Definite spin 4-manifolds bounding homology spheres

定値偶形式をもつ４次元多様体とその境界

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2014/11/23

Problem

Let Y a homology sphere. Do there exist a spin negative definite W^4.

Correction term d

If W^4 is a negative definite bounding of a homology sphere Y, then the following holds:

$$c_1^2(\mathfrak{s}) + b_2(W) \leq 4d(Y)$$

for any spin^c structure $\mathfrak{s}$.
In particular, if W is negative definite spin, then

$$b_2(W) \leq 4d(Y).$$

NS-invariant $\bar{\mu}$ (Ue)

Let Y be a rational Seifert homology sphere with spin structure c.
If W has a negative-defnite spin bounding W, then

$$-\frac{8\overline{\mu}(Y,c)}{9} \leq b_2(W) \leq -8\bar{\mu}(Y,c)$$

$\Sigma(2,3,r)$

	μ	$\bar{\mu}$	d	Definite spin bounding
$\Sigma(2,3,12k-5)$	1	1	0	*No.*
$\Sigma(2,3,12k-1)$	0	0	2	*No.*
$\Sigma(2,3,12k+1)$	0	0	0	must be $b_2 = 0$
$\Sigma(2,3,12k+5)$	1	-1	2	must be $b_2 = 8$

$$4d(\Sigma(2,3,12k+5)) = -8\bar{\mu}(\Sigma(2,3,12k+5)) = 8$$

$\Sigma(2,3,13), \Sigma(2,3,25)$ have contractible bounding.
Do $\Sigma(2,3,12k+1)$ have any contractible bounding?
Do $\Sigma(2,3,12k+5)$ have any $-E_8$-bounding?

Theorem

Let $Y_n = \Sigma(2,3,12n+5)$ *(*$0 \leq n \leq 12, 14$*). Then there exists* W_n *with* $\partial W_n = \Sigma(2,3,12n+5)$ *and* $Q_{W_n} \cong -E_8$*.*
In particular

$$\mathfrak{ds}(\Sigma(2,3,6n-1)) = g_8(\Sigma(2,3,6n-1)) = 1.$$

Proof

Corollary

$E(1)$ has the following decomposition:

$$E(1) = W_n \cup N_n$$

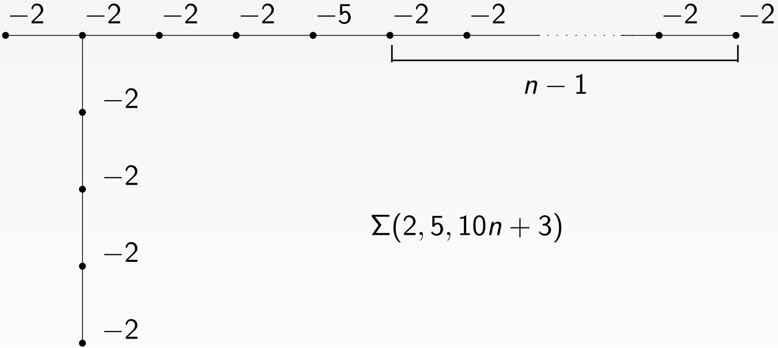
N_{-n} : the Gompf Nuclei with $\begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}$

$N_{-n} = \chi(\text{trefoil}, \textit{meridian}; 0, -n)$

$\Sigma(2, 5, r)$

	μ	$\bar{\mu}$	d	Definite spin bounding
$\Sigma(2, 5, 20k - 11)$	1	-1	2	must be $b_2 = 8$
$\Sigma(2, 5, 20k - 1)$	0	0	2	*No.*
$\Sigma(2, 5, 20k + 11)$	1	1	0	*No.*
$\Sigma(2, 5, 20k + 1)$	0	0	0	must be $b_2 = 0$
$\Sigma(2, 5, 20k + 3)$	1	-1	2	must be $b_2 = 8$
$\Sigma(2, 5, 20k + 13)$	0	0	2	*No.*
$\Sigma(2, 5, 20k - 3)$	1	1	0	*No.*
$\Sigma(2, 5, 20k - 13)$	0	0	0	must be $b_2 = 0$

Minimal resolution



$\Sigma(2, 5, 10n + 3)$

$Q_{R_{2k}} \cong -E_8 \oplus^{2k} \langle -1 \rangle . (n = 2k)$
The -1 classes in R_{2k} cannot be blow-downed.

Indefinite invariant

Definition (Bounding genus)

Let Y be a homology 3-sphere. Then the bounding genus $|Y|$ of Y is defined to be

$$|Y| := \begin{cases} \min\{n | \partial X = Y, Q_X = nH\} & \mu(Y) = 0, \\ \infty & \mu(Y) = 1, \end{cases}$$

where the bounding 4-manifold X is restricted to homologically 1-connected 4-manifold.

$$|\cdot| : \Theta^3_{\mathbb{Z}} \to \mathbb{N} \cup \{0, \infty\}.$$

Definite spin invariants

Definition ($\mathfrak{ds}, \overline{\mathfrak{ds}}$)

If Y has a definte spin bounding X, then

$$\mathfrak{ds}(Y) := \max\left\{\frac{b_2(X)}{8} | b_2(X) = |\sigma(X)|,\ w_2(X) = 0 \ \&\ \partial X = Y\right\}$$

$$\overline{\mathfrak{ds}}(Y) := \min\left\{\frac{b_2(X)}{8} | b_2(X) = |\sigma(X)|,\ w_2(X) = 0 \ \&\ \partial X = Y\right\}$$

If Y has no definte spin bounding,

$$\mathfrak{ds}(Y) = \overline{\mathfrak{ds}}(Y) = \infty$$

We assume homologically 1-connected bounding as X.

Definition (ϵ)

$$\epsilon(Y) = \begin{cases} 1 & \partial X = Y; X \textit{ positive definite spin with } b_2(X) > 0 \\ -1 & \partial X = Y; X \textit{ negative definite spin with } b_2(X) > 0 \\ 0 & \partial X = Y; b_2(X) = 0 \\ \infty & \textit{no definite spin bound} \end{cases}$$

Proposition

ϵ is well-defined.

Definition ($g_8, \overline{g_8}$)

Let Y be a homology 3-sphere with finite $\epsilon(Y)$. If Y has an E_8-bounding, then we define the E_8-genera as follows:

$$g_8(Y) = \max\{|n| \mid Y = \partial X \text{ and }, w_2(X) = 0, Q_X = nE_8\}$$

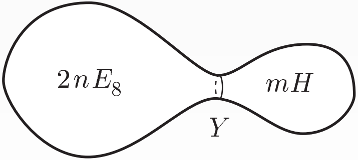
$$\overline{g_8}(Y) = \min\{|n| \mid Y = \partial X \text{ and }, w_2(X) = 0, Q_X = nE_8\},$$

11/8 conjecture

Proposition

The following is equivalent to each other

- Any closed spin smooth 4-manifold X satisfies $b_2(X) \geq \frac{11}{8}|\sigma(X)|$.
- Any colsed homology 3-sphere Y with $\mu(Y) = 0$ and $\mathfrak{ds}(Y) < \infty$ satisfies $2|Y| \geq 3\mathfrak{ds}(Y)$.



Similar invariants

Definition (Manolescu's ξ.)

$$\xi(Y) = \max\{p - q | p, q \in \mathbb{Z}, q > 0, p(-E_8) \oplus qH = Q_X$$

$$, \partial X = Y \textit{ and } w_2(X) = 0\}.$$

$$\xi(Y) \leq \kappa(Y) - 1$$

Definition (Bohr and Lin's $m, \overline{m}$)

$$m(Y) = \max\left\{\frac{5}{4}\sigma(X) - b_2(X) | p, q, \in \mathbb{Z}, \partial X = Y, \textit{ and } w_2(X) = 0\right\}$$

$$\overline{m}(Y) = \min\left\{\frac{5}{4}\sigma(X) - b_2(X) | p, q, \in \mathbb{Z}, \partial X = Y, \textit{ and } w_2(X) = 0\right\}$$

$$m(-Y)/2 = \max\left\{ \frac{b_2(N)}{8} - q \,|\, \partial X = Y, Q_X \cong N \oplus qH \right.$$

$$\left. \text{and } N: \text{even negative-definite form}, w_2(X) = 0 \right\}.$$

Thus we have

$$m(-Y)/2 \leq \xi(Y) + 1,$$

Question

Can the Seiberg-Witten invariant or Donaldson Invarinats contribute to $\mathfrak{Ds}$?

Fundamental properties

Theorem (Properties of $\mathfrak{ds}$)

Let $\mathfrak{ds}'$ be one of $\mathfrak{ds}, \overline{\mathfrak{ds}}, g_8, \overline{g}_8$.

1. *The $\mathfrak{ds}'$ and g_8' are h-cobordism invariants i.e.,* $\mathfrak{ds}' : \Theta^3_{\mathbb{Z}} \to \mathbb{N} \cup \{0, \infty\}$.
2. $\overline{\mathfrak{ds}}(Y) = 0$ *or* $\overline{g_8}(Y) = 0$, *if and only if* $[Y] = 0$ *in* $\Theta^3_{\mathbb{Z}}$.
3. *If* $\mathfrak{ds}(Y), g_8(Y) < \infty$, *then* $\mu(Y) \equiv \mathfrak{ds}'(Y) \equiv g_8'(Y) \equiv 0 \bmod 2$
4. *If* $\epsilon(Y_1)\epsilon(Y_2) = 1$, *then* $\mathfrak{ds}(Y_1) + \mathfrak{ds}(Y_2) \leq \mathfrak{ds}(Y_1 + Y_2)$.
5. *If* $\epsilon(Y_1)\epsilon(Y_2) = 1$, *then* $\overline{\mathfrak{ds}}(Y_1 + Y_2) \leq \overline{\mathfrak{ds}}(Y_1) + \overline{\mathfrak{ds}}(Y_2)$.
6. *If* $\mathfrak{ds}(Y) = 1$, *then* $g_8(Y) = 1$.
7. $\mathfrak{ds}(-Y) = \mathfrak{ds}(Y)$ *and* $\overline{\mathfrak{ds}}(-Y) = \overline{\mathfrak{ds}}(Y)$.
8. $g_8(-Y) = g_8(Y)$ *and* $\overline{g_8}(-Y) = \overline{g_8}(Y)$.

9. If $0 < \mathfrak{ds}(Y) < \infty$, then $\epsilon(Y)d(Y) < 0$ and $\mathfrak{ds}(Y) \leq |d(Y)|/2$.
10. If $\mathfrak{ds}'(Y)$ or $g_8'(Y)$ is odd, then $|Y| = \infty$.
11. If $\mathfrak{ds}(Y)$ is even, then we have $\mathfrak{ds}(Y) + 1 \leq |Y|$.
12. If $|Y| = 1, 2$, then $\mathfrak{ds}(Y) = \infty$.
13. If $\epsilon(Y) \neq \infty$, then $\mathfrak{ds}(Y) - 1 \leq m(-Y)/2 - 1$.
14. Suppose that Y is a Seifert homology 3-sphere. If $\mathfrak{ds}(Y) < \infty$, then $\bar{\mu}(Y)\epsilon(Y) > 0$ and $\mathfrak{ds}(Y) \leq |\bar{\mu}(Y)|$.
15. Can the values of $\mathfrak{ds}$ can give examples with $2|Y| < 3\mathfrak{ds}(Y)$.(11/8-conjecture)

Qestions

Question

1. *Find more general constructions of positive (or negative) E_8-boundings for many homology 3-spheres.*
2. *When* $\mathfrak{ds}$ *or* $\overline{\mathfrak{ds}}$ *is additive? For two homology 3-spheres with* $\mathfrak{ds}(X_i) < \infty$ $(i = 1, 2)$*, Let denote* $\tilde{\mathfrak{ds}}(Y) = \epsilon(Y)\mathfrak{ds}(Y)$*. Then when does the equality*

$$\tilde{\mathfrak{ds}}(X_1) + \tilde{\mathfrak{ds}}(X_2) = \tilde{\mathfrak{ds}}(X_1 \# X_2)$$

   hold?
3. *Let Y be a Brieskorn homology 3-sphere. If* $4d(Y) = -8\overline{\mu}(Y) > 0$*, then is* $\mathfrak{ds}(Y) = \frac{d(Y)}{2}$ *true?*
4. *If* $\mathfrak{ds}(Y) < \infty$*, then does Y have an E_8-bounding?*

Question

1. *When the equality* $m(-Y)/2 = \mathfrak{ds}(Y)$ *or* $\overline{m}(-Y)/2 = \overline{\mathfrak{ds}}(Y)$ *hold?*
2. *Are there exist any homology 3-spheres* $g_8(Y) < \overline{g_8}(Y)$*,* $\mathfrak{ds}(Y) \neq g_8(Y)$ *or* $\overline{\mathfrak{ds}}(Y) \neq \overline{g}_8(Y)$*?*

Definite spin buondings

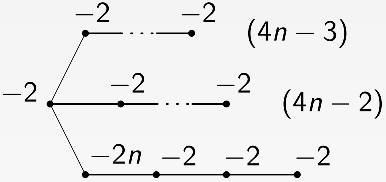
Construction

Minimal resolution

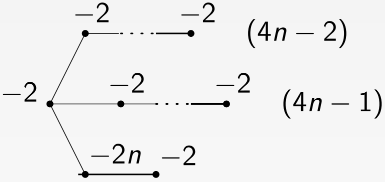
$$\Sigma(4n-2, 4n-1, 8n-3),\ \Sigma(4n-1, 4n, 8n-1)$$

$$\Sigma(4n-2, 4n-1, 8n^2-4n+1),\ \Sigma(4n-1, 4n, 8n^2-1)$$

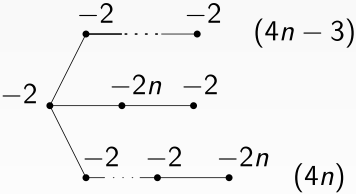
have $\mathfrak{ds} = n$.



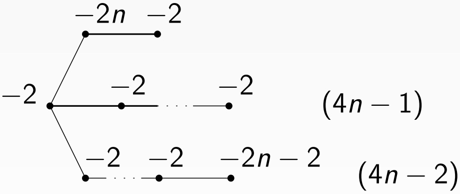
$\Sigma(4n-2, 4n-1, 8n-3)$



$\Sigma(4n-1, 4n, 8n-1)$



$\Sigma(4n-2, 4n-1, 8n^2-4n+1)$



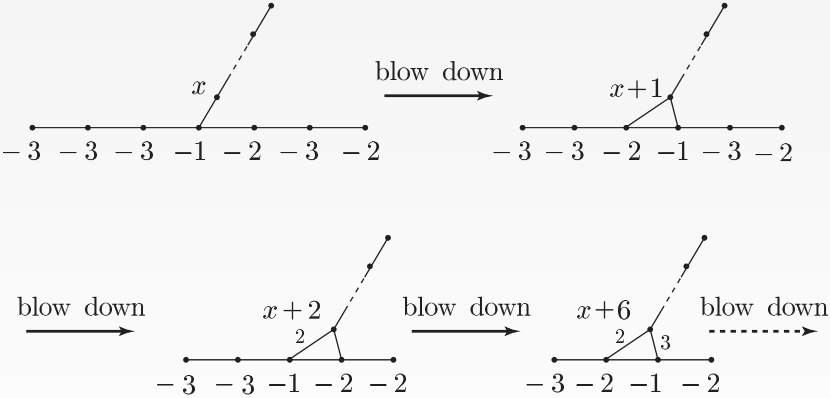
$\Sigma(4n-1, 4n, 8n^2-1)$

Theorem

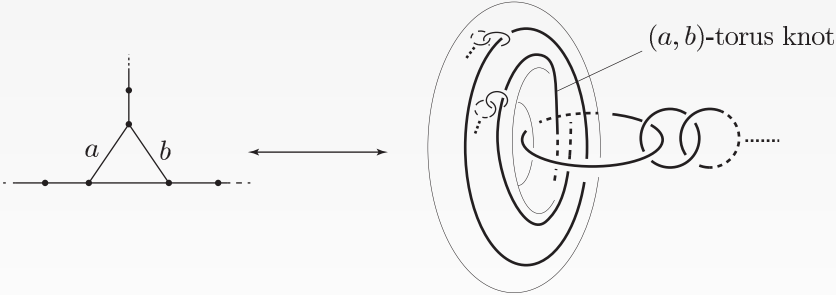
If Brieskorn homology spheres $\Sigma(a_1, a_2 \cdots, a_n)$ has the minimal resolution with intersection form $-E_8$, then it is one of the following:

$$\Sigma(2,3,5), \Sigma(3,4,7), \Sigma(2,3,7,11), \Sigma(2,3,7,23), \Sigma(3,4,7,43)$$

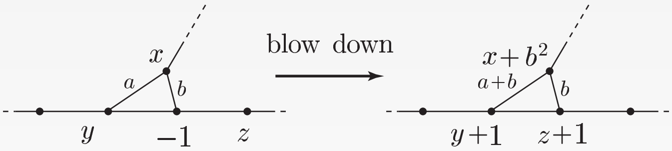
Blow-down of minimal resolution



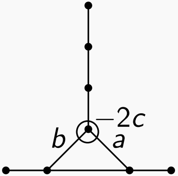
$$\begin{pmatrix} x+2 & 2 & 1 & 0 & \cdots \\ 2 & -1 & 1 & 0 & \cdots \\ 1 & 1 & -2 & 1 & \cdots \\ 0 & 0 & 1 & -3 & \cdots \end{pmatrix} \to \begin{pmatrix} x+6 & 3 & 2 & \cdots \\ 3 & -1 & 1 & \cdots \\ 2 & 1 & -2 & \cdots \\ \cdots & & & \end{pmatrix}$$



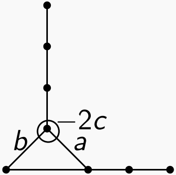
a
b
(a, b)-torus knot



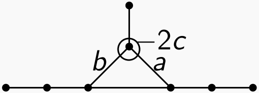
$$\begin{pmatrix} x & a & b & 0 & \cdots \\ a & y & 1 & 0 & \cdots \\ b & 1 & -1 & 1 & \cdots \\ 0 & 0 & 1 & z & \cdots \end{pmatrix} \to \begin{pmatrix} x+b^2 & a+b & b & \cdots \\ a+b & y+1 & 1 & \cdots \\ b & 1 & z+1 & \cdots \\ \cdots & & & \end{pmatrix}$$



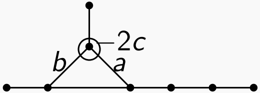
(1)



(2)



(3)



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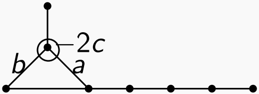
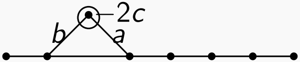
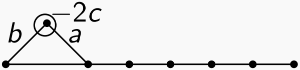


(5)



(6)



(7)

G	a	b	c
(1)	$3k-2\ell\pm 2$	$-2k+3\ell\mp 2$	$3k^2-4k\ell+3\ell^2\pm 2(2k-2\ell)+2$
(2)	$4k-\ell\pm 2$	$-3k+2\ell\mp 2$	$6k^2-3k\ell+\ell^2\pm 2(3k-\ell)+2$
(3)	$4k-3\ell\pm 2$	$-3k+4\ell\mp 2$	$6k^2-9k\ell+6\ell^2\pm 2(3k-3\ell)+2$
(4)	$5k-2\ell\pm 2$	$-4k+3\ell\mp 2$	$10k^2-8k\ell+3\ell^2\pm 2(4k-2\ell)+2$
(5)	$6k-\ell\pm 2$	$-5k+2\ell\mp 2$	$15k^2-5k\ell+\ell^2\pm 2(5k-\ell)+2$
(6)	$12k-4\ell\pm 3$	$-10k+6\ell\mp 3$	$60k^2-40k\ell+12\ell^2\pm 6(5k-2\ell)+4$
(6)	$12k-4\ell\pm 5$	$-10k+6\ell\mp 5$	$60k^2-40k\ell+12\ell^2\pm 10(5k-2\ell)+11$
(6)	$12k-4\ell\pm 1$	$-10k+6\ell$	$60k^2-40k\ell+12\ell^2\pm 10k+1$
(6)	$12k-4\ell\pm 3$	$-10k+6\ell\mp 2$	$60k^2-40k\ell+12\ell^2\pm 2(15k-4\ell)+4$
(7)	$14k-2\ell\pm 3$	$-12k+4\ell\mp 3$	$84k^2-24k\ell+4\ell^2\pm 6(6k-\ell)+4$
(7)	$14k-2\ell\pm 5$	$-12k+4\ell\mp 5$	$84k^2-24k\ell+4\ell^2\pm 10(6k-\ell)+11$
(7)	$14k-2\ell\pm 2$	$-12k+4\ell\mp 1$	$84k^2-24k\ell+4\ell^2\pm 12(2k-\ell)+2$
(7)	$14k-2\ell\pm 4$	$-12k+4\ell\mp 3$	$84k^2-24k\ell+4\ell^2\pm 6(8k-\ell)+7$

The negative-definite E_8-boundings for $(G; a, b, c)$

p	q	r
$10i+7$	$15i+8$	$120i^2+148i+45$
$10i+3$	$15i+2$	$120i^2+52i+5$
$20i-8$	$30i-17$	$480i^2-464i+109$
$20i+8$	$30i+7$	$480i^2+304i+45$
$30i-13$	$45i-27$	$1080i^2-1116i+281$
$30i-7$	$45i-18$	$1080i^2-684i+101$
$30i+7$	$45i+3$	$1080i^2+324i+17$
$30i+13$	$45i+12$	$1080i^2+756i+125$
$20i+2$	$30i-7$	$480i^2-64i-11$
$20i-2$	$30i-23$	$480i^2-256i+21$
$10i+7$	$15i-2$	$120i^2+68i-365$
$10i+13$	$15i+7$	$120i^2+212i+73$
$60i-28$	$90i-57$	$4320i^2-4752i+1277$
$60i-8$	$90i-27$	$4320i^2-1872i+173$
$60i+8$	$90i-3$	$4320i^2+432i-19$
$60i+28$	$90i+27$	$4320i^2+3312i+605$

$\Sigma(p,q,r)$ with (1) and $1 \leq a \leq 6$

Theorem

Any Σ *in this table,* $\mathfrak{ds}(\Sigma) = g_8(\Sigma) = 1$.

$\because$ The simple computation of $\bar{\mu}(\Sigma) = -1$.

Other examples

Theorem (Milnor fibration)

$$\mathfrak{ds}(\Sigma(2,3,6n-1)\#(-\Sigma(2,3,6n-5)))$$
$$= g_8(\Sigma(2,3,6n-1)\#(-\Sigma(2,3,6n-5))) = 1$$

K. Sato's examples

Let K be a knot. $K_{2,4q\pm1}$: Cable knot of K.
$Y = S^3_{\mathrm{sgn}(q)}(K_{2,4q\pm1})$ $(q \neq 0)$. Then

$$\overline{\mathfrak{ds}}(Y) \leq q \leq \mathfrak{ds}(Y)$$

In particular, suppose that K bound null-homologous disk in a punctured $\#^n\overline{\mathbb{C}P^2}$ and $q \neq 0$. e.g., K torus knot or figure-8, then

$$\mathfrak{ds}(Y) = |q|$$

$$\epsilon(Y) = \mathrm{sgn}(q)$$

Minimal genus in $E(1)$

$E(n)$: elliptic fibration with $\chi = 12n$ without no multiple fibers
$E(1) = \mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2}$
The minimal genus problem of non-negative classes in $\mathbb{C}P^2 \#^n \overline{\mathbb{C}P^2}$ $(7 \leq n < 9)$ or surface bundle is completely solved by Gauge theory.
For the negative classes in $\mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2}$, not much is known.

Fact (Li^2)

If $\xi \in H_2(\mathbb{C}^2 \# ^n \overline{\mathbb{C}P^2})$*, then all classes with* $0 > \xi^2 > -(n+7)$ *have minimal genus* 0*.*

In the case of $\xi^2 \geq -16$, the orbit of $\mathrm{Aut}(H_2(\mathbb{C}P^2 \# ^9 \overline{\mathbb{C}P^2}))$ is unique. But $\xi^2 = -17$, they are not unique.

Finashin-Mikalkin

There exists a smooth embedding of S^2 into a K3-surface X with the normal Euler number equal to n for any negative even $n \geq -86$.

F: the fiber class
S: the section class
The class $n \cdot F - S$

$$(n \cdot F - S)^2 = -(2n + 2)$$

Theorem

Suppose that $x = n \cdot F - S \in H_2(E(1)) = H_2(\mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2})$. *If* $x^2 \geq -31$, *then* $G(x) = 0$.

$\mathbf{x} \in H_2(X)$,

$$G(\mathbf{x}) = \min\{g(\Sigma) | [\Sigma] = \mathbf{x}\}$$

Proposition

Suppose that $x = n \cdot F - S \in H_2(E(1)) = H_2(\mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2})$. *If* $x \leq -25$ *or* $x^2 = -29$, *then* $G(x) = 0$.

Proof of Proposition

$\because$ By using decomposition $E(1) = W_n \cup_{Y_n} N_n$ $E(1) = W_n \cup N_n$ for $n \equiv 1(2)$, $1 \leq n \leq 25$ or $n = 29$.

Proposition

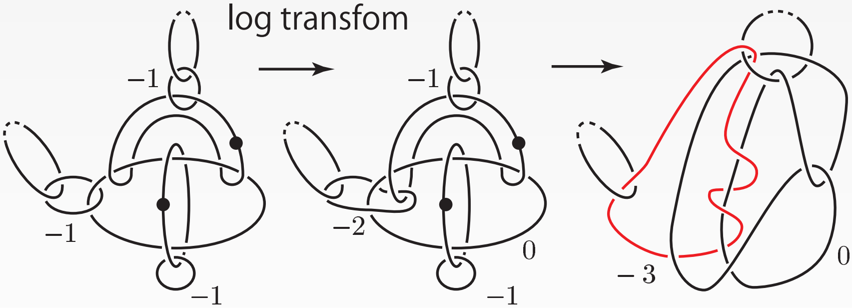
Suppose that $x = n \cdot F - S \in H_2(E(1)) = H_2(\mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2})$. *If* $x^2 = -27$ *or* -31, *then* $G(x) = 0$.

Proof of Proposition

$E(1) \rightsquigarrow E(1)_1 = W_1 \cup N_{-1}(1) = E(1)$ 1-log transform.

$\partial D^2 \to \partial D^2 + \alpha$

Proof of Proposition



$F - S$

Use the same handle slide in the first theorem.

Theorem (Estimate of genus)

Let $n = -(p^2 + p + 2s + 1)$*, where* $0 \leq s \leq 15$*. Suppose that* $\mathbf{x}_n = n \cdot F - S$*. Then* $G(\mathbf{x}_n) \leq \frac{p(p-1)}{2}$*.*

$\because$ $p-$ log-transform $E(1)_p = E(1)$ gives a surface with genus $\frac{p(p-1)}{2}$.

Conjecture

For $\mathbf{x}_n = n \cdot F - S$

$$G(\mathbf{x}_n) \sim \frac{n}{2}$$

What about classes $\mathbf{x}$ in $H_2(\mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2})$?
The orbit set with $\mathbf{x}^2 = -n$ is finite.