HEEGAARD FLOER HOMOLOGY OF MATSUMOTO'S MANIFOLDS

MOTOO TANGE

Abstract. We consider a homology sphere $M_n(K_1, K_2)$ presented by two knots K_1, K_2 with linking number 1 and framing $(0, n)$. We call the manifold *Matsumoto's manifold*. We show that there exists no contractible bound of $M_n(T_{2,3}, K_2)$ if $n < 2\tau(K_2)$ holds. We also give a formula of Ozsváth-Szabó's τ-invariant as the total sum of the Euler numbers of the reduced filtration. We compute the δ-invariants of the twisted Whitehead doubles of torus knots and correction terms of the branched covers of the Whitehead doubles. By using Owens and Strle's obstruction we show that the 12-twisted Whitehead double of the $(2, 7)$-torus knot and the 20-twisted Whitehead double of the $(3, 7)$-torus knot are not slice but the double branched covers bound rational homology 4-balls. These are the first examples having a gap between sliceness and rational 4-ball bound-ness of the double branched cover.

1. Introduction and computational results.

1.1. Matsumoto's manifold and the contractible bound-ness. Let K_1, K_2 be two knots. We define to be $M_n(K_1, K_2)$ a homology 3-sphere presented by K_1, and K_2 with geometrically linking number one and framing $(0, n)$ respectively. See Figure 1. Let $W_n(K_1, K_2)$ be a 4-manifold described

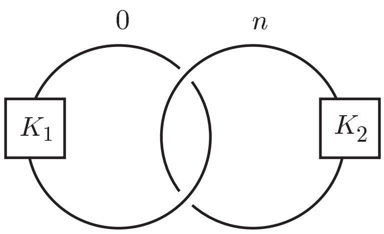

Figure 1. Matsumoto manifold $M_n(K_1, K_2)$.

by the same picture. The 4-manifold $W_n(K_1, K_2)$ is a homology $S^2 \times S^2 -$

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$\nu(B^4)$, where B^4 is the standard 4-ball. Y. Matsumoto asked in [1] whether two generators in $H_2(W_0(K_1, K_2))$ can be realized by an embedded $S^2 \vee S^2$ (the one-point union of two 2-spheres) or not. If yes, then $M_0(K_1, K_2)$ bounds a contractible 4-manifold. If no, then the generators are realized by two embedded Casson handles in $W_0(K_1, K_2)$, and it is an exotic 2-handle homeomorphic to $D^2 \times \mathbb{R}^2$, by Freedman's theorem [6]. Following Matsumoto, we call $M_n(K_1, K_2)$ *Matsumoto's manifold* in this paper.

If K_1 is a slice knot, then $M_n(K_1, K_2)$ bounds a contractible 4-manifold, because $M_n(K_1, K_2)$ is the boundary of the slice disk complement K_1 together with a 2-handle. Thus, the attachment has $\pi_1 = e$ and $H_2 = H_3 = 0$, hence it is a contractible 4-manifold. If K_1 is not slice, the problem of the contractible bound-ness is unclear.

Let K be a knot in S^3. Then we have

$$M_n(T_{2,3}, K) = S^3_1(D_+(K, n)),$$

where $T_{r,s}$ is the positive (r, s)-torus knot and $D_+(K, n)$ is the n-twisted (positive-clasped) Whitehead double of K. $S^3_p(K')$ is p-surgery of K' in S^3. For example, FIGURE 2 is the picture of $D_+(T_{2,3}, n)$. Let F be the figure-8 knot. Then we have $M_n(F, K) = S^3_{-1}(D_+(K, n))$. We argue the existence of contractible bounds of $M_n(T_{2,3}, K)$ in the present paper.

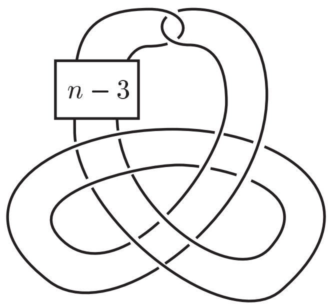

FIGURE 2. The n-twisted Whitehead double of the trefoil.

If K_2 is a slice knot, then $D_+(K_2, 0)$ is slice as in [12]. Thus, $M_0(T_{2,3}, K_2)$ has a contractible bound.

The Alexander polynomial of the n-twisted Whitehead double is as follows:

$$\Delta_{D_+(K,n)}(t) = -nt + 2n + 1 - nt^{-1}.$$

The result in [7] says that if K' is slice, then the Alexander polynomial is of form $\Delta_{K'}(t) = f(t)f(t^{-1})$, where $f(t)$ is a polynomial with integer coefficients. Hence, if $D_+(K, n)$ is slice, then $\Delta_{D_+(K,n)}$ must be of form $f(t)f(t^{-1})$. This condition is equivalent to $n = m(m + 1)$ for some integer m.

Table 1 is a list of well-known facts about sliceness of $D_+(K, n)$ and contractible bound-ness of $M_n(K_1, K_2)$. We here state Ozsváth-Szabó's τ-invariant formula of $D_+(K, n)$ by Hedden. The τ-invariant by Ozsváth and

1976	Casson, Rolfsen	$D_+(T_{2,3},6)$ is slice. [21, 12]
1984	Maruyama	$M_6(T_{2,3},T_{2,3})$ bounds a contractible 4-manifold. [15]
1997	Akbulut	$M_0(T_{2,3},T_{2,3})$ bounds no contractible 4-manifold. [2]
2006	Bar-Natan	$D_+(T_{2,3},2)$ is not slice. [3]
2007	Hedden	The computation of $\tau(D_+(K,n))$. [8]
2012	Collins	$D_+(T_{p,q},n)$ is slice for any relatively prime (p,q) for at most one n. [5]
2013	Tsuchiya	$M_{2n+1}(T_{2,3},T_{2,3})$ bounds no contractible 4-manifold. [24]

Table 1. The well-known results for the sliceness of $D_+(T_{p,q},n)$ and the contractible bound-ness of $M_n(T_{2,3},K)$.

Szabó is a homomorphism from the smooth knot concordance group to integers, i.e., $\tau:\mathcal{C}_{\mathrm{sm}}\to\mathbb{Z}$.

Theorem 1 ([8]). *Let K be a knot in S^3.*

$$\tau(D_+(K,n))=\begin{cases}0 & n\geq 2\tau(K)\\ 1 & n<2\tau(K).\end{cases}$$

In particular, if $n<2\tau(K)$, then $D_+(K,n)$ is not slice.

We can easily compute the Casson invariant by using the Dehn surgery formula as follows:

$$\lambda(M_n(T_{2,3},K))=\frac{1}{2}\cdot\Delta''_{D_+(K,n)}(t)|_{t=1}=-n. \tag{1}$$

One of the main purposes of the present paper is to compute of HF^+ of $S^3_1(D_+(T_{2,3},n))$ and to generalize to $S^3_1(D_+(K,n))$ and to discuss the contractible bound-ness of $M_n(T_{2,3},K)$. Let $M_n(K)$ denote $M_n(T_{2,3},K)$.

Theorem 2. *The Heegaard Floer homology of $M_n(T_{2,3})$ is computed as follows:*

$$HF^+(M_n(T_{2,3}))=\begin{cases}\mathcal{T}^+_{(0)}\oplus HF_{\mathrm{red}}(M_n(T_{2,3})) & n\geq 2\\ \mathcal{T}^+_{(-2)}\oplus HF_{\mathrm{red}}(M_n(T_{2,3})) & n<2,\end{cases}$$

and further

$$HF_{\mathrm{red}}(M_n(T_{2,3}))\cong\begin{cases}\mathbb{F}^{n-2}_{(-1)}\oplus\mathbb{F}^2_{(-3)} & n\geq 2\\ \mathbb{F}^{1-n}_{(-2)}\oplus\mathbb{F}^2_{(-3)} & n<2.\end{cases}$$

This computation is generalized to the case of any knot K as well as $T_{2,3}$ as follows:

Theorem 3. *Let K be a knot in S^3 with genus g. The Heegaard Floer homology of $M_n(K)$ is computed as follows:*

$$HF^+(M_n(K))\cong\begin{cases}\mathcal{T}^+_{(0)}\oplus HF_{\mathrm{red}}(M_n(K)) & n\geq 2\tau(K)\\ \mathcal{T}^+_{(-2)}\oplus HF_{\mathrm{red}}(M_n(K)) & n<2\tau(K)\end{cases}$$

and further,

$$HF_{\mathrm{red}}(M_n(K)) \cong \begin{cases} \mathbb{F}_{(-1)}^{n-2\tau(K)} \overset{g}{\underset{i=-g}{\oplus}} H_{*+1}(\widetilde{\mathcal{F}}(K,i))^2 & n \geq 2\tau(K) \\ \mathbb{F}_{(-2)}^{2\tau(K)-n-1} \overset{g}{\underset{i=-g}{\oplus}} H_{*+1}(\widetilde{\mathcal{F}}(K,i))^2 & n < 2\tau(K). \end{cases}$$

The reduced knot filtration $\widetilde{\mathcal{F}}(K,i)$ will be defined in the Section 2.2. $\widetilde{\mathcal{F}}(K,i)$ is a sub-filtration of the knot filtration

This theorem means that from Theorem 1 we have

$$\tau(D_+(K,n)) = -2d(S_1^3(D_+(K,n))) = -2d(M_n(K)).$$

In the same way as the one of the case of $M_n(T_{2,3},K)$ one can also compute the Heegaard Floer homology of $M_n(F,K) = S_{-1}^3(D_+(K,n))$. Here we state the correction term formula of $M_n(F,K)$.

Theorem 4. *Let K be a knot in S^3. Then we have*

$$d(M_n(F,K)) = 0.$$

From Theorem 3, the following holds naturally:

Corollary 1. *If $n < 2\tau(K)$, then $M_n(K)$ bounds no negative-definite 4-manifold.*

In particular, if $n < 2\tau(K)$, then $M_n(K)$ bounds no contractible 4-manifold, hence, $D_+(K,n)$ is not slice.

Proof. Since in the case of $n < 2\tau(K)$, the correction term $d(M_n(K))$ is a negative integer. The non-negativity of $d(Y^3)$ for a homology 3-sphere Y^3 is a necessary condition to have a negative definite bound for Y^3 (see [18]). Hence, in particular, $M_n(K)$ has no contractible bound. Therefore $D_+(K,n)$ is not slice. □

The following question is to be solved.

Question 1. *Let n be an integer with $n \geq 2\tau(K)$. Does $M_n(K)$ bound a contractible 4-manifold?*

For any integer n, does $M_n(F,K)$ bound a contractible 4-manifold?

By using the Casson invariant computation (1) of $M_n(K)$, we can give a formula of $\tau(K)$ as the whole sum of Euler numbers of the reduced filtration of K.

Corollary 2. *Let K be a knot in S^3 with genus g. Then the τ-invariant of K is computed by the following formula:*

$$\tau(K) = \sum_{i=-g}^{g} \chi(\widetilde{\mathcal{F}}(K,i)),$$

where $\widetilde{\mathcal{F}}(K,i)$ is the reduced knot filtration of K.

1.2. **Rational 4-ball bound-ness of $\Sigma_2(D_+(K,n))$ and the δ-invariant.** The next purpose of this paper is to consider the rational 4-ball bound-ness of the double branched cover $\Sigma_2(K')$ of a knot K'. This is related to the sliceness of K'. The following always holds:

$$K' \text{ is slice} \Rightarrow \Sigma_2(K') \text{ bounds a rational 4-ball.}$$

A rational 4-ball B is a 4-manifold with $H_*(B,\mathbb{Q}) \cong H_*(B^4,\mathbb{Q})$, where B^4 is the standard 4-ball. The δ-invariant by Manolescu and Owens is a related invariant to this relationship. The invariant is defined to be

$$\delta(K') = 2d(\Sigma_2(K'),\mathfrak{c}_0),$$

where $\mathfrak{c}_0$ is the canonical Spin^c structure for the unique spin structure on $\Sigma_2(K')$. The δ-invariant gives a homomorphism

$$\delta : \mathcal{C}_{\mathrm{sm}} \to \mathbb{Z}.$$

Thus, we have the following

$$\Sigma_2(K') \text{ bounds a rational 4-ball} \Rightarrow \delta(K') = 0.$$

They also computed the δ-invariant for the untwisted Whitehead double of any knot or any alternating knot.

Theorem 5 ([14])**.** *For any knot K we have $\delta(D_+(K,0)) \leq 0$ and inequality is strict, if $\tau(K) > 0$. If K is alternating, then $\delta(D_+(K,0)) = -4\max\{\tau(K),0\}$.*

Hence, the δ-invariant of the untwisted Whitehead double of $T_{2,2p+1}$ is as follows:

$$\delta(D_+(T_{2,2p+1},0)) = -4p.$$

Lisca in [13] proved that when K' is a 2-bridge knot, conversely if $\Sigma_2(K')$ bounds a rational 4-ball, then K' is slice. Namely, she proved the following theorem:

Theorem 6 ([13])**.** *Suppose that K' is a 2-bridge knot. Then we have*

$$K' \textit{ is slice} \Leftrightarrow \Sigma_2(K') \textit{ bounds a rational 4-ball.}$$

How about the case where K' is the n-twisted Whitehead double of a knot K?

Question 2. *Suppose that K' is the n-twisted Whitehead double. Then, does the following equivalence*

$$K' \textit{ is slice} \Leftrightarrow \Sigma_2(K') \textit{ bounds a rational 4-ball.}$$

hold?

We give a negative answer for Question 2

Theorem 7. *Let K' be $D_+(T_{2,7},12)$ or $D_+(T_{3,7},20)$. Then K' is not slice but $\Sigma_2(K')$ bounds a rational 4-ball.*

We compute the values of δ in the cases of $K' = D_+(T_{2,2p+1},n)$ and $D_+(T_{3,3p+1},n)$.

Theorem 8. *Let n be a non-negative integer and p a positive integer. Then we have*
$$\delta(D_+(T_{2,2p+1}, n)) = -4 \max\left\{p - \left\lfloor \frac{n}{2} \right\rfloor, 0\right\}$$
and we have
$$\delta(D_+(T_{3,3p+1}, n)) = -4 \max\left\{2p - \left\lfloor \frac{n}{3} \right\rfloor, 0\right\}$$

We define t_s, t_τ, t_δ and t_{d_1} as follows:
$$t_s(K) = \min\{t \in \mathbb{Z} | s(D_+(K, t)) = 0\}$$
$$t_\tau(K) = \min\{t \in \mathbb{Z} | \tau(D_+(K, t)) = 0\}$$
$$t_\delta(K) = \min\{t \in \mathbb{Z} | \delta(D_+(K, t)) = 0\}.$$
$$t_{d_1}(K) = \min\{t \in \mathbb{Z} | d(S^3_1(D_+(K, t))) = 0\}.$$
Hedden in [8] showed that $t_\tau(K) = 2\tau(K)$ and our result says that $t_{d_1}(K) = 2\tau(K)$. Hedden and Ording conjectured $t_s(T_{2,2p+1}) = 3p - 1$ in [10]. Theorem 8 shows that
$$t_\delta(T_{2,2p+1}) = t_\tau(T_{2,2p+1}) = 2p$$
and
$$t_\delta(T_{3,3p+1}) = t_\tau(T_{3,3p+1}) = 6p$$

We say K to be a positive L-space knot, if the positive integral Dehn surgery of K is an L-space.

Theorem 9. *If K is a positive torus knot (or in general, any positive L-space knot), then we have*
$$t_\delta(K) = 2\tau(K).$$

We ask the following question.

Question 3. *Does there exist a non-L-space knot K with $t_\delta(K) \neq 2\tau(K)$?*

Theorem 10. *Suppose that n is a non-negative integer and $s = 2, 3$. If $\Sigma_2(D_+(T_{s,sp+1}, n))$ bounds a rational 4-ball, then $(s, p, n) = (2, 1, 6)$, $(2, 3, 12)$, $(3, 1, 12)$ or $(3, 2, 20)$.*

These results are due to Owens and Strle's refinement of the δ-invariant in [20]. Thus we have the following corollary:

Corollary 3. *Let n be a non-negative integer and p a positive integer. Then we have:*

$\Sigma_2(D_+(T_{2,2p+1}), n)$ *bounds a rational 4-ball* $\Leftrightarrow (p, n) = (1, 6)$ *or* $(3, 12)$.

$\Sigma_2(D_+(T_{3,3p+1}), n)$ *bounds a rational 4-ball* $\Leftrightarrow (p, n) = (1, 12)$ *or* $(2, 20)$.

Proof. The two cases $(1, 6)$ and $(1, 12)$ are immediate because $D_+(T_{2,3}, 6)$ and $D_+(T_{3,4}, 12)$ are slice (by Casson). The second two cases are due to Theorem 7. □

We can easily extend the proof (by Casson) of the sliceness of $D_+(T_{2,3}, 6)$ by Kauffman in [12] to the sliceness of $D_+(T_{n,n+1}, n(n+1))$. We raise a question of the rational 4-ball bound-ness of the double branched cover of $D_+(T_{p,q}, m(m+1))$. Collins' obstruction in [5] denies the sliceness of $D_+(T_{p,q}, m(m+1))$ unless $\{p, q\} = \{m, m+1\}$.

Question 4. *Let n, p and q be integers with $n = m(m+1)$ and $\{p, q\} \neq \{m, m+1\}$ for some integer m. When does $\Sigma_2(D_+(T_{p,q}, n))$ bound a rational 4-ball?*

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2. Heegaard Floer homology of $M_n(T_{2,3})$ and $M_n(K)$.

We prove Theorem 2. Let M_n denote $M_n(T_{2,3})$.

Proof of Theorem 2. Ozsváth-Szabó's τ-invariant of $T_{2,3}$ is 1. From the equality $M_n = S^3_1(D_+(T_{2,3}, n))$ and Proposition 7.2 in [8], we have

$$\begin{aligned}\widehat{HF}(M_n) &= \widehat{HF}(S^3_1(D_+(T_{2,3}, n))) \\ &\cong \begin{cases} \mathbb{F}^{n-4}_{(-1)} \oplus \mathbb{F}^{n-3}_{(0)} \oplus V & n \geq 2 \\ \mathbb{F}^{-1-n}_{(-1)} \oplus \mathbb{F}^{2-n}_{(-2)} \oplus \mathbb{F}^{-2}_{(0)} \oplus V & n < 2. \end{cases}\end{aligned}$$

The negative exponent means the quotient operation in place of the direct sum operation. The summand V is isomorphic to

$$V = \bigoplus_{i=-1}^{1} [H_*(\mathcal{F}(T_{2,3}, i))]^2 \bigoplus_{i=-1}^{1} [H_{*+1}(\mathcal{F}(T_{2,3}, i))]^2.$$

The chain complex $CFK^\infty(T_{2,3})$ is as in Figure 3. Therefore, we have

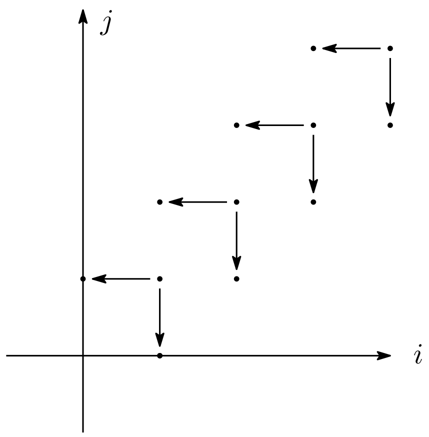


Figure 3. $CFK^\infty(T_{2,3})$.

$$H_*(\mathcal{F}(T_{2,3},i)) \cong \begin{cases} \mathbb{F}_{(-2)} & i=-1 \\ \mathbb{F}_{(0)} & i=1 \\ 0 & \textit{otherwise.} \end{cases} \tag{2}$$

As a result, the V-part is computed by

$$V = \mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(0)} \oplus \mathbb{F}^2_{(-3)} \oplus \mathbb{F}^2_{(-1)}.$$

Therefore, we have

$$\widehat{HF}(M_n) = \begin{cases} \mathbb{F}^{n-1}_{(0)} \oplus \mathbb{F}^{n-2}_{(-1)} \oplus \mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(-3)} & n \geq 2 \\ \mathbb{F}^{1-n}_{(-1)} \oplus \mathbb{F}^{4-n}_{(-2)} \oplus \mathbb{F}^2_{(-3)} & n < 2. \end{cases} \tag{3}$$

Here we use the exact triangle

$$\cdots \to \widehat{HF}(M_n) \overset{j_*}{\to} HF^+(M_n(T_{2,3})) \overset{\times U}{\to} HF^+(M_n) \overset{\delta}{\to} \widehat{HF}(M_n) \to \cdots .$$

The map j_* is the one induced by the natural injection. The multiplication of U is Ozsváth-Szabó's usual action lowering the degree by two. The connecting homomorphism δ shifts the degree by 1.

We compute the correction term $d(M_n)$.

$$HF^+(S^3_1(D_+(T_{2,3},n))) \cong H_*(C\{\max(i,j) \geq 0\}),$$

where the chain complex C is $CFK^\infty(D_+(T_{2,3},n))$.

The case of $n \geq 2$. $C\{i=0\}$ is filtered chain homotopic to

$$C\{(0,j)\} \simeq \begin{cases} \mathbb{F}^{n-4}_{(1)} \bigoplus_{i=-1}^{1} H_{*-1}(\mathcal{F}(T_{2,3},i))^2 & j=1 \\ \mathbb{F}^{2n-7}_{(0)} \bigoplus_{i=-1}^{1} H_*(\mathcal{F}(T_{2,3},i))^4 & j=0 \\ \mathbb{F}^{n-4}_{(-1)} \bigoplus_{i=-1}^{1} H_{*+1}(\mathcal{F}(T_{2,3},i))^2 & j=-1, \end{cases}$$

due to the result in [8]. The boundary maps:

$$\partial^k_1 : C\{(0,k)\} \to C\{(0,k-1)\},$$

on $CFK^\infty(D_+(T_{2,3},n))$ consists of following the notation in [8]. We denote the map:

$$C\{(k,0)\} \to C\{(k-1,0)\}$$

obtained by exchanging i and j by δ^k_1. The boundary map ∂^0_1 induces a surjective map on the group $\widehat{HFK}$ (Lemma 6.1 in [8])

The component in $\widehat{HF}(M_n)$ attaining the minimal degree in the non-torsion part in $HF^+(M_n)$ is located at $(i,j)=(0,0)$, that is, $x \in \mathbb{F}_{(0)} \subset C\{(0,0)\}$. This generator x vanishes by the boundary map $\partial^0_1 : C\{(0,0)\} \to C\{(0,-1)\}$, because it is the generator of $\widehat{HF}(S^3)$. It also vanishes by the map $\delta^0_1 : C\{(0,0)\} \to C\{(-1,0)\}$. Hence x is a generator in $HF^+(M_n)$ and it becomes clearly $U \cdot x = 0$ and $\mathrm{gr}(x) = 0$, where gr stands for the absolute grading on C. This means $d(M_n) = 0$.

The case of $n < 2$**.** $C\{i=0\}$ is filtered chain homotopic to

$$C\{(0,j)\} \simeq \begin{cases} \mathbb{F}^{2-n}_{(0)} \oplus \left(\underset{i=-1}{\overset{1}{\oplus}} H_{*-1}(\mathcal{F}(T_{2,3},i))^2 \right) / \mathbb{F}^2_{(1)} & j=1 \\ \mathbb{F}^{3-2n}_{(-1)} \oplus \left(\underset{i=-1}{\overset{1}{\oplus}} H_{*}(\mathcal{F}(T_{2,3},i))^4 \right) / \mathbb{F}^4_{(0)} & j=0 \\ \mathbb{F}^{2-n}_{(-2)} \oplus \left(\underset{i=-1}{\overset{1}{\oplus}} H_{*+1}(\mathcal{F}(T_{2,3},i))^2 \right) / \mathbb{F}^2_{(-1)} & j=-1 \end{cases}$$

$$\cong \begin{cases} \mathbb{F}^{2-n}_{(0)} \oplus H_{*-1}(\mathcal{F}(T_{2,3},-1))^2 & j=1 \\ \mathbb{F}^{3-2n}_{(-1)} \oplus H_{*}(\mathcal{F}(T_{2,3},-1))^4 & j=0 \\ \mathbb{F}^{2-n}_{(-2)} \oplus H_{*+1}(\mathcal{F}(T_{2,3},-1))^2 & j=-1, \end{cases}$$

due to the result in [8]. The generator x in $\widehat{HF}(S^3)$ lies in $C\{(0,1)\}$. That is, $x \in \mathbb{F}_{(0)} \subset C\{(0,1)\}$. The boundary map

$$\delta^0_1 : C\{(0,0)\} \to C\{(-1,0)\}$$

is surjective due to [8]. The U-action $C\{(-1,0)\} \ni U \cdot x \neq 0$, namely, $U \cdot x$ is the minimal generator in $HF^+(M_n)$. Thus we have $d(M_n) = \mathrm{gr}(U \cdot x) = -2$.

We put $HF^+(M_n) \cong \mathcal{T}^+_{(d(M_n))} \underset{i=1}{\overset{m}{\oplus}} W_i$, where $W_i = \mathbb{F}[n_i]_{(d_i)} \cong \mathbb{F}[U]/U^{n_i}$ with minimal degree d_i. Then we have

$$\widehat{HF}(M_n) = \mathbb{F}_{(d(M_n))} \underset{i=1}{\overset{m}{\oplus}} (\mathbb{F}_{(d_i)} \oplus \mathbb{F}_{(d_i+2n_i-1)}).$$

From the computation of $\widehat{HF}(M_n)$, the number of the components is

$$m = \begin{cases} n & n \geq 2 \\ 3-n & n < 2. \end{cases}$$

If some integer i has $n_i > 1$, then there exists in the $\widehat{HF}(M_n)$ a pair of two summands with the difference of $2n_i - 1 \geq 3$. The pair is just the case $n_i = 2$ and the only pair is $\mathbb{F}_{(0)}$ and $\mathbb{F}_{(-3)}$ in the case of $n \geq 2$, due to (3). Hence, the number of the pairs is at most two.

Proposition 1. *There exists no such a pair. Therefore, in the case of* $n \geq 2$*, we have*

$$HF^+(M_n) = \mathcal{T}^+_{(0)} \oplus \mathbb{F}^{n-2}_{(-1)} \oplus \mathbb{F}^2_{(-3)}$$

and in the case of $n < 2$*, we have*

$$HF^+(M_n) = \mathcal{T}^+_{(-2)} \oplus \mathbb{F}^{1-n}_{(-2)} \oplus \mathbb{F}^2_{(-3)}.$$

Proof. We may consider the $n \geq 2$ case. Suppose that there exist such two pairs in $\widehat{HF}(M_n)$. Then the components are $\mathbb{F}[2]^2_{(-3)}$ and the remaining part is

$$\mathcal{T}^+_{(0)} \oplus \mathbb{F}^{n-4}_{(-1)} \oplus \mathbb{F}^2_{(-2)}.$$

The Casson invariant becomes $\lambda(M_n) = -4 - (n-4) + 2 = -n + 2$. This is contradiction about (1).

Suppose that there exists such a single pair $\mathbb{F}_{(0)}, \mathbb{F}_{(-3)}$ in $\widehat{HF}(M_n)$. Then the component is $\mathbb{F}[2]_{(-3)}$ and the remaining part is

$$\mathcal{T}^+_{(0)} \oplus \mathbb{F}^{n-3}_{(-1)} \oplus \mathbb{F}_{(-2)} \oplus \mathbb{F}_{(-3)}.$$

The Casson invariant is $\lambda = -2-(n-3)+1-1 = -n+1$. This is contradiction about (1). Therefore since we have $n_i = 1$ for any i, we get the required computation of $HF^+(M_n)$. $\square$

This proposition means Theorem 2. In the case of $n = 6$, we can also check our computation (Theorem 2) by Némethi's algorithm ([16]) on any plumbed 3-manifold with at most one bad vertex. In fact for M_6 we can construct the negative definite bound as in Figure 4. The multiplicity -1 vertex is the only bad vertex. Then $HF^+(-M_6)$ can be computed as follows:

$$HF^+(-M_6) = \mathcal{T}^+_{(0)} \oplus \mathbb{F}^4_{(0)} \oplus \mathbb{F}^2_{(2)}.$$

For example, use HFNem by MAGMA code in [11]. By reversing the orientation, we get

$$HF^+(M_6) = \mathcal{T}^+_{(0)} \oplus \mathbb{F}^4_{(-1)} \oplus \mathbb{F}^2_{(-3)}.$$

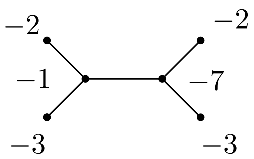


FIGURE 4. The negative definite plumbing of M_6.

2.1. **The Heegaard Floer homology of $M_n(K)$.** We define $\widehat{CF}(K) = \bigcup_{i\in\mathbb{Z}}\mathcal{F}(K,i)$, i.e., it is chain homotopy equivalent to $\widehat{CF}(K) \simeq \widehat{CF}(S^3)$. Here we define to be ϵ_i the composition

$$\epsilon_i : \mathcal{F}(K,i) \hookrightarrow \widehat{CF}(K) \to \mathbb{F}_{(0)}$$

for any i, where the last map is the one that the homological generator is mapped to 1 and any other elements to 0. Furthermore, since the map $\widehat{CF}(K) \to \mathbb{F}_{(0)}$ is splittable, we obtain the natural decomposition $\widehat{CF}(K) \cong \widetilde{CF}(K) \oplus \mathbb{F}$. We denote the kernel of φ by

$$\widetilde{\mathcal{F}}(K,i) := \ker(\epsilon_i).$$

Then $\widetilde{\mathcal{F}}(K,i)$ is a filter on $\widetilde{CF}(K) := \bigcup_{i\in\mathbb{Z}}\widetilde{\mathcal{F}}(K,i)$. The chain complex $\widetilde{CF}(K)$ is acyclic, because $\widehat{CF}(K) \to \mathbb{F}_{(0)}$ induces an isomorphism on the homology. We say $\widetilde{\mathcal{F}}(K,i)$ to be a *reduced knot filtration*.

Proof of Theorem 3. In the same way as the case where K is the right-handed trefoil,

$$d(M_n(K)) = \begin{cases} 0 & n \geq 2\tau(K) \\ -2 & n < 2\tau(K). \end{cases}$$

$$HF_{\rm red}(M_n(K))\cong\begin{cases}\mathbb{F}_{(-1)}^{n-2g-2}\bigoplus_{i=-g}^{g}H_{*+1}(\mathcal{F}(K,i))^2 & n\ge 2\tau(K)\\ \mathbb{F}_{(-2)}^{2\tau(K)-n-1}\oplus\mathbb{F}_{(-1)}^{2\tau(K)-2g-2}\bigoplus_{i=-g}^{g}H_{*+1}(\mathcal{F}(K,i))^2 & n<2\tau(K)\end{cases}$$

For each integer i with $i\ge\tau(K)$, ϵ_i is a non-trivial map and $H_{*+1}(\mathcal{F}(K,i))$ contains at least one summand $\mathbb{F}_{(-1)}$, which is non-trivial via the map ϵ_i. For each integer i with $i<\tau(K)$, ϵ_i is the 0-map. We thus have $\widetilde{\mathcal{F}}(K,i)\cong\mathcal{F}(K,i)$ in such an integer i. Thus, we have the following:

$$\begin{aligned}&\mathbb{F}_{(-1)}^{2\tau(K)-2g-2}\bigoplus_{i=-g}^{g}H_{*+1}(\mathcal{F}(K,i))^2\\ \cong\ &\bigoplus_{i=-g}^{\tau(K)-1}H_{*+1}(\mathcal{F}(K,i))^2\bigoplus_{i=\tau(K)}^{g}\left(H_{*+1}(\mathcal{F}(K,i))/\mathbb{F}_{(-1)}\right)^2\\ \cong\ &\bigoplus_{i=-g}^{\tau(K)-1}H_{*+1}(\widetilde{\mathcal{F}}(K,i))^2\bigoplus_{i=\tau(K)}^{g}H_{*+1}(\widetilde{\mathcal{F}}(K,i))^2\\ \cong\ &\bigoplus_{i=-g}^{g}H_{*+1}(\widetilde{\mathcal{F}}(K,i))^2.\end{aligned}$$

□

We prove Theorem 4.

Proof of Theorem 4. The Heegaard Floer homology of -1-surgery of a knot K' is computed by using the quotient complex:

$$CFK^\infty(K')\{i\ge 0\text{ and }j\ge 0\},$$

Since the double chain complex $CFK^\infty(D_+(K,n))$ can be seen in the proof of Theorem 3.

Let $K'=D_+(K,n)$ and $C:=CFK^\infty(K')$. Suppose that $n\ge 2\tau(K)$. Let $x_0\in C\{i=0\}$ be an element with non-zero $[x_0]\in H_*(C\{i\ge 0\})$. This element x_0 lies in $C\{(0,0)\}$ and clearly survives in $x_0\in C\{i\ge 0\text{ and }j\ge 0\}$ as a non-zero element in $\mathbb{F}_{(0)}\subset H_*(C\{(0,0)\})$. This is the bottom class in the $\mathcal{T}^+$-component in $H_*(C\{i\ge 0\text{ and }j\ge 0\})$. Hence, we have $d(M_n(F,K))=0$.

Suppose that $n<2\tau(K)$. The same element in $C\{i\ge 0\}$ satisfying the same condition as above is homologous to an element x_0 in $C\{(0,1)\}$. The element is the bottom class in the $\mathcal{T}^+$-component in $H_*(C\{i\ge 0\text{ and }j\ge 0\})$ with absolute grading 0. Thus $d(M_n(F,K))=0$. □

2.2. **The total sum of Euler numbers of the reduced knot filtration.** As a corollary of the Casson invariant formula (1) and Heegaard Floer homology formula (Theorem 3) we give a new formula of the τ-invariant:

Corollary 4. *The total Euler number of the reduced knot filtration is $\tau(K)$, namely we have:*

$$\tau(K)=\sum_{i=-g}^{g}\chi(\widetilde{\mathcal{F}}(K,i))=\sum_{i\in\mathbb{Z}}\chi(H_*(\widetilde{\mathcal{F}}(K,i))).$$

Proof. If $n \geq 2\tau(K)$, then we have

$$\lambda(M_n(K)) = -n = -(n - 2\tau(K)) + \sum_{i=-g}^{g} \chi(H_{*+1}(\widetilde{\mathcal{F}}(K,i))^2).$$

By using this equality, the sum of Euler numbers of the chain complex $\widetilde{\mathcal{F}}(K,i)$ is $\tau(K)$. In the case of $n < 2\tau(K)$, by the same argument, we get the same result.

Since $\widetilde{\mathcal{F}}(K,i)$ is acyclic in the case of $|i| \gg 0$, the last equality holds. □

3. The rational 4-ball bound-ness of $\Sigma_2(D_+(T_{2,2p+1},n))$ and $\Sigma_2(D_+(T_{3,3p+1},n))$.

If a rational homology 3-sphere Y is the boundary of a rational 4-ball, then Y is said to be that Y *bounds a rational 4-ball*. As mentioned in Section 1.2, the sliceness of $K' = D_+(K,n)$ is a sufficient condition for $\Sigma_2(K')$ to bound a rational 4-ball. We show that the sliceness is *not* a necessary condition.

First, we compute the δ-invariant (smooth knot concordance invariant) by Manolescu-Owens. To prove Theorem 8, we use the following fact:

Proposition 2. *For integer n we have*

$$\Sigma_2(D_+(K,n)) = S^3_{\frac{4n+1}{2}}(K\#K^r)$$

where K^r is the knot K with the reverse orientation.

If $\Sigma_2(D_+(K,n))$ bounds a rational 4-ball, then $n = m(m+1)$ holds for some integer m.

We notice that the condition $n = m(m+1)$ is also a necessary condition for $S^3_{\frac{4n+1}{2}}(K\#K^r)$ to bound a rational 4-ball. For example, one can see the condition in [4].

Proof. The former assertion is an application of the Montesinos trick. By using the result by Casson and Gordon [4], the order of the first homology group of the double branched cover is $4n+1$ and it is an odd square number. Thus we have $n = m(m+1)$ for some integer m. □

Before proving Theorem 8, we introduce the correction term formula for rational Dehn surgery. Here we give a brief review of the invariants V_k and H_k in [17].

Let C be a double chain complex $C := CFK^\infty(K)$ for a knot K. Let A_k^+ denote

$$C\{i \geq 0 \text{ or } j \geq k\}$$

and B^+ denote

$$C\{i \geq 0\}.$$

The maps $\mathfrak{v}_k : A_k^+ \to B^+$ are defined to be the natural projections (the forgetting map of the Alexander grading)

$$\mathfrak{v}_k^+ : A_k^+ \to B^+$$

and the maps $\mathfrak{h}_k : A_k^+ \to B^+$ are defined to be the compositions of the horizontal natural projections and the identification

$$\mathfrak{h}_k : A_k^+ \to C\{j \geq k\} \to B^+.$$

$\mathfrak{A}_k^T \subset H_*(A_k^+)$ is the sub-$\mathbb{F}[U]$-module isomorphic to $U^n H_*(A_k^+)$ for $n \gg 0$. The homomorphisms $\mathfrak{v}_k^T, \mathfrak{h}_k^T$ are the restriction maps on $\mathfrak{A}_k^T$

$$\mathfrak{v}_k^T, \mathfrak{h}_k^T : \mathfrak{A}^T \cong \mathcal{T}^+ \to H_*(B^+) \cong \mathcal{T}^+.$$

The maps are equivalent to the multiplication by U^m for some $m \geq 0$. We define the exponent m to be V_k or H_k respectively.

The correction term formula by Ni and Wu in [17] is the following:

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}.$$

If q is an even integer, then the canonical Spin^c structure of $S^3_{p/q}(K)$ has $i_0 = \frac{p+q-1}{2}$ (see [22]). Then, since $V_{\lfloor \frac{i_0}{q} \rfloor} = H_{\lfloor \frac{i_0-p}{q} \rfloor}$, we have

$$d(S^3_{p/q}(K), i_0) = d(L(p,q), i_0) - 2V_{\lfloor \frac{p+q-1}{2q} \rfloor}.$$

Proof of Theorem 8. Suppose that $s = 2$ or 3. Let $C_{s,p}$ denote the minimal generating complex that $C_{s,p}[U, U^{-1}]$ is a chain complex of $CFK^\infty(T_{s,sp+1})$. The chain complex $C_{2,1}$ is presented by the left of FIGURE 5. In general, the differentials of $C_{2,p}$ are stair-shape (see the left in FIGURE 7). We define $C_{2,p}^{(2)}$ to be the chain complex which is isomorphic to $C_{2,p} \otimes C_{2,p}$ and that the (i,j)-coordinate of the element x_0 with the i-coordinate minimal is $(-p, p)$ as in FIGURE 5. Then the knot Floer chain complex of $T_{s,sp+1} \# T^r_{s,sp+1}$ is as follows:

$$CFK^\infty := CFK^\infty(T_{s,sp+1} \# T^r_{s,sp+1}) = C^{(2)}_{s,p}[U, U^{-1}].$$

To compute the values of $V_{s,p,k}$ for $T_{s,sp+1} \# T^r_{s,sp+1}$ we investigate $C^{(2)}_{s,p}$. FIGURE 5 presents $C_{2,1}$ and $C^{(2)}_{2,1}$. The number written nearby each of lattice points is the dimension of the vector space corresponding to the point. Each of elements surrounded by circles is the unique generator in $H_*(C^{(2)}_{s,p}) \cong \mathbb{F}_{(-2p)}$ namely all the elements are homologous to each other. The generator represents a non-trivial element in $HF^\infty(S^3) \cong \mathcal{T}^\infty \cong \mathbb{F}[U, U^{-1}] \cdot [x_0]$.

Here we define $A^+_{s,p,k}$ and B^+ as follows:

$$C^{(2)}_{s,p}\{i \geq 0 \text{ or } j \geq k\} =: A^+_{s,p,k}$$

$$C^{(2)}_{s,p}\{i \geq 0\} =: B^+.$$

The chain complexes $C_{2,p}$ and $C^{(2)}_{2,p}$ are FIGURE 7. Each of the elements surrounded by the circles represents a unique homological generator in $H_*(C^{(2)}_{2,p})$.

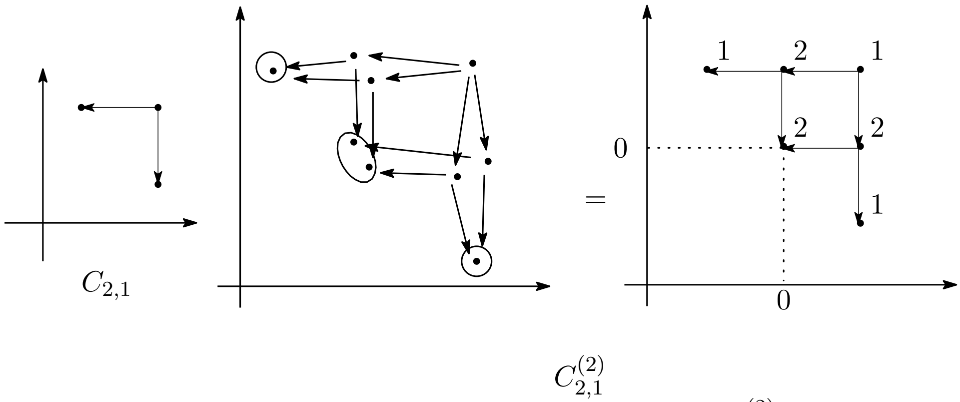


FIGURE 5. The chain complexes $C_{2,1}$ and $C^{(2)}_{2,1}$.

Let $F_k = CFK^\infty\{i < 0 \text{ and } j < k\}$. The chain complex $A^+_{s,p,k}$ means CFK^∞/F_k. Here we define a map $\phi_{i,k}$ to be

$$\phi_{i,k} : C^{(2)}_{2,p} \overset{\varphi_i}{\hookrightarrow} CFK^\infty \to A^+_{2,p,k}.$$

The map φ_i is defined to be $\varphi_i(x) = U^{-i}x$ for any element $x \in C^{(2)}_{2,p}$. The induced map $(\phi_{i,k})_* : H_*(C^{(2)}_{2,p}) = \mathbb{F} \to H_*(A^+_{2,p,k})$ is as follows:

(1) The case of $i \geq p$ (the left in FIGURE 6).
Then $(\phi_{i,k})_*$ is injective.

(2) The case of $|i| < p$ (the center in FIGURE 6).
- If $2i+1 \leq k$ then $(\phi_{i,k})_*$ is injective.
- If $2i+1 > k$ then $(\phi_{i,k})_*$ is the 0-map.

(3) The case of $i \leq -p$ (the right in FIGURE 6).
- If $k \leq i-p$ then $(\phi_{i,k})_*$ is injective.
- If $k > i-p$ then $(\phi_{i,k})_*$ is the 0-map.

Indeed, $(\phi_{i,k})_*$ is the 0-map, if and only if $F_k \cap \{(i-\ell, i+\ell) \in \mathbb{Z}^2 | |\ell| \leq p\} \neq \emptyset$.

The values $V_{2,p,k}$ can be interpreted as follows:

$$V_{2,p,k} = \max\{p - i | (\phi_{i,k})_* \text{ is injective}\}.$$

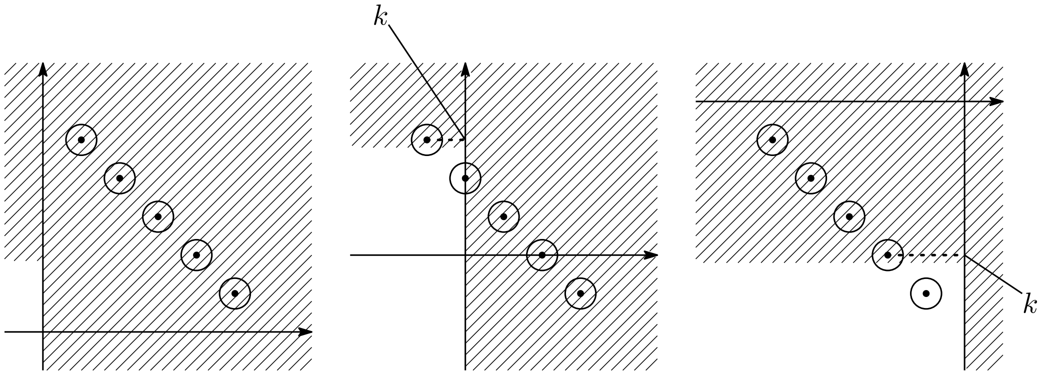
Hence, we can compute values $V_{2,p,k}, H_{2,p,k}$ to become the table below.

k	0	1	2	3	$\cdots$	$2p-2$	$2p-1$	$k \geq 2p$
$V_{2,p,k}$	p	p	$p-1$	$p-1$		1	1	0
$H_{2,p,k}$	p	$p+1$	$p+1$	$p+2$	$\cdots$	$2p-1$	$2p$	k

k	$k \leq -2p$	$-2p+1$	$\cdots$	-2	-1	0
$V_{2,p,k}$	$\lvert k\rvert$	$2p$	$\cdots$	$p+1$	$p+1$	p
$H_{2,p,k}$	0	1	$\cdots$	$p-1$	p	p

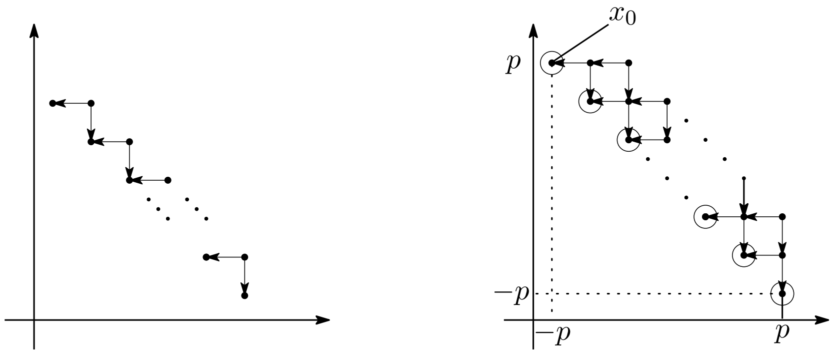
Thus, we obtain the following values:

$$V_{2,p,k} = \begin{cases} p - \lfloor \frac{k}{2} \rfloor & |k| < 2p \\ 0 & k \geq 2p \\ -k & k \leq -2p. \end{cases}$$



(1) $i \geq p$ (2) $|i| < p$ (3) $i \geq -p$

FIGURE 6. The images which represent the non-trivial element in the homology of $\varphi_i(C^{(2)}_{2,p})$. Examples of F_k: (2) the case of $k \leq 2i+1$. (3) the case of $i - p < k$.



$C_{2,p}$ $C^{(2)}_{2,p}$

FIGURE 7. The chain complexes $C_{2,p}$, $C^{(2)}_{2,p}$ and an element $x_0 \in C^{(2)}_{2,p}\{(-p,p)\}$ which is non-vanishing in $H_*(C^{(2)}_{2,p})$.

$$H_{s,p,k} = V_{s,p,-k}$$

We move on to the case of $s = 3$. The pictures in FIGURE 8 are examples of the chain complexes $C_{3,p}$ and $C^{(2)}_{3,p}$. The construction of the chain complex $C_{3,p}$ is due to [19]. The differentials of $C_{3,p}$ is the stair-shape whose steps are p-steps with slope $(1,-2)$ and consecutively p-steps with slope $(2,-1)$ in the positive direction of i. The subcomplex $C^{(2)}_{3,p} \subset CFK^\infty(T_{3,p}\#T^r_{3,p})$ is the chain complex which satisfies $C^{(2)}_{3,p} \cong C_{3,p} \otimes C_{3,p}$ and whose left most element x_0 as in FIGURE 8 has the coordinate $(-2p, 4p)$.

The elements surrounded by circles represent the one which makes the unique generator in $H_*(C_{3,p})$ or $H_*(C^{(2)}_{3,p})$. Notice that any torus knot is a lens space knot. In the same way as the one of the $s = 2$ case we obtain the values $V_{3,p,k}$ as follows:

$$V_{3,p,k} = \begin{cases} 2p - \lfloor \frac{k}{3} \rfloor & k < 6p \\ 2p - \lfloor \frac{2k}{3} \rfloor & -6p < k < 0 \\ 0 & k \geq 6p \\ -k & k \leq -6p. \end{cases}$$

By using the correction term formula of lens spaces in [22], we have

$$d(L(2r+1,2),j) = \begin{cases} \frac{(2k-r-2)^2}{2(2r+1)} & j = 2k-1 \\ \frac{4k^2-4kr-4k+r^2}{2(2r+1)} & j = 2k. \end{cases} \tag{4}$$

In the case of $r = 2n$, we have $d(L(4n+1,2),i_0) = 0$. Thus, we have

$$\begin{aligned} \delta(D_+(T_{2,2p+1},n)) &= -4V_{2,p,n} = \begin{cases} 0 & n \geq 2p \\ -4(p - \lfloor \frac{n}{2} \rfloor) & 0 \leq n < 2p \end{cases} \\ &= -4 \max \left\{ p - \left\lfloor \frac{n}{2} \right\rfloor, 0 \right\} \end{aligned}$$

and

$$\begin{aligned} \delta(D_+(T_{3,3p+1},n)) &= -4V_{3,p,n} = \begin{cases} 0 & n \geq 6p \\ -4(2p - \lfloor \frac{n}{3} \rfloor) & 0 \leq n < 6p \end{cases} \\ &= -4 \max \left\{ 2p - \left\lfloor \frac{n}{3} \right\rfloor, 0 \right\}. \end{aligned}$$

□

Remark 1. *This computation of the values of V can be also done in the cases of more general torus knots, however to compute the such chain complexes derails our aims, and we do not write down it here. It, however, seems that the determination of the rational 4-ball bound-ness for the double branched covers of twisted Whitehead double of any torus knot or more general knots becomes a challenging work.*

As a corollary we give a sufficient condition to satisfy $\delta(D_+(K,n)) = 0$ for a knot K with non-negative $\tau(K)$.

Corollary 5. *Let K be a knot in S^3 with $\tau(K) \geq 0$. If $n \geq 2\tau(K)$, then*

$$\delta(D_+(K,n)) = 0.$$

Proof. We claim that if $k = 2\tau(K) = \tau(K\#K^r)$, then we have $V_k = 0$. Then, by the decreasing property $V_k \geq V_{k+1} \geq 0$, the assertion required holds.

Let C denote $CFK^\infty(K\#K^r)$ and let k denote $2\tau(K)$. There exist a generator $x \in C\{(0,k)\}$ and some element $\alpha \in C\{\max\{i, j-k\} \geq 0\}$ such that a non-zero class $[x+\alpha] \in H_*(A_k^+)$, and its image by $\mathfrak{v}_k^+ : H_*(A_k^+) \to H_*(B^+) = \mathcal{T}^+$ is the bottom generator. Thus this means that $[x+\alpha] \neq 0$. Clearly, $U \cdot [x+\alpha] \neq 0$, $[x+\alpha]$ is the bottom generator in A_k^T. This means $V_k = 0$. See Figure 9.

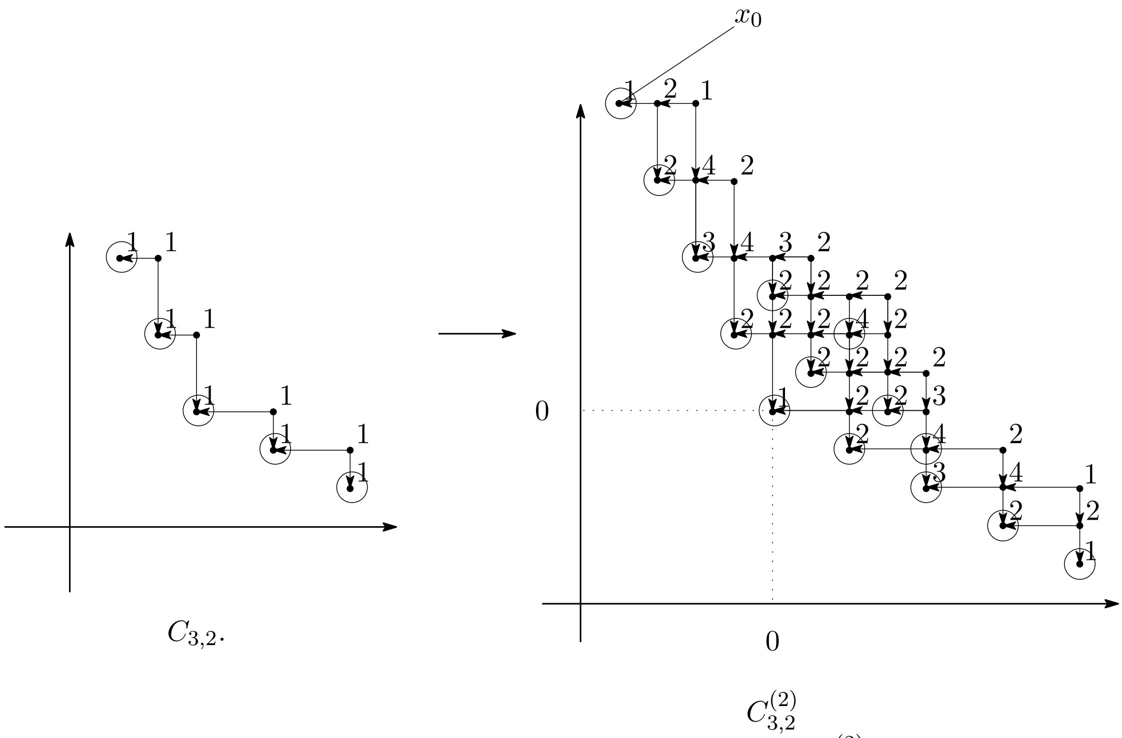


FIGURE 8. The generators and differentials of $C_{3,2}$ and $C^{(2)}_{3,2}$ and the element x_0.

Therefore, for any $n \geq 2\tau(K)$, we have

$$\delta(D_+(K,n)) = 2d(S^3_{\frac{4n+1}{2}}(K\#K^r), i_0) = 2d(L(4n+1,2), i_0) - 4V_n = 0 - 0 = 0.$$

$\square$

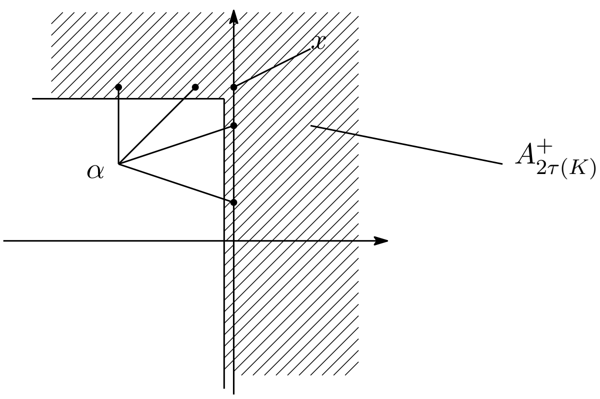


FIGURE 9. The generator x and some element α in $A^+_{2\tau(K)}$.

If $0 \leq n \leq 2\tau(K)$, then the behavior of V_n depends on the filtered chain complex with respect to K.

Proof of Theorem 9. Let K be a positive L-space knot. Then the genus $g(K)$ coincides with $\tau(K)$. We choose the minimal chain complex CFK^∞, which the dimension of $CFK^\infty\{(i,j)\}$ is at most one. Since $CFK^\infty\{(0,\tau(K)-1)\}$ has dimension one due to [9, 23], the chain complex $CFK^\infty(K)$ is as in Figure 10.

As described in Figure 10, $V_{\tau(K)} = 0$ and $V_{\tau(K)-1} = 1$ hold. Thus, from Corollary 5, if $n \geq 2\tau(K)$ holds, then $\delta(D_+(K,n)) = 0$. From Figure 10 we have

$$\delta(D_+(K, 2\tau(K)-1)) = -4V_{2\tau(K)-1)} = -4.$$

This means $t_\delta(K) = 2\tau(K)$. □

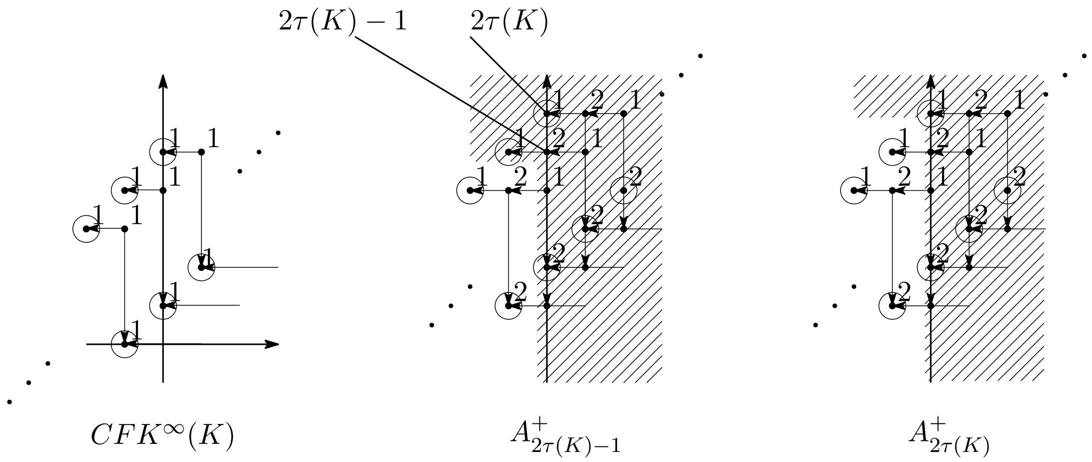


Figure 10. The minimal $CFK^\infty(K)$ of an L-space knot K (the left picture). $A^+_{2\tau(K)-1}$, $A^+_{2\tau(K)}$ for $K\#K^r$ (the central and right pictures). The points indicated by the circles represent the unique generator for each of homologies.

3.1. **An obstruction by Owens and Strle.** If K is a slice knot, then the double branched cover $\Sigma_2(K)$ must bound a rational 4-ball. To show Theorem 10 (the rational 4-ball bound-ness), we use the following refinement of the δ-invariant by Owens and Strle.

Proposition 3 ([20])**.** *Let Y be a rational homology 3-sphere bounding a rational 4-ball X. If the order of $H^2(Y)$ is $h = t^2$, then*

$$d(Y, \mathfrak{t}_0 + \beta) = 0$$

for any $\beta \in \mathcal{T} \subset H^2(Y)$, where t_0 is a Spin^c structure, $\mathcal{T}$ is the image of the map $H^2(X) \to H^2(Y)$, and $|\mathcal{T}| = t$.

By using Proposition 2 and 3 for the half-integer surgery of $T_{2,2p+1}\#T^r_{2,2p+1}$ we prove Theorem 10.

Proof of Theorem 10. Suppose that

$$X_{s,p,m} := \Sigma_2(D_+(T_{s.sp+1}, m(m+1)))$$

bounds a rational 4-ball.

Since the canonical Spin^c structure corresponds to $i_0 = 2m(m+1)+1$, Owens and Strle's subset $\mathfrak{t}_0 + \mathcal{T}$ is $\{i_0 + \ell(2m+1) | 0 \leq |\ell| \leq m\}$, because the subgroup of order $2m+1$ in $H^2(X_{s,p,m}, \mathbb{Z}) \cong \mathbb{Z}/(2m+1)^2\mathbb{Z}$ is unique.

By using the formula (4), we have

$$d(L((2m+1)^2, 2), i_0 + \ell(2m+1)) = \begin{cases} 2\ell_1(\ell_1 - 1) & \ell = 2\ell_1 - 1 \\ 2\ell_1^2 & \ell = 2\ell_1, \end{cases}$$

$$V_{s,p,\lfloor \frac{i_0+\ell(2m+1)}{2} \rfloor} = V_{s,p,m(m+1+\ell)+\lfloor \frac{\ell+1}{2} \rfloor}$$

$$H_{s,p,\lfloor \frac{i_0+\ell(2m+1)-(2m+1)^2)}{2} \rfloor} = V_{s,p,m(m+1-\ell)-\lfloor \frac{\ell}{2} \rfloor}$$

and

$$\max\{V_{s,p,\lfloor \frac{i_0+\ell(2m+1)}{2} \rfloor}, H_{s,p,\lfloor \frac{i_0+\ell(2m+1)-(2m+1)^2)}{2} \rfloor}\} = V_{s,p,m(m+1)-m\ell-\lfloor \frac{\ell}{2} \rfloor}.$$

Thus we have

$$d(X_{s,p,m}, i_0 + \ell(2m+1)) = \begin{cases} 2\ell_1(\ell_1 - 1) - 2V_{s,p,m(m+1-\ell)-\ell_1+1} & \ell = 2\ell_1 - 1 \\ 2\ell_1^2 - 2V_{s,p,m(m+1-\ell)-\ell_1} & \ell = 2\ell_1. \end{cases}$$

Suppose that $\ell = 2 \leq m$ holds. Then we have $V_{s,p,m^2-m-1} = 1$. In the case of $s = 2$, since $m^2 - m - 1$ is odd, we obtain $m^2 - m - 1 = 2p - 1$. In the case of $s = 3$, since $m^2 - m - 1 \equiv \pm 1 \bmod 6$, we have $m^2 - m - 1 = 6p - 1$. If $m = 2$, then $(s, p) = (2, 1)$ holds.

Suppose that $m \geq 3$ and $\ell = 3 \leq m$. Then we have $V_{s,p,m^2-2m-1} = 2$. In the case of $s = 2$, we have $m^2 - 2m - 1 = 2s - 3$, or $2s - 4$, thus $(m, p) = (3, 3)$. In the case of $s = 3$, since $m^2 - 2m - 1 \neq 3, 6 \bmod 6$, we have $m^2 - 2m - 1 = 6p - 5, 6p - 4$ holds. Thus $(p, m) = (2, 4), (1, 3)$.

Therefore, we obtain $(s, p, m) = (2, 1, 2)$, $(2, 3, 3)$, $(3, 1, 3)$, and $(3, 2, 4)$. □

The obstruction (Theorem 10) by Owens and Strle [20] for $K = T_{2,2p+1}$ and $T_{3,3p+1}$ coincides with the condition for $\Sigma_2(D_+(K, n))$ to bound a rational 4-ball. It would be unlikely that the coincidence can be extended to other torus knots version. For example, the double branched covers of $D_+(T_{3,5}, 12)$, $D_+(T_{3,11}, 30)$ and $D_+(T_{3,20}, 56)$ satisfy the condition by Owens and Strle, however we do not know whether rational 4-ball bounds of the manifolds exist or not.

Question 5. *Let K' be $D_+(T_{3,5}, 12)$, $D_+(T_{3,11}, 30)$ or $D_+(T_{3,20}, 56)$. Then does the double branched covers of K' bound a rational 4-ball?*

Proof of Theorem 7. The knots $K' = D_+(T_{2,7}, 12)$ and $D_+(T_{3,7}, 20)$ are not slice by Collins' result [5]. We prove that each of the double branched cover of K' actually bounds a rational 4-ball. Each of the double branched covers is $\Sigma_2(K') = S^3_{49/2}(T_{2,7} \# T^r_{2,7})$, $S^3_{81/2}(T_{3,7} \# T^r_{3,7})$. They are diffeomorphic to each of the last diagrams in FIGURE 11 and 12. The 0-framed components in the last diagrams consist of slice knots S_1 and S_2 respectively. Indeed, the knot S_1 (in the $K' = D_+(T_{2,7}, 12)$ case) is 10_{155} in the Rolfsen table. It is a well-known slice knot. The sliceness of S_2 (in the

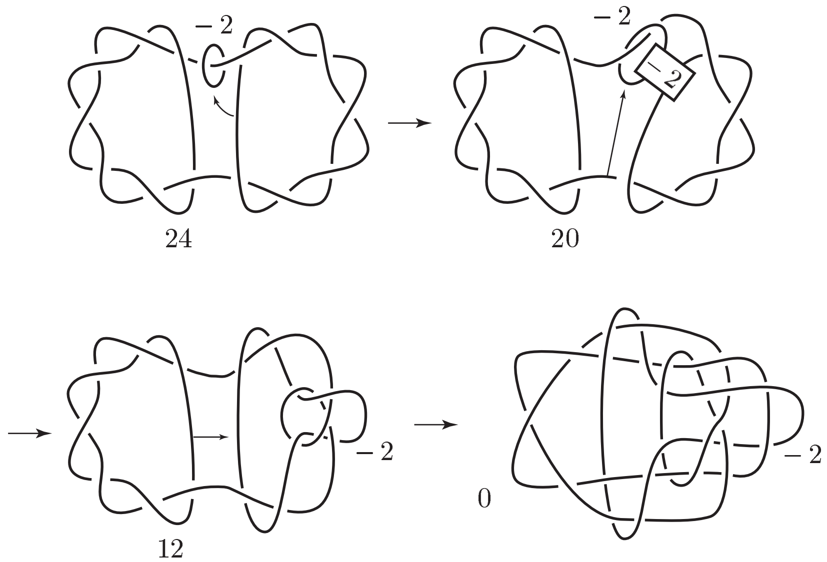


FIGURE 11. A diffeomorphism of $S^3_{49/2}(T_{2,7}\#T^r_{2,7})$.

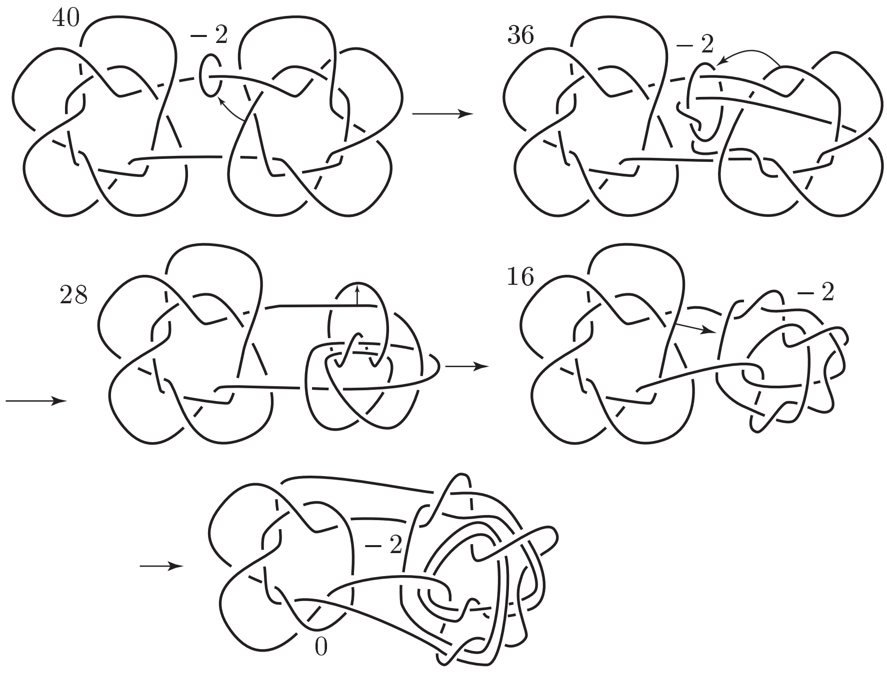


FIGURE 12. A diffeomorphism of $S^3_{81/2}(T_{3,7}\#T^r_{3,7})$.

$K' = D_+(T_{3,7}, 20)$ case) is proven in FIGURE 13. Let B_7, and B_9 be the slice disk complements of S_1 and S_2 together with the -2-framed 2-handles. See FIGURE 14 and 15. By computing H_1 and H_2 of B_7 and B_9 from the

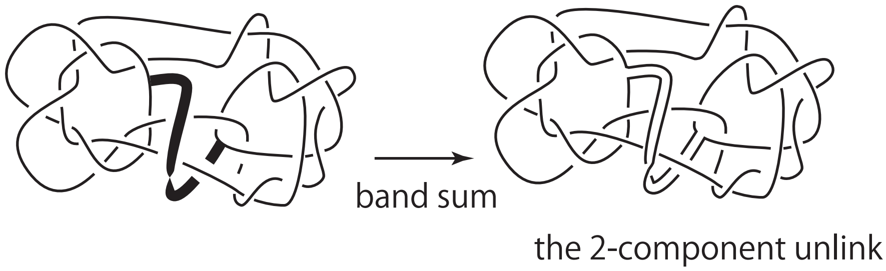


FIGURE 13. The sliceness of S_2.

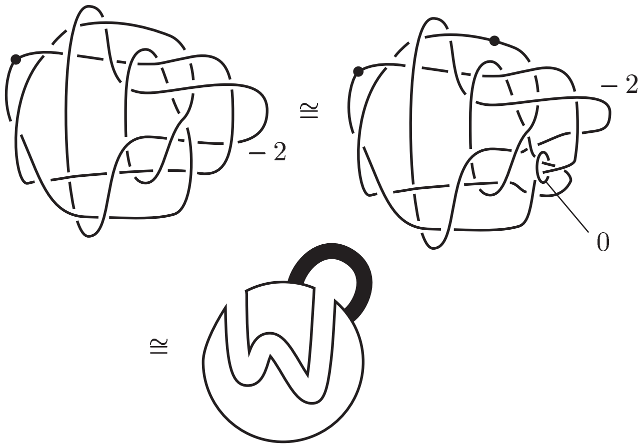


FIGURE 14. A rational 4-ball B_7: A slice disk complement of $S_1 = 10_{155}$ attaching -2-framed 2-handle.

FIGURE 15. A rational 4-ball B_9: A slice disk complement of S_2 attaching -2-framed 2-handle.

last pictures in FIGURE 14 and 15 respectively, we obtain immediately

$$H_1(B_7) \cong \mathbb{Z}/7\mathbb{Z} \text{ and } H_2(B_7) \cong 0$$

and

$$H_1(B_9) \cong \mathbb{Z}/9\mathbb{Z} \text{ and } H_2(B_9) \cong 0.$$

Hence, B_7 and B_9 are rational 4-balls bounding the double branched covers of $D_+(T_{2,7}, 12)$ and $D_+(T_{3,7}, 20)$ respectively. $\square$

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University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571 Japan

E-mail address: tange@math.tsukuba.ac.jp