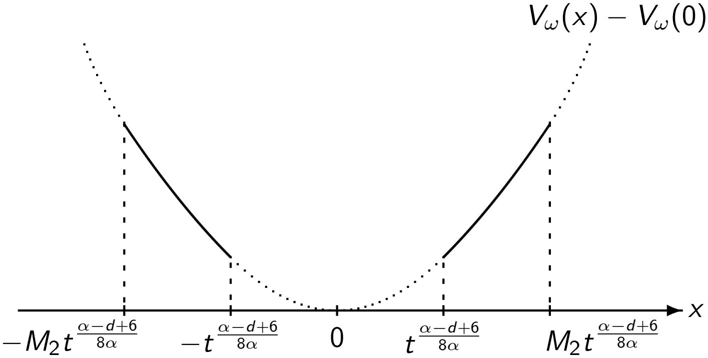

By "penalizing a crossing",

$$Q_t \left(B_{[0,t]} \subset B \left(0, M_2 t^{\frac{\alpha-d+6}{8\alpha}} \right) \right) \to 1.$$

4.3 Fine control on the potential

The "crude control on the trajectory" is good enough to yield

$$Q_t\Big(V_\omega(x) - V_\omega(0) \sim C(d,\alpha) t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \text{ in } B(0, t^{\frac{\alpha-d+2}{4\alpha}+\epsilon})\Big) \to 1.$$

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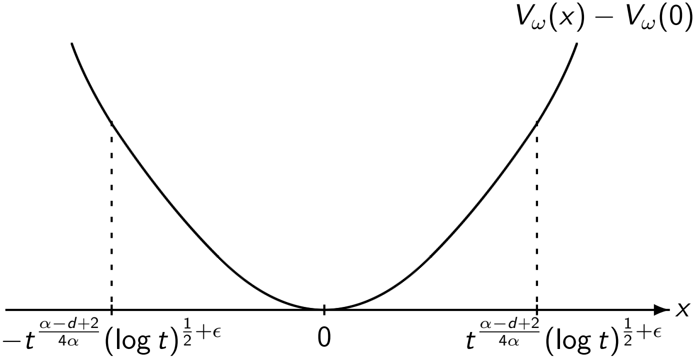

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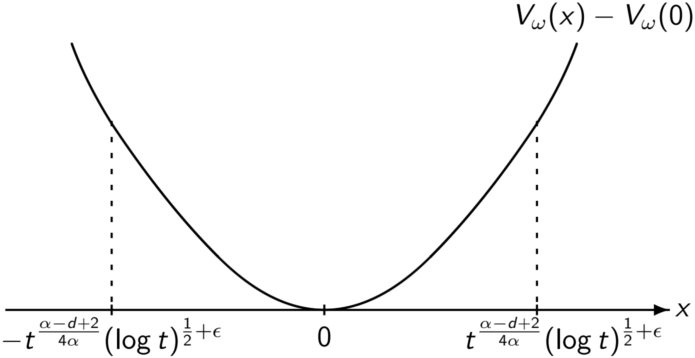

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Thank you!