Hyperelliptic Lefschetz fibrations and the Dirac braid

joint work with Seiichi Kamada

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1 Introduction

4-manifolds $\underset{\text{total space}}{\longleftarrow}$ Lefschetz fibrations $\underset{\text{monodromy}}{\longrightarrow}$ mapping class groups

Elliptic surfaces w/o multiple tori $\underset{\text{natural generalization}}{\longrightarrow}$ Hyperelliptic Lefschetz fibrations

In this talk . . .

★ **We define a new invariant w for hyperelliptic Lefschetz fibrations**

★ **We employ Kamada's chart description to introduce w**

★ **A detailed proof of invariance of w is given**

Reference: H. Endo and S. Kamada,
Counting Dirac braids and hyperelliptic Lefschetz fibrations, arXiv:1508.07687.

2 Hyperelliptic Lefschetz fibrations

- **Lefschetz fibrations**

M, B : closed oriented smooth 4-manifold and 2-manifold
Σ_g : a closed surface of genus g , $f : M \to B$: a smooth map

Def. : f is a (achiral) Lefschetz fibration (LF) of genus g

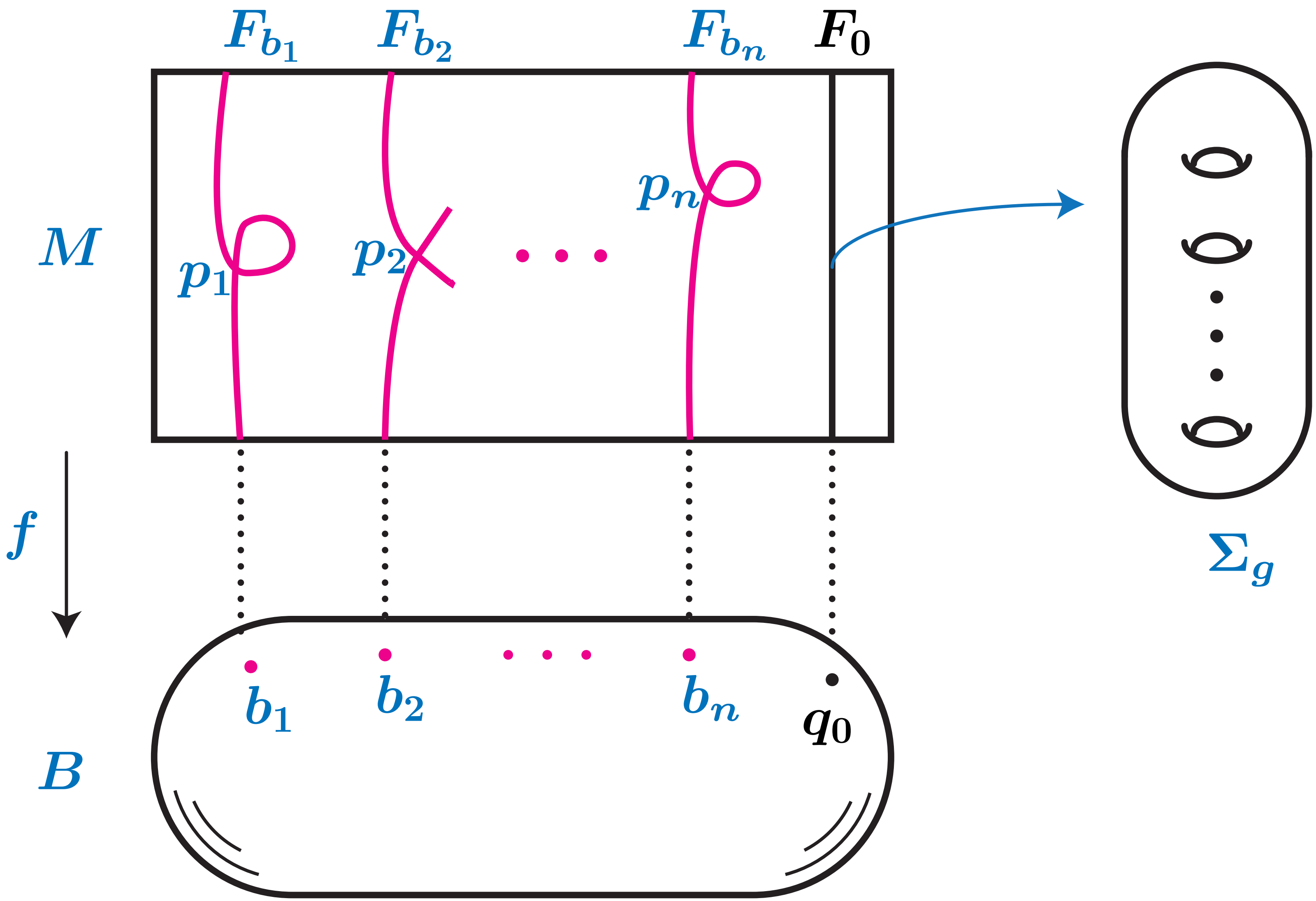
$\Longleftrightarrow$ (1) $\Delta \subset B$: the set of the critical values of f,
f is a fiber bundle with fiber Σ_g over $B - \Delta$

(2) there exists a unique critical point p on $F_b := f^{-1}(b)$,
f is written as $(z_1, z_2) \mapsto z_1 z_2$ or $z_1 \bar{z}_2$ about p and b

(3) no fiber contains a (± 1)-sphere

M : total space, B : base space, f : projection, Σ_g : fiber,
F_b : singular fiber $(b \in \Delta)$

- **Anatomy of an LF**



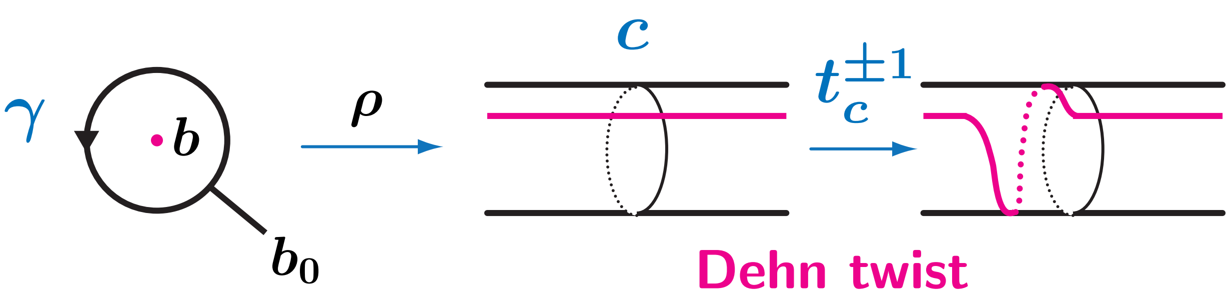
- **Monodromy**

$\Phi : \Sigma_g \to F_0 := f^{-1}(b_0)$: an orientation-preserving diffeomorphism
$\gamma : [0,1] \to B - \Delta$: a loop based at b_0

$\rightsquigarrow$ the pull-back $\gamma^* f : \gamma^* M \to [0,1]$ of f is a trivial bundle

$\rightsquigarrow$ $\exists$ a natural "bundle map" $\varphi : [0,1] \times \Sigma_g \to M$ extending Φ

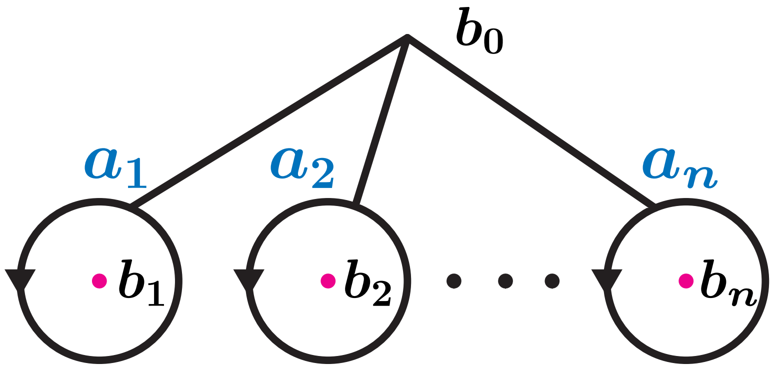
$\rightsquigarrow$ $\rho : \pi_1(B - \Delta, b_0) \to \pi_1(\mathrm{BDiff}_+\Sigma_g) \cong \pi_0(\mathrm{Diff}_+\Sigma_g) =: \mathcal{M}_g$
$: [\gamma] \mapsto [\Phi \circ \varphi_1]$: monodromy representation of f w.r.t. Φ



★ The mapping calss group $\mathcal{M}_g$ of Σ_g acts on the right.

- **Hurwitz system of LF over S^2**

$$\pi_1(S^2 - \Delta, b_0) = \langle a_1, \cdots, a_n \mid a_1 \cdots a_n = 1 \rangle$$



$\rightsquigarrow$ $\rho : a_1 \cdots a_n = 1 \mapsto t_{a_1}^{\varepsilon_1} \cdots t_{a_n}^{\varepsilon_n} = 1 \in \mathcal{M}_g$ $(\varepsilon_i = \pm 1)$

$\rightsquigarrow$ $(\rho(a_1), \ldots, \rho(a_n)) = (t_{a_1}^{\varepsilon_1}, \ldots, t_{a_n}^{\varepsilon_n})$: a **Hurwitz system** of f

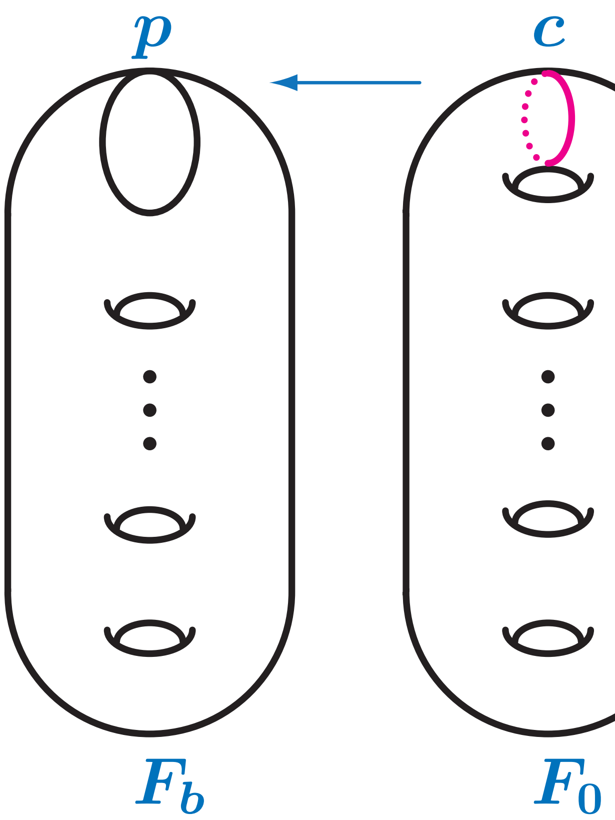
Theorem (Kas, cf. Matsumoto) : $g \geqq 2$

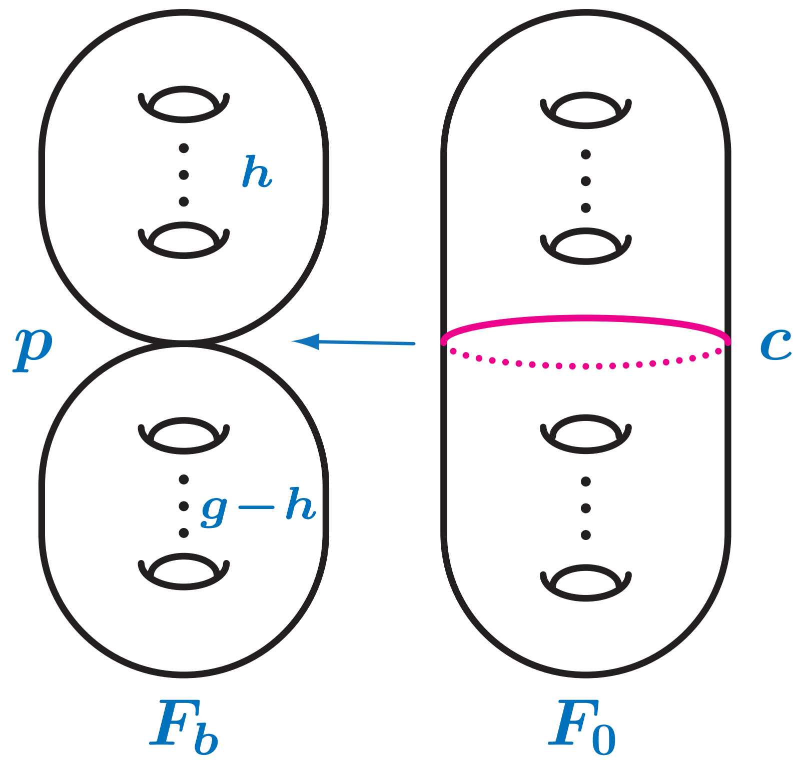
{ LF $f : M \to B$ of genus g }/ isomorphism $\cong$ of LF

$\overset{1:1}{\longleftrightarrow}$ { homomorphism $\rho : \pi_1(B - \Delta, b_0) \to \mathcal{M}_g$ with $\rho(a_i)$ a Dehn twist }/ equivalence

$\underset{B=S^2}{\overset{1:1}{\longleftrightarrow}}$ { Hurwitz system $(t_{a_1}^{\varepsilon_1}, \ldots, t_{a_n}^{\varepsilon_n})$ }/ **Hurwitz equiv.** & conj.

- **Singular fibers and vanishing cycles**





non-separating, type $\mathrm{I}^{\pm}$

$\# = n_0^{\pm}(f)$

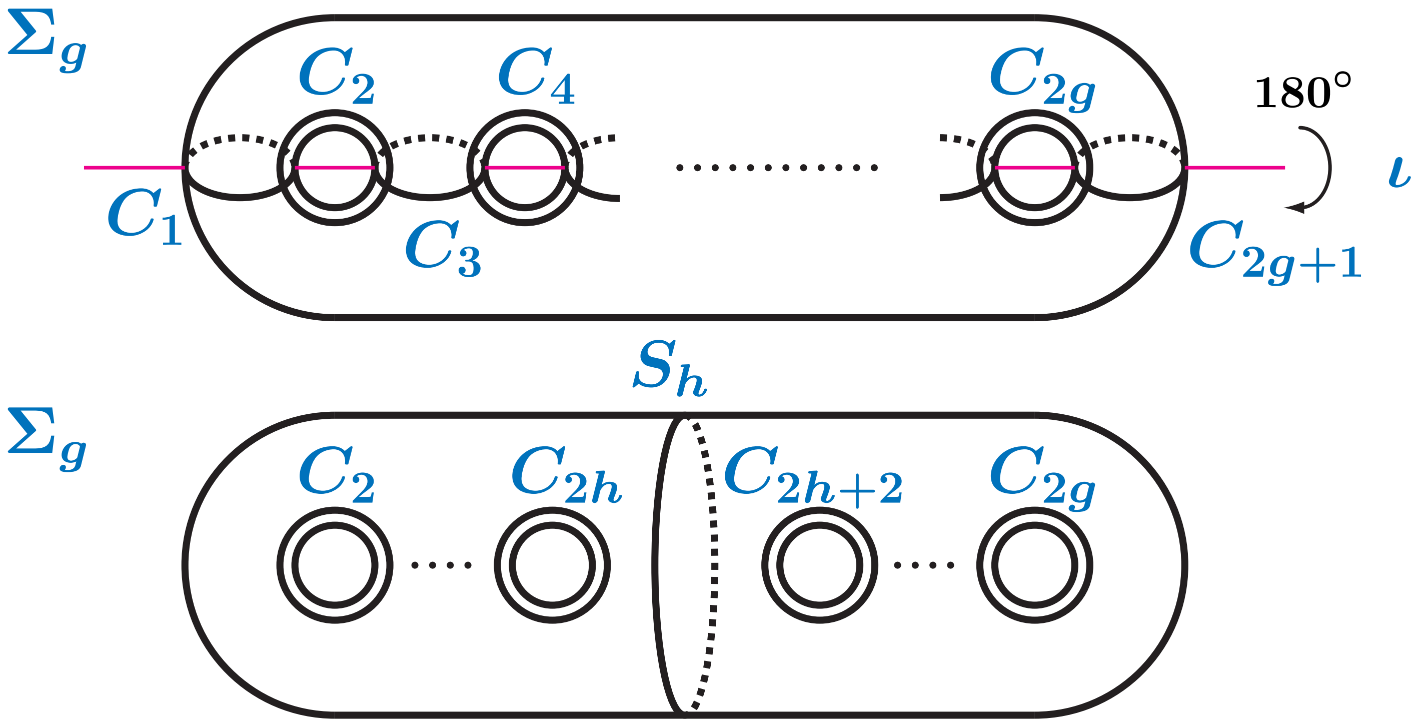
separating, type $\mathrm{II}_h^{\pm}$

$\# = n_h^{\pm}(f)$

- **Hyperelliptic mapping class group**

$\iota : \Sigma_g \to \Sigma_g$: an involution of Σ_g with $2g+2$ fixed points

$\mathcal{H}_g := \{\varphi \in \mathcal{M}_g \mid \varphi\iota = \iota\varphi\}$: hyperelliptic mapping class group



★ $t_C \in \mathcal{H}_g \Leftrightarrow t_{\iota(C)} = t_C \Leftrightarrow \iota(C) = C$

★ $\zeta_i := t_{C_i}$, $\sigma_h := t_{S_h}$ $(i = 1, \ldots, 2g+1,\ h = 1, \ldots, [g/2])$

- **Hyperelliptic Lefschetz fibrations**

$f : M \to B$: an LF of genus g
$\Phi : \Sigma_g \to F_0 := f^{-1}(b_0)$: an orientation-preserving diffeomorphism
$\rho : \pi_1(B - \Delta, b_0) \to \mathcal{M}_g$: monodromy of f w.r.t. Φ

Def. : (f, Φ) is a hyperelliptic LF $\Longleftrightarrow$ $\mathrm{Im}\,\rho \subset \mathcal{H}_g$

★ We can define an isomorphism $\cong_H$ of two hyperelliptic LFs.

(1) $(f, \Phi) \cong_H (f', \Phi')$ $\Leftrightarrow$ ρ, ρ' is equiv. in $\mathcal{H}_g$ up to conj.

(2) $(f, \Phi) \cong_H (f', \Phi')$ $\Rightarrow$ $f \cong f'$ as LFs

(3) $\mathrm{Im}\,\rho = \mathcal{H}_g$ and $f \cong f'$ $\Rightarrow$ $(f, \Phi) \cong_H (f', \Phi')$

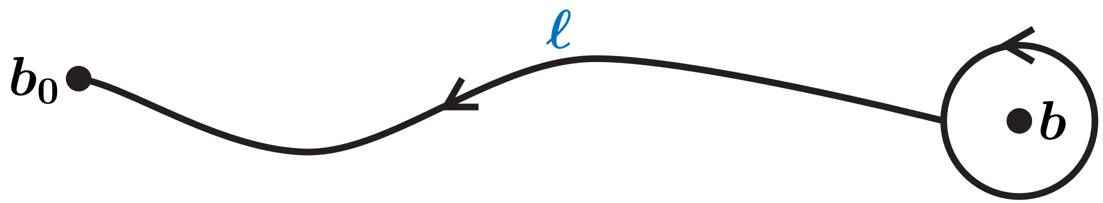
★ We often denote (f, Φ) by f for short.

3 Chart descriptions

- G-monodromy representations

B : a closed oriented smooth 2-manifold
Δ : a finite subset of B , b_0 : a base point of $B - \Delta$
$\mathcal{X}$: a set , $\mathcal{R}, \mathcal{S}$: sets of words in $\mathcal{X} \cup \mathcal{X}^{-1}$
G : a group with presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$
$\mathcal{C} := (\mathcal{X}, \mathcal{R}, \mathcal{S})$

$\mathcal{M}(B, \Delta, b_0; \mathcal{C}) := \{\rho : \pi_1(B - \Delta, b_0) \to G$: homomorphism
$\mid \rho([\ell]) \sim [s]\ (\exists s \in \mathcal{S})$ for every meridional loop $\ell\}$



$b \in \Delta$

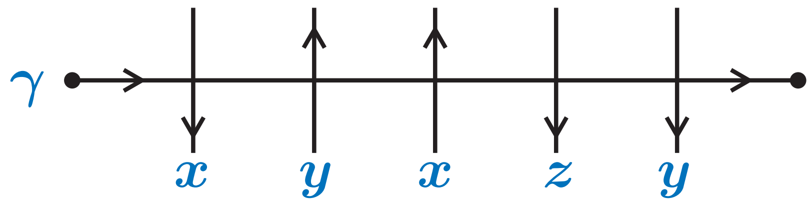
★ We can define an equivalence of two such ρ s.

- **Charts**

Γ : a finite graph in B, edges oriented and labeled with $x \in \mathcal{X}$

Def. : the **intersection word** $w_\Gamma(\gamma)$ of a simple path γ w.r.t. Γ

$\Longleftrightarrow$



$w_\Gamma(\gamma) = xy^{-1}x^{-1}zy$

Def. : Γ is a $\mathcal{C}$-**chart** in B

$\Longleftrightarrow$ Γ saitisfies (1) and (2):

(1) vertices of Γ : **white vertices** and **black vertices**

(2) for each **white** vertex v, $w_\Gamma(m_v) \in \mathcal{R} \cup \mathcal{R}^{-1}$;

for each **black** vertex v, $w_\Gamma(m_v) \in \mathcal{S}$

m_v : a (counterclockwise) meridian loop of v

v : white vertex of **type** r $\Longleftrightarrow$ $w_\Gamma(m_v)^{-1} = r \in \mathcal{R}$

v : black vertex of **type** s $\Longleftrightarrow$ $w_\Gamma(m_v) = s \in \mathcal{S}$

- **Charts and monodromies**

Γ : a $\mathcal{C}$-chart in B with base point b_0
Δ_Γ : the set of black vertices of Γ

Def. : the homomorphism determined by Γ
$\Longleftrightarrow$ $\rho_\Gamma : \pi_1(B - \Delta_\Gamma, b_0) \to G : [\eta] \mapsto [w_\Gamma(\eta)]$

Theorem (Kamada, Hasegawa) :
For any $\rho \in \mathcal{M}(B, \Delta, b_0; \mathcal{C})$, there exists a $\mathcal{C}$-chart Γ with $\rho_\Gamma = \rho$.

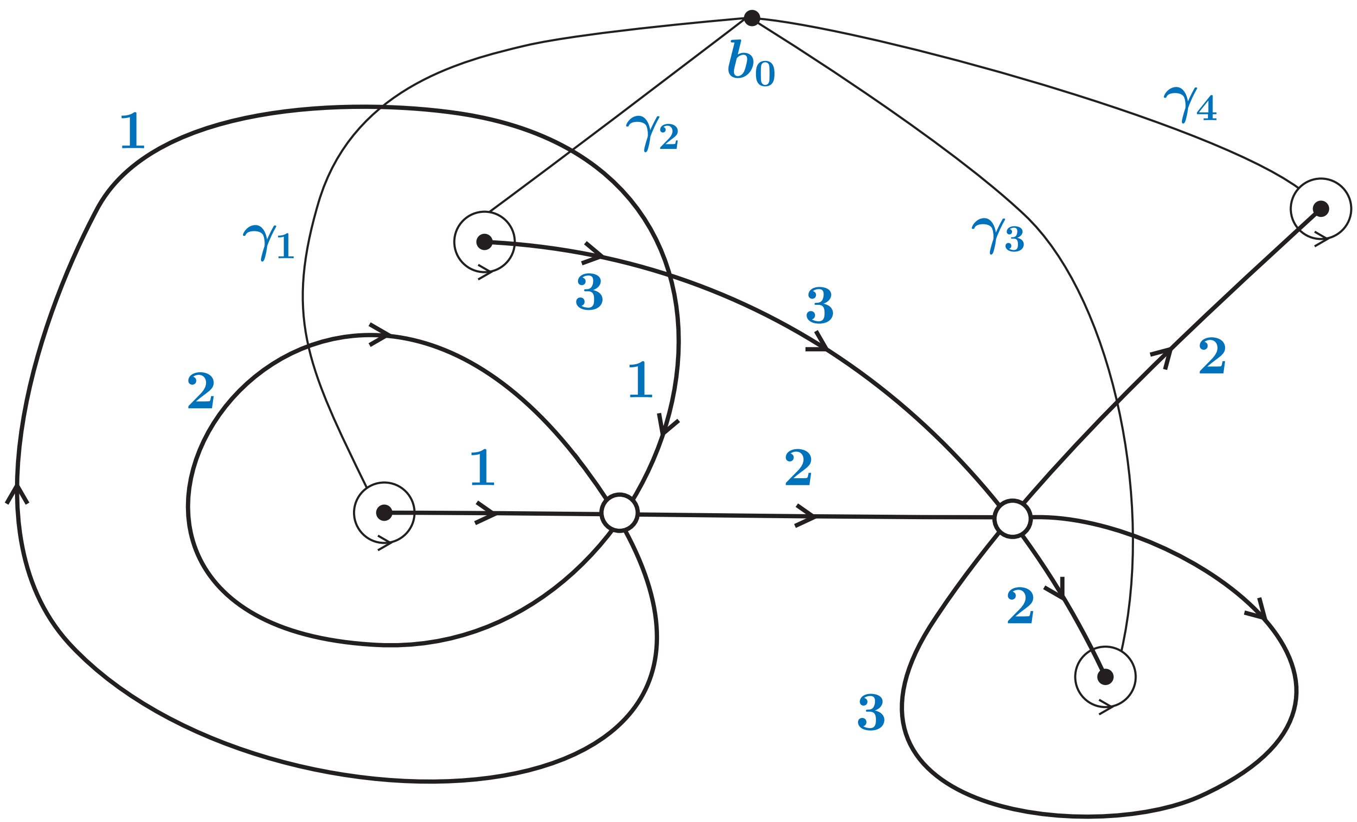
Theorem (Kamada, Hasegawa) :

$$\mathcal{M}(B, \Delta, b_0; \mathcal{C}) / \text{ equivalence of } G\text{-monodromies}$$
$$\overset{1:1}{\longleftrightarrow} \{ \ \mathcal{C}\text{-charts } \Gamma \text{ in } B \ \} / \text{ chart moves}$$

★ We use the terminology of chart description in Kamada's paper:
S. Kamada, Topology Appl. 154 (2007)

- **An example of $\mathcal{C}$-chart Γ**

$$G := B_4,\ \mathcal{C} := (\mathcal{X}, \mathcal{R}, \mathcal{S}),\ \mathcal{X} := \mathcal{S} := \{\sigma_1, \sigma_2, \sigma_3\},$$
$$\mathcal{R} := \{\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}, \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}, \sigma_2\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\}$$



$\rightsquigarrow$ $w_\Gamma(\gamma_1) = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_1$, $w_\Gamma(\gamma_2) = \sigma_1^{-1}\sigma_3\sigma_1, \ldots$ **etc.**

- **Chart moves**

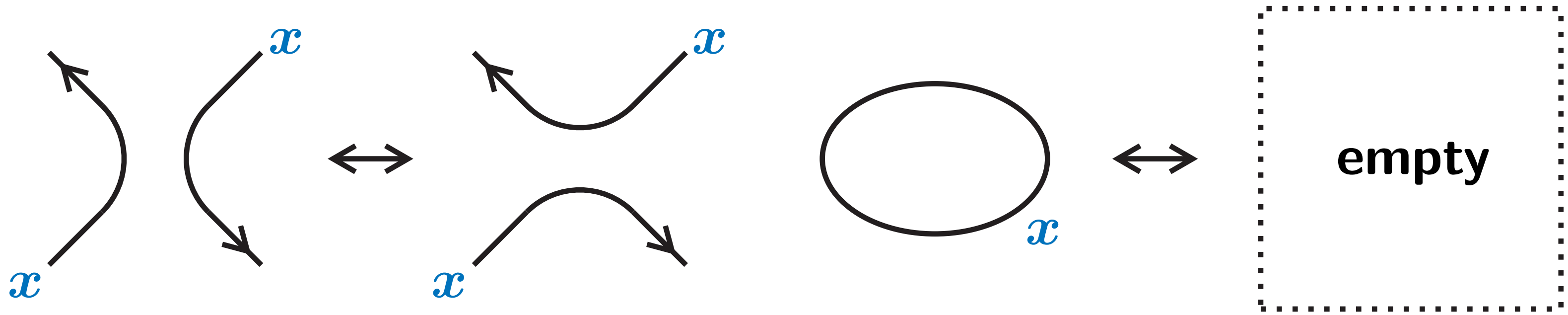
Γ, Γ' : $\mathcal{C}$-charts in B, b_0 : a base point of B

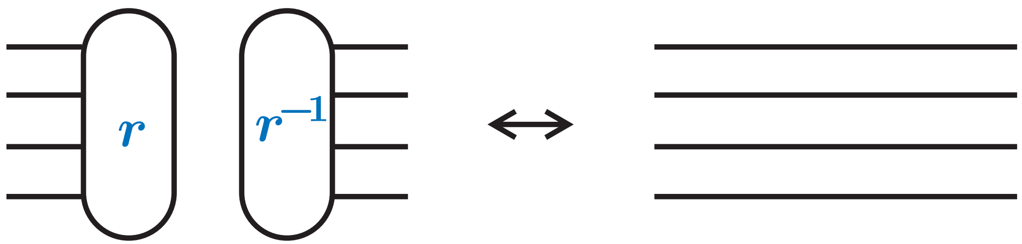
Def. : Γ' is obtained from Γ by **chart move of type W**

$\Longleftrightarrow$ • $\Gamma \cap (B - \mathrm{Int}\, D) = \Gamma' \cap (B - \mathrm{Int}\, D)$

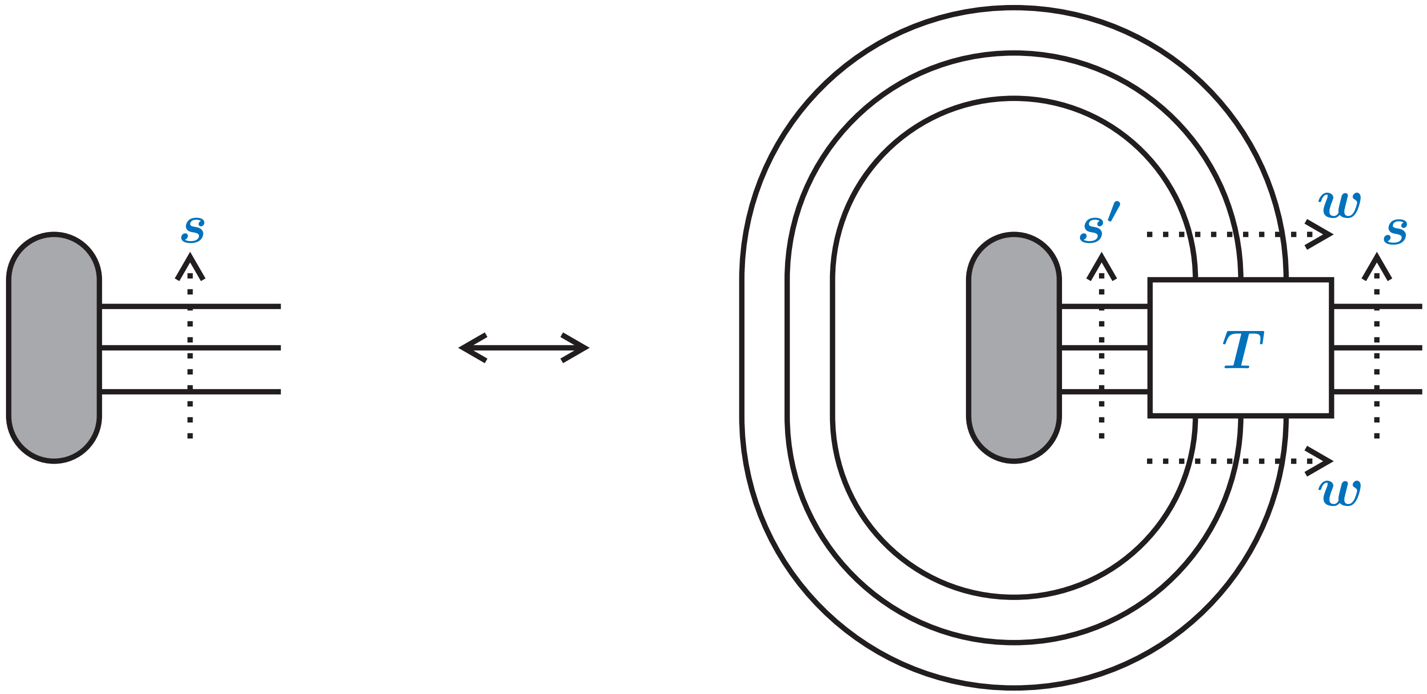
• both $\Gamma \cap D$ and $\Gamma' \cap D$ have **no black vertices**

for a disk D embedded in $B - \{b_0\}$





Def. : Γ' is obtained from Γ by **chart move of transition**
$\Longleftrightarrow$ Γ' is obtained from Γ by a local replacement depicted below



where $s, s' \in \mathcal{S}$, $w \in \mathcal{X} \cup \mathcal{X}^{-1}$, and

- s' and wsw^{-1} determine the **same** element of G
- the box labeled T is filled only by **edges** and **white vertices**

Def. : Γ' is obtained from Γ by **chart move of conjugacy type**
$\Longleftrightarrow$ Γ' is obtained from Γ by a local replacement depicted below

Def. : **chart moves** for $\mathcal{C}$-charts
$\Longleftrightarrow$ the following **four** kinds of moves:

- chart moves of type W
- chart moves of transition
- chart moves of conjugacy type
- sending by orientation preserving diffeomorphisms of B

4 An invariant w

- Three explicit $\mathcal{C}$s — ❶ $\mathcal{C}$ for $\mathcal{M}_{0,2g+2}$

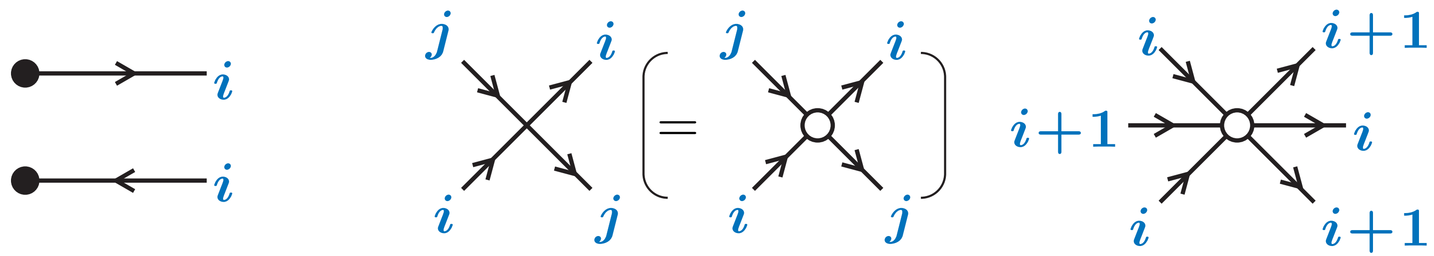
$G = \mathcal{M}_{0,2g+2}$: the mapping class group of S^2 with $2g+2$ points

$$\mathcal{C}_0 := (\mathcal{X}_0, \mathcal{R}_0, \mathcal{S}_0),\ \mathcal{X}_0 := \{\xi_1, \xi_2, \ldots, \xi_{2g+1}\},$$
$$\mathcal{R}_0 := \{r_1(i,j) \mid |i-j| > 1\} \cup \{r_2(i) \mid i = 1, \ldots, 2g\} \cup \{r_3, r_4\},$$
$$\mathcal{S}_0 := \{\ell_0(i)^{\pm 1} \mid i = 1, \ldots, 2g+1\} \cup \{\ell_h^{\pm 1} \mid h = 1, \ldots, [g/2]\}$$

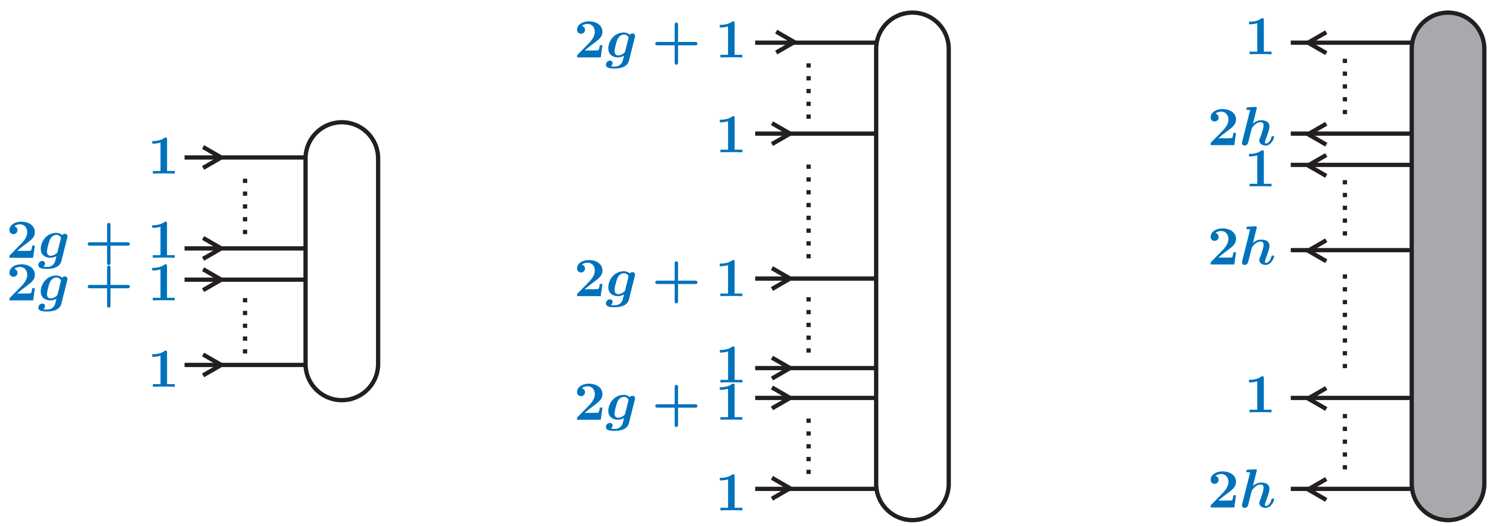
$$r_1(i,j) := \xi_i \xi_j \xi_i^{-1} \xi_j^{-1} \quad (|i-j| > 1),$$
$$r_2(i) := \xi_i \xi_{i+1} \xi_i \xi_{i+1}^{-1} \xi_i^{-1} \xi_{i+1}^{-1} \quad (i = 1, \ldots, 2g),$$
$$r_3 := \xi_1 \xi_2 \cdots \xi_{2g+1} \xi_{2g+1} \cdots \xi_2 \xi_1,$$
$$r_4 := (\xi_1 \xi_2 \cdots \xi_{2g+1})^{2g+2},$$
$$\ell_0(i) := \xi_i \quad (i = 1, \ldots, 2g+1),$$
$$\ell_h := (\xi_1 \xi_2 \cdots \xi_{2h})^{4h+2} \quad (h = 1, \ldots, [g/2]).$$

★ $\mathcal{M}_{0,2g+2} = \langle \mathcal{X}_0 \mid \mathcal{R}_0 \rangle$ **(Magnus)**

- **Vertices of types $\ell_0(i)^{\pm 1}, r_1(i,j), r_2(i)$**



- **Vertices of types r_3, r_4, ℓ_h**



- Three explicit $\mathcal{C}$s — ❷ $\mathcal{C}$ for $B_{2g+2}(S^2)$

$G = B_{2g+2}(S^2)$: the braid group of S^2 with $2g+2$ strands

$\tilde{\mathcal{C}} := (\tilde{\mathcal{X}}, \tilde{\mathcal{R}}, \tilde{\mathcal{S}})$, $\tilde{\mathcal{X}} := \{x_1, x_2, \ldots, x_{2g+1}\}$,
$\tilde{\mathcal{R}} := \{\tilde{r}_1(i,j) \mid |i-j| > 1\} \cup \{\tilde{r}_2(i) \mid i = 1, \ldots, 2g\} \cup \{\tilde{r}_3\}$,
$\tilde{\mathcal{S}} := \{\tilde{\ell}_0(i)^{\pm 1} \mid i = 1, \ldots, 2g+1\} \cup \{\tilde{\ell}_h^{\pm 1} \mid h = 1, \ldots, [g/2]\}$

$\tilde{r}_1(i,j) := x_i x_j x_i^{-1} x_j^{-1} \ (|i-j| > 1)$,
$\tilde{r}_2(i) := x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} \ (i = 1, \ldots, 2g)$,
$\tilde{r}_3 := x_1 x_2 \cdots x_{2g+1} x_{2g+1} \cdots x_2 x_1$,
$\tilde{\ell}_0(i) := x_i \ (i = 1, \ldots, 2g+1)$,
$\tilde{\ell}_h := (x_1 x_2 \cdots x_{2h})^{4h+2} \ (h = 1, \ldots, [g/2])$.

★ $B_{2g+2}(S^2) = \langle \tilde{\mathcal{X}} \mid \tilde{\mathcal{R}} \rangle$ (Fadell–Van Buskirk)

★ Vertices of types $\tilde{\ell}_0(i)^{\pm 1}, \tilde{r}_1(i,j), \tilde{r}_2(i), \tilde{r}_3, \tilde{\ell}_h$ in $\tilde{\mathcal{C}}$–charts are similar to those of types $\ell_0(i)^{\pm 1}, r_1(i,j), r_2(i), r_3, \ell_h$ in $\mathcal{C}_0$–charts

- **Three explicit $\mathcal{C}$s — ❸ $\mathcal{C}$ for $\mathcal{H}_g$**

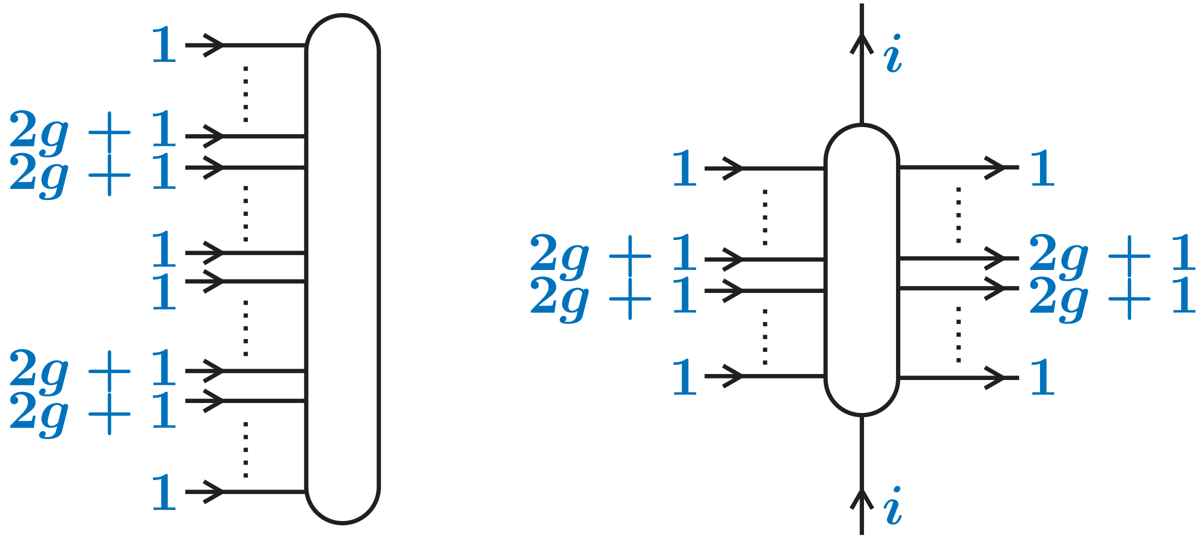
$G = \mathcal{H}_g$: **the hyperelliptic mapping class group of Σ_g**

$$\hat{\mathcal{C}} := (\hat{\mathcal{X}}, \hat{\mathcal{R}}, \hat{\mathcal{S}}), \ \hat{\mathcal{X}} := \{\zeta_1, \zeta_2, \ldots, \zeta_{2g+1}\},$$
$$\hat{\mathcal{R}} := \{\hat{r}_1(i,j) \mid |i-j| > 1\} \cup \{\hat{r}_2(i), \hat{r}_5(i) \mid i = 1, \ldots, 2g\} \cup \{\hat{r}_3, \hat{r}_4\},$$
$$\hat{\mathcal{S}} := \{\hat{\ell}_0(i)^{\pm 1} \mid i = 1, \ldots, 2g+1\} \cup \{\hat{\ell}_h^{\pm 1} \mid h = 1, \ldots, [g/2]\}$$

$$\hat{r}_1(i,j) := \zeta_i \zeta_j \zeta_i^{-1} \zeta_j^{-1} \quad (|i-j| > 1),$$
$$\hat{r}_2(i) := \zeta_i \zeta_{i+1} \zeta_i \zeta_{i+1}^{-1} \zeta_i^{-1} \zeta_{i+1}^{-1} \quad (i = 1, \ldots, 2g),$$
$$\hat{r}_3 := (\zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1)^2,$$
$$\hat{r}_4 := (\zeta_1 \zeta_2 \cdots \zeta_{2g+1})^{2g+2},$$
$$\hat{r}_5(i) := [\zeta_i, \zeta_1 \zeta_2 \cdots \zeta_{2g+1} \zeta_{2g+1} \cdots \zeta_2 \zeta_1] \quad (i = 1, \ldots, 2g+1),$$
$$\hat{\ell}_0(i) := \zeta_i \quad (i = 1, \ldots, 2g+1),$$
$$\hat{\ell}_h := (\zeta_1 \zeta_2 \cdots \zeta_{2h})^{4h+2} \quad (h = 1, \ldots, [g/2]).$$

★ $\mathcal{H}_g = \langle \hat{\mathcal{X}} \mid \hat{\mathcal{R}} \rangle$ **(Birman–Hilden)**

- **Vertices of types $\hat{r}_3, \hat{r}_5(i)$**



★ **Vertices of types $\hat{\ell}_0(i)^{\pm 1}, \hat{r}_1(i,j), \hat{r}_2(i), \hat{r}_4, \hat{\ell}_h$ in $\hat{\mathcal{C}}$–charts are similar to those of types $\ell_0(i)^{\pm 1}, r_1(i,j), r_2(i), r_4, \ell_h$ in $\mathcal{C}_0$–charts**

★ $1 \to \mathbb{Z}_2 \longrightarrow B_{2g+2}(S^2) \longrightarrow \mathcal{M}_{0,2g+2} \to 1$ **: central extension**

★ $1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{H}_g \xrightarrow{\pi} \mathcal{M}_{0,2g+2} \longrightarrow 1$ **: central extension**

- **Charts and Hyperelliptic Lefschetz fibrations**

Γ : a $\hat{\mathcal{C}}$-chart in B with base point b_0
Δ_Γ : the set of black vertices of Γ
$\rho_\Gamma : \pi_1(B - \Delta_\Gamma, b_0) \to \mathcal{H}_g$: the homomorphism determined by Γ

Propositon (Kamada–E.) :
(f, Φ) : hyperelliptic LF of genus g over B, $g \geqq 2$
$\rho : \pi_1(B - \Delta, b_0) \to \mathcal{H}_g$: a monodromy representation of (f, Φ)
$\Longrightarrow$ there exists a $\hat{\mathcal{C}}$-chart Γ which satisfies $\rho_\Gamma = \rho$.

Theorem (Kamada–E.) : $g \geqq 2$

$\{$ hyperelliptic LFs (f, Φ) of genus g over $B\}/ \cong_H$
$\overset{1:1}{\longleftrightarrow}$ $\{$ monodromies $\rho : \pi_1(B - \Delta) \to \mathcal{H}_g$ $\}/$ equivalence for $\mathcal{H}_g$
$\overset{1:1}{\longleftrightarrow}$ $\{$ $\hat{\mathcal{C}}$-charts Γ $\}/$ chart moves

- An invariant w

(f, Φ) : hyperelliptic LF of genus g over B
$\rho : \pi_1(B - \Delta, b_0) \to \mathcal{H}_g$: a monodromy representation of f
$\hat{\Gamma}$: a $\hat{\mathcal{C}}$-chart in B which satisfies $\rho_{\hat{\Gamma}} = \rho$.

Def. : $w(\hat{\Gamma})$: # of white vertices of type $\hat{r}_4^{\pm 1}$ included in $\hat{\Gamma}$ mod 2
$w(f, \Phi) := w(\hat{\Gamma})$

Theorem (Kamada–E.): w is invariant under chart moves for odd g

Γ : a $\mathcal{C}_0$-chart in B with base point b_0

Def. : $w(\Gamma)$: # of white vertices of type $r_4^{\pm 1}$ included in Γ mod 2

$(f, \Phi), \rho, \hat{\Gamma}$: as above
$\rightsquigarrow$ We can construct a $\mathcal{C}_0$-chart Γ by changing $\hat{\Gamma}$ locally
- Γ corresponds to $\rho_0 := \pi \circ \rho : \pi_1(B - \Delta, b_0) \to \mathcal{M}_{0,2g+2}$
- $w(\Gamma) = w(\hat{\Gamma})$ in $\mathbb{Z}_2$

- **Examples**

Hurwitz system (for odd g)

$W := (\zeta_1, \ldots, \zeta_{2g+1}, \zeta_{2g+1}, \ldots, \zeta_1)^{g+1}$

$W' := (\zeta_1, \ldots, \zeta_{2g+1})^{2g+2}$

$\rightsquigarrow$ **We obtain hyperelliptic LFs f, f' of genus g over S^2**

Both f and f' have $2(g+1)(2g+1)$ non-separating fibers, and they do not have separating fibers

$\rightsquigarrow$ **they have the same Euler characteristic and the same signature**

Drawing $\hat{\mathcal{C}}$-charts corresponding to f and f', we obtain $w(f) = 0$ and $w(f') = 1$

$\rightsquigarrow$ **f and f' are not isomorphic.**

★ **By a theorem of Usher, the total spaces of f and f' are homeomorphic but not diffeomorphic to each other.**

5 Proof of invariance

We will show the invariance of w for $\mathcal{C}_0$-charts.

$\rightsquigarrow$ **It is obvious that w is invariant under chart moves of conjugacy and orientation preserving diffeomorphisms**

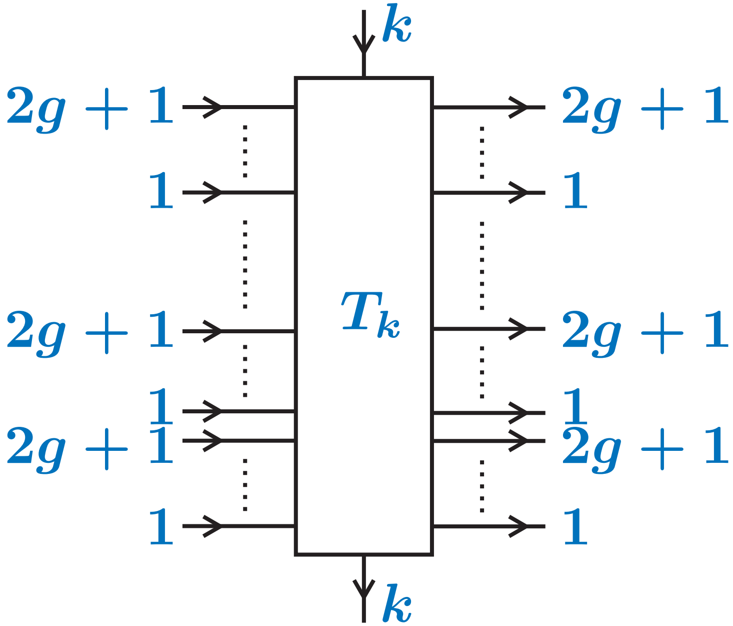
$\rightsquigarrow$ **It suffices to show the invariance under chart moves of type W and chart moves of transition**

- **Invariance under chart move of type W**

Prop. : w is invariant under chart moves of type W

We need a lemma.

Lem. : For a $\mathcal{C}_0$-chart below, the box labeled T_k can be filled only with edges and white vertices of types $r_1(i,j)^{\pm 1}, r_2(i)^{\pm 1}, r_3^{\pm 1}$.



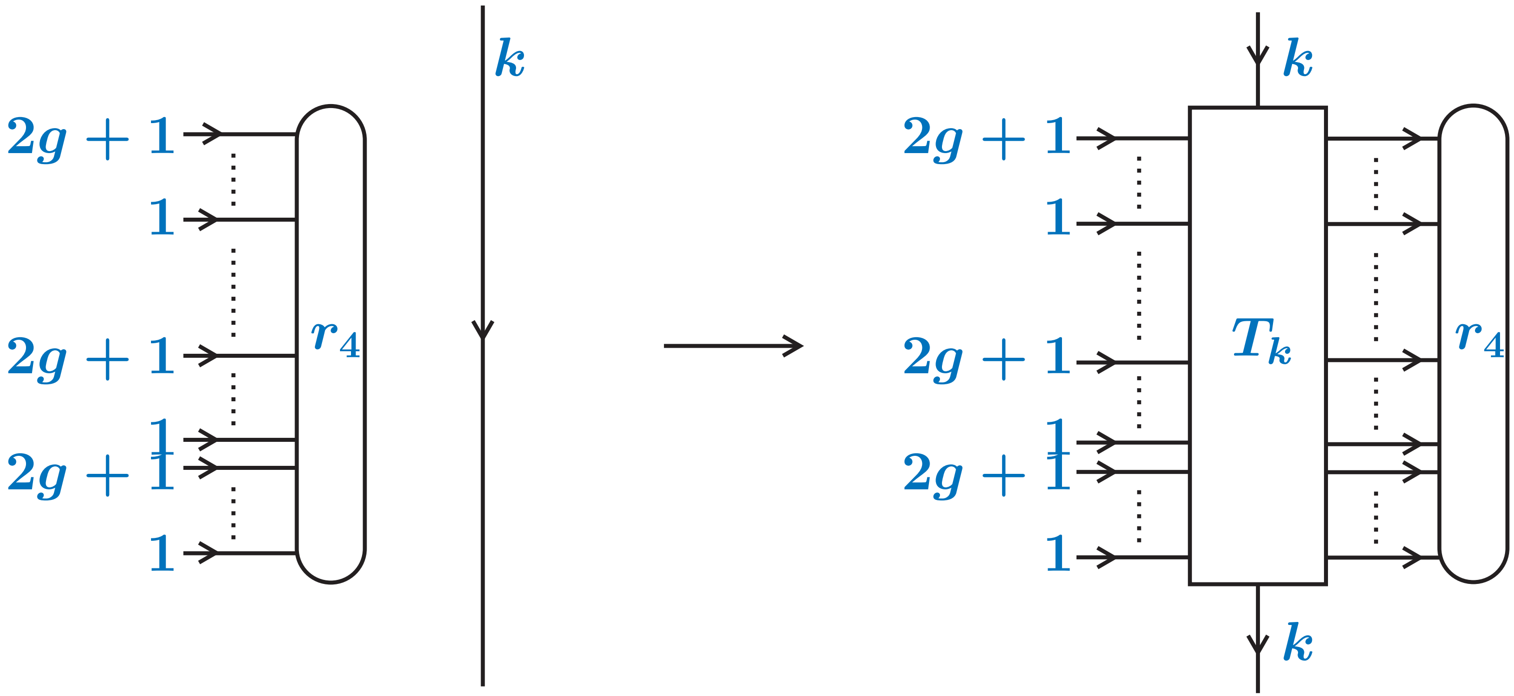
Proof : The Dirac braid $\Delta := (x_1 x_2 \cdots x_{2g+1})^{2g+2}$ is included in the center of of $B_{2g+2}(S^2)$

$\rightsquigarrow$ We can fill the box labeled T_k with $\tilde{\mathcal{C}}$-chart

$\rightsquigarrow$ Change all labels x_i of edges into ξ_i to obtain a $\mathcal{C}_0$-chart $\square$

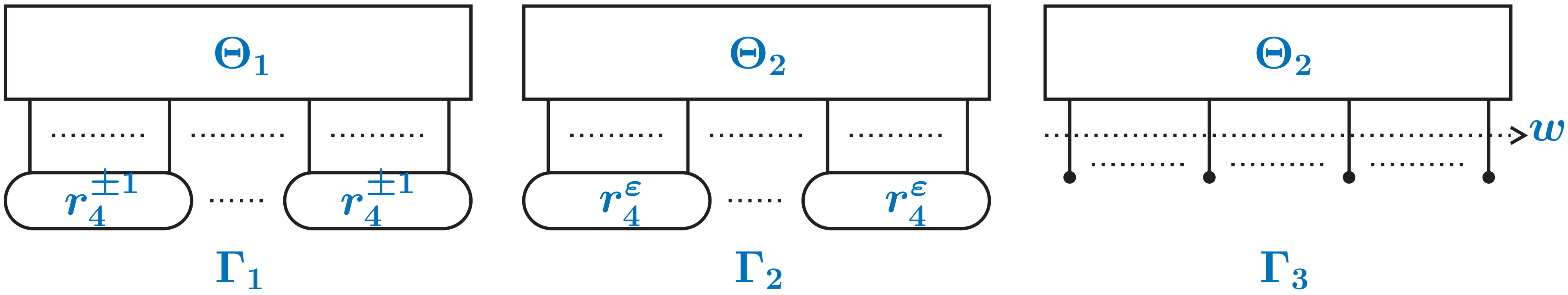
Proof of Prop. : It suffices to show that $w(\Gamma) = 0$ for every $\mathcal{C}_0$-chart Γ in S^2 **without** black vertices.

$\rightsquigarrow$ Consider the chart move of type W below



$\rightsquigarrow$ By **Lemma**, the box labeled T_k can be filled only with edges and white vertices of types $r_1(i,j)^{\pm 1}, r_2(i)^{\pm 1}, r_3^{\pm 1}$.

$\rightsquigarrow$ **Use this move repeatedly to obtain Γ_1 below from Γ**



$\rightsquigarrow$ **Cancel pairs of white vertices of type $r_4^{\pm 1}$ to obtain Γ_2 from Γ_1**

n **:= # of white vertices of type r_4^{ε} in Γ_2**

$\rightsquigarrow$ **Replace all white vertices of type r_4^{ε} with black vertices of type $\ell_0(i)^{-\varepsilon}$ to obtain Γ_3 from Γ_2**

★ **the box labeled Θ_1, Θ_2 is filled only with edges and white vertices of types $r_1(i,j)^{\pm 1}, r_2(i)^{\pm 1}, r_3^{\pm 1}$.**

$\rightsquigarrow$ **Change all labels ξ_i of edges into x_i to obtain a $\tilde{\mathcal{C}}$-chart $\tilde{\Gamma}_3$**

$\rightsquigarrow$ **The intersection word w is $\Delta^{\varepsilon n} = (x_1 x_2 \cdots x_{2g+1})^{\varepsilon(2g+2)n}$, which represents 1 of $B_{2g+2}(S^2)$**

$\rightsquigarrow$ **n must be even because Δ represents an element of order two**

Thus we have

$$w(\Gamma) = w(\Gamma_1) = w(\Gamma_2) = n = 0$$

and this completes the proof. $\square$

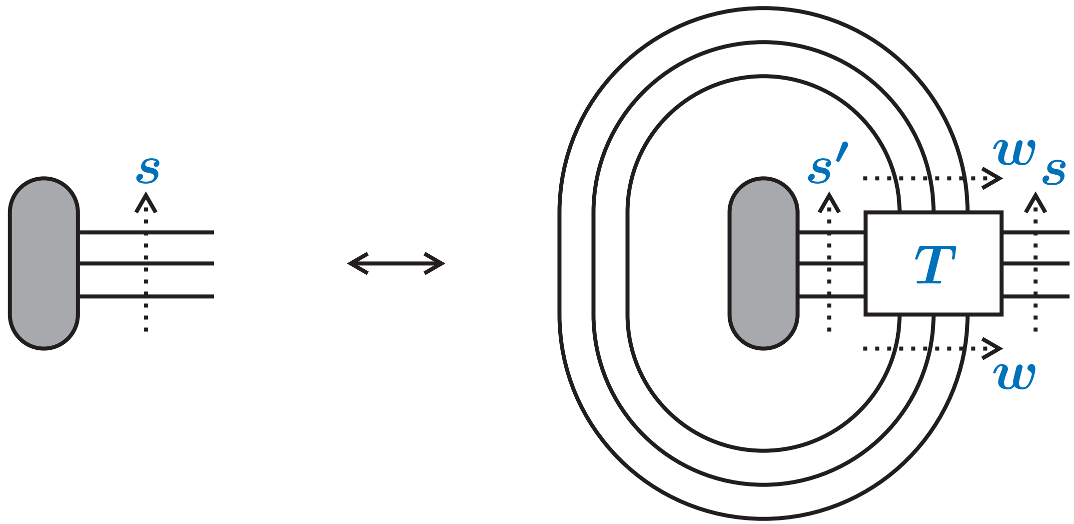
- **Invariance under chart move of transition**

Prop. : w is invariant under **chart moves of transition** if g is **odd**

We divide chart moves of transition for $\mathcal{C}_0$-charts into two cases.

❶ $s, s' \in \{\ell_0(i)^{\pm 1} \mid i = 1, \ldots, 2g+1\}$ $(\ell_0(i) := \xi_i)$

❷ $s, s' \in \{\ell_h^{\pm 1} \mid h = 1, \ldots, [g/2]\}$ $(\ell_h := (\xi_1 \xi_2 \cdots \xi_{2h})^{4h+2})$

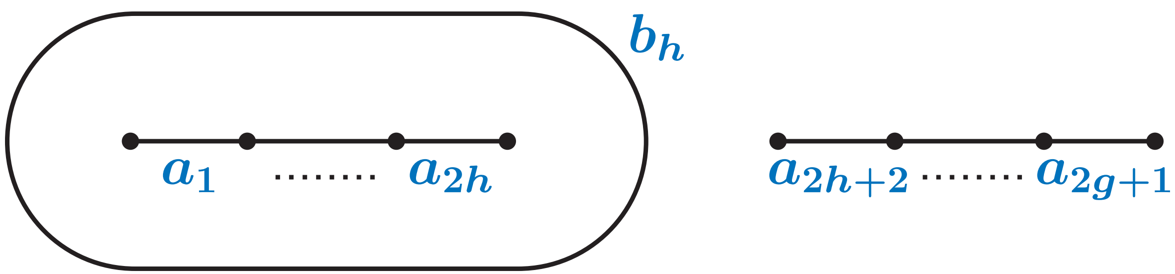


We will give a proof of **Prop.** only for Case ❷.
The proof for Case ❶ is omitted.

Proof for Case ❷ : Assume $s = s' = (\xi_1\xi_2\cdots\xi_{2h})^{4h+2}$.

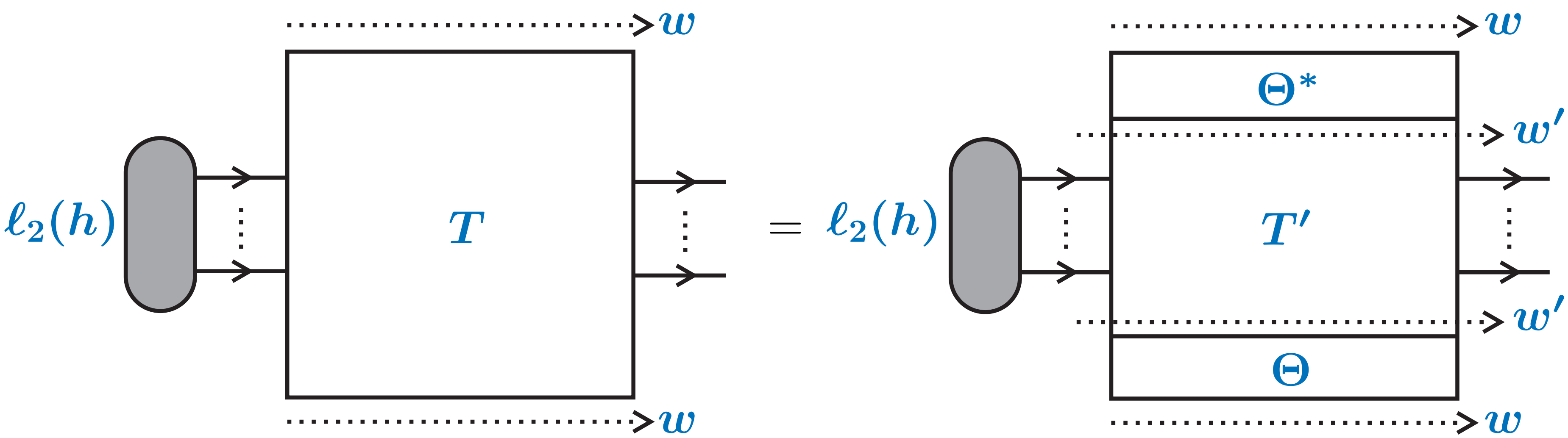
$\varphi := [w]$, $\tau_h := [(\xi_1\xi_2\cdots\xi_{2h})^{4h+2}] \in \mathcal{M}_{0,2g+2}$
$\rightsquigarrow$ a relation $\varphi\tau_h\varphi^{-1} = \tau_h$ in $\mathcal{M}_{0,2g+2}$ from the chart above

$\mathrm{pr} : \Sigma_g \longrightarrow \Sigma_g/I = S^2$: the projection
$a_i := \mathrm{pr}(C_i)$, $b_h := \mathrm{pr}(S_h)$
$F : S^2 \to S^2$: an orientation preserving diffeo. representing φ
$\rightsquigarrow$ $(b_h)F$ is isotopic to b_h
$\rightsquigarrow$ We can assume F fixes b_h pointwise because g is odd



$\rightsquigarrow$ φ is represented by a word w' in $\xi_1^{\pm 1}, \ldots, \xi_{2h}^{\pm 1}$, $\xi_{2h+2}^{\pm 1}, \ldots, \xi_{2g+1}^{\pm 1}$

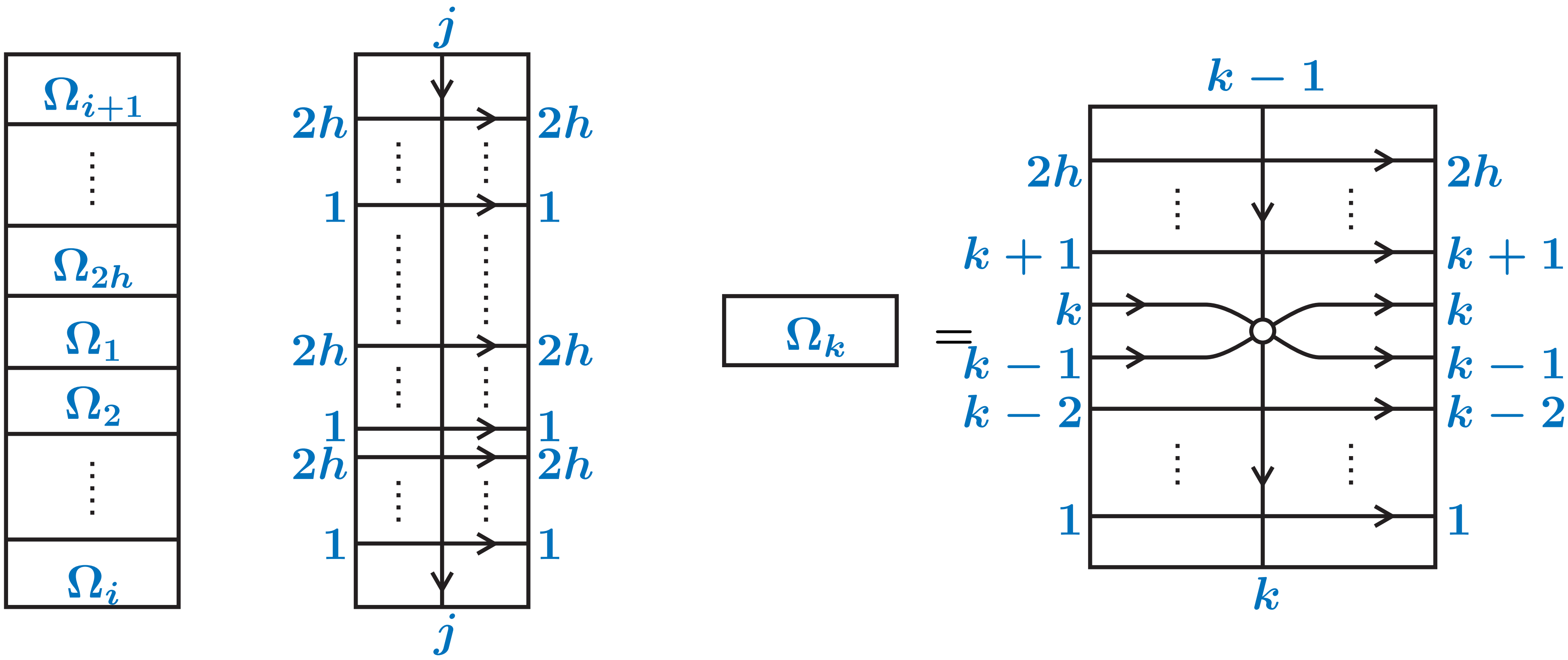
We divide the box labeled T into three boxes labeled T', Θ, Θ^*



★ The box labeled Θ is filled only with edges and white vertices

★ The box labeled Θ^* is the mirror image of Θ

★ Since w' is a word in $\xi_1^{\pm 1}, \ldots, \xi_{2h}^{\pm 1}, \ \xi_{2h+2}^{\pm 1}, \ldots, \xi_{2g+1}^{\pm 1}$,
the box labeled T' is filled with copies of (a) and (b) below



(a) (b)

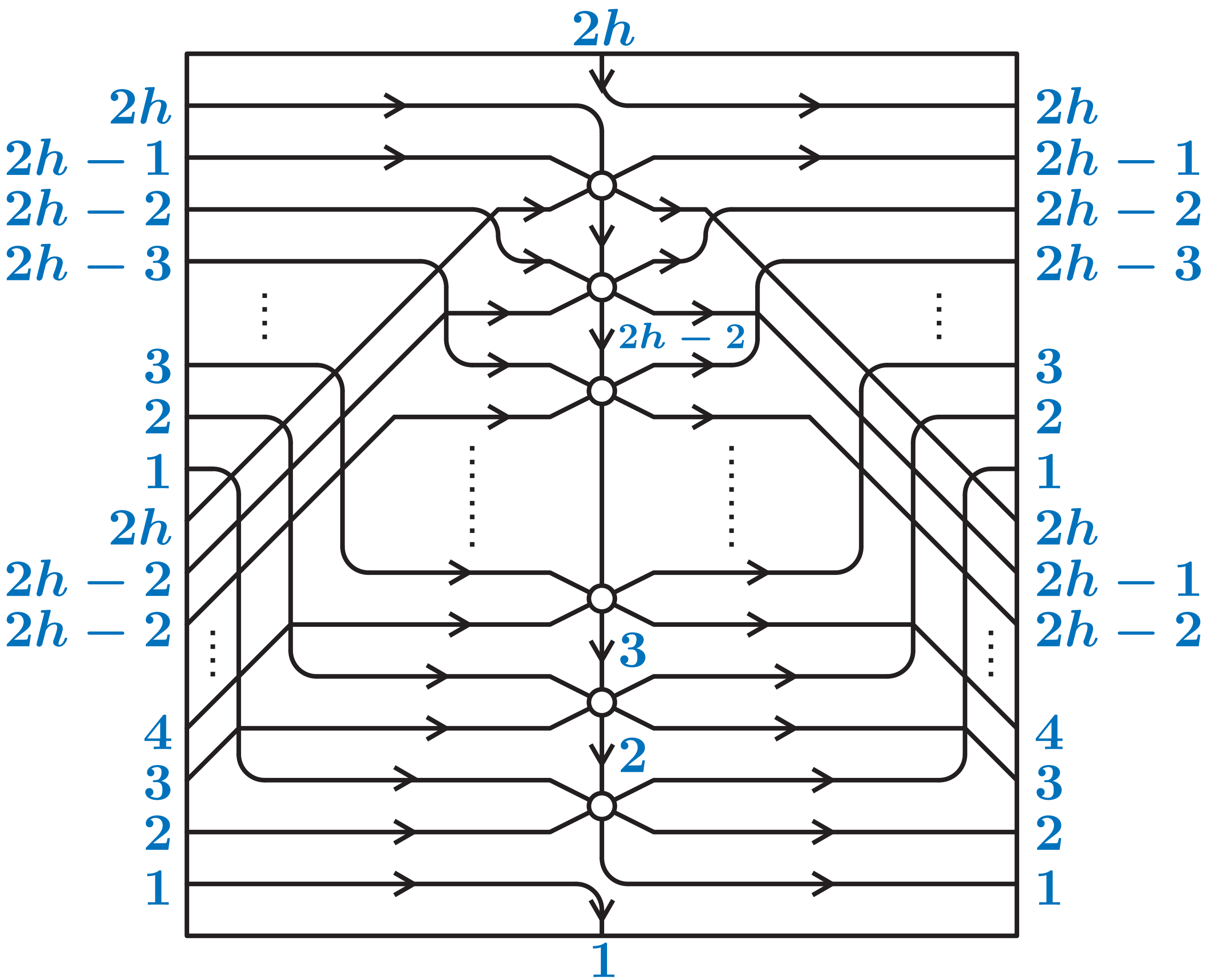
★ (a) corresponds to ξ_i $(i = 1, \ldots, 2h)$

★ (b) corresponds to ξ_j $(j = 2h+2, \ldots, 2g+1)$

★ Ω_k for $k = 2, \ldots, 2h$ is depicted above

★ Ω_1 is depicted below

$\boxed{\Omega_1} \quad =$



⇝ **The box labeled T' is filled only with edges and white vertices of types $r_1(i,j)^{\pm 1}$ and $r_2(i)^{\pm 1}$**

$\rightsquigarrow$ **# of white vertices of type $r_4^{\pm 1}$ included in the box labeled Θ^* is equal to that for the box labeled Θ**

$\rightsquigarrow$ **The box labeled T is filled with edges, white vertices of types $r_1(i,j)^{\pm 1}, r_2(i)^{\pm 1}, r_3^{\pm 1}$, and an even number of white vertices of types $r_4^{\pm 1}$**

By virtue of the invariance under chart moves of type W, any subchart filling the box labeled T has this property $\square$

We thus proved our main theorem.

– Owari –

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