1 Primitive relations

1.1 Braid relation

For any two simple closed curves α and β, we have

$$t_\beta t_\alpha t_\beta^{-1} = t_{t_\beta(\alpha)} \ : \text{ the } \textit{braid relation} \text{ (in the wider sense),}$$

or equivalently,

$$t_\beta t_\alpha = t_{t_\beta(\alpha)} t_\beta, \quad t_\beta t_\alpha = t_\alpha t_{t_\alpha^{-1}(\beta)}.$$

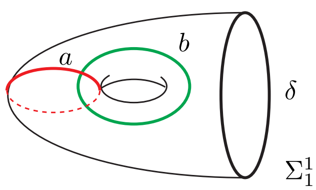
Special cases of braid relation :
If $\alpha \cap \beta = \emptyset$, then

$$t_\alpha t_\beta = t_\beta t_\alpha \ : \text{ the } \textit{far commutativity relation.}$$

If $\alpha \pitchfork \beta = \{*\}$, then

$$t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta \ : \text{ the } \textit{braid relation} \text{ (in the strict sense).}$$

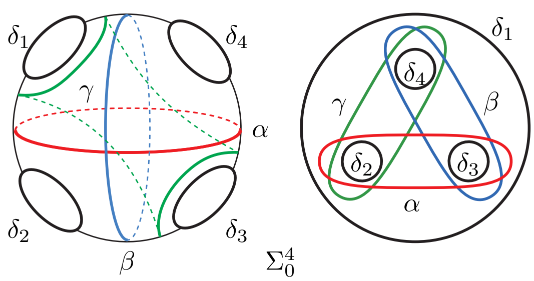
1.2 2-chain relation



$$(t_a t_b)^6 = t_\delta$$

: the 2-*chain relation*, or the *chain relation.*

1.3 Lantern relation



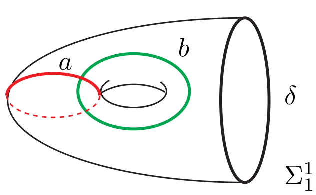
$$t_\alpha t_\beta t_\gamma = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}$$

: the *lantern relation.*

2 Useful relations in low genus MCG's

2.1 k-holed torus relation

2.1.1 1-holed torus relation



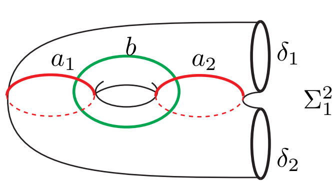
$$(t_a t_b)^6 = t_\delta$$

: the 1-*holed torus relation*

(the same as the 2-chain relation).

$(\kappa_c, \kappa_l) = (1, 0)$.

2.1.2 2-holed torus relation



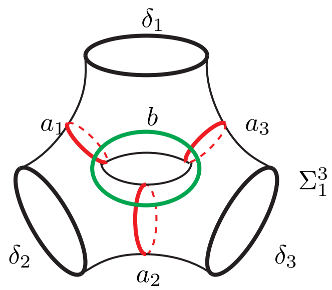
$$(t_{a_1} t_{a_2} t_b)^4 = t_{\delta_1} t_{\delta_2}$$

: the 2-*holed torus relation*

(the same as the 3-chain relation modulo braid relations).

$(\kappa_c, \kappa_l) = (1, 1)$.

2.1.3 3-holed torus relation



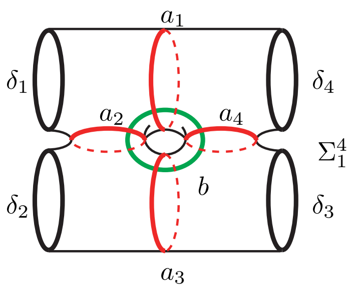
$$(t_{a_1}t_{a_2}t_{a_3}t_b)^3 = t_{\delta_1}t_{\delta_2}t_{\delta_3}$$

: the 3-*holed torus relation*,

also known as the *star relation*.

$(\kappa_c, \kappa_l) = (1, 2)$.

2.1.4 4-holed torus relation

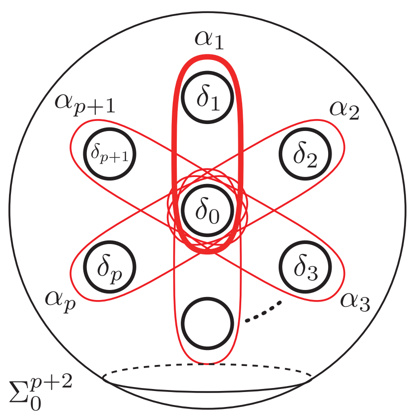


$$(t_{a_1}t_{a_3}t_bt_{a_2}t_{a_4}t_b)^2 = t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}$$

: the 4-*holed torus relation*.

$(\kappa_c, \kappa_l) = (1, 3)$.

2.2 Daisy relation

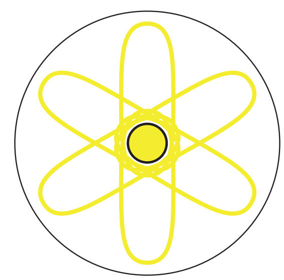


$$t_{\alpha_1}t_{\alpha_2}\cdots t_{\alpha_{p+1}} = t_{\delta_0}^{p-1}t_{\delta_1}t_{\delta_2}\cdots t_{\delta_{p+1}}$$

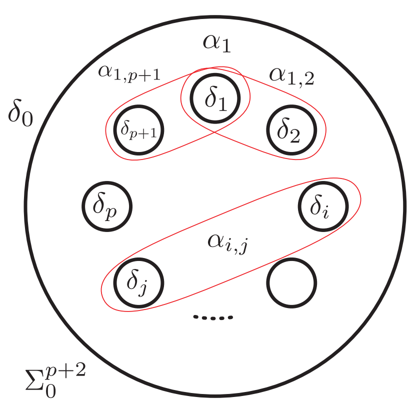
: the *daisy relation*.

Note: coincides with the lantern relation when $p = 2$.

$(\kappa_c, \kappa_l) = (0, p-1)$.



2.3 Rose relation



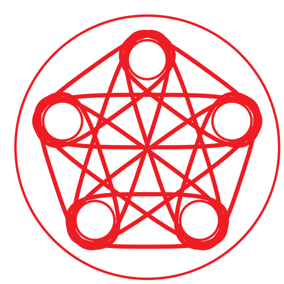
$$\prod_{i=1}^{p}\left(\prod_{j=i+1}^{p+1} t_{\alpha_{i,j}}\right) = t_{\alpha_{1,2}}t_{\alpha_{1,3}}t_{\alpha_{1,4}}\cdots t_{\alpha_{1,p+1}}\cdot t_{\alpha_{2,3}}t_{\alpha_{2,4}}\cdots t_{\alpha_{2,p+1}}\cdots t_{\alpha_{p,p+1}}$$

$$= t_{\delta_0}t_{\delta_1}^{p-1}t_{\delta_2}^{p-1}\cdots t_{\delta_{p+1}}^{p-1}$$

: the *rose relation*.

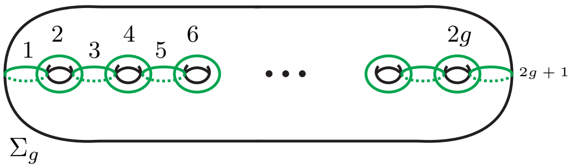
Note: coincides with the lantern relation when $p = 2$.

$(\kappa_c, \kappa_l) = (0, (p-1)p/2)$.



3 Representative Lefschetz fibrations and pencils

3.1 The hyperelliptic relation (the Birman-Hilden relation)



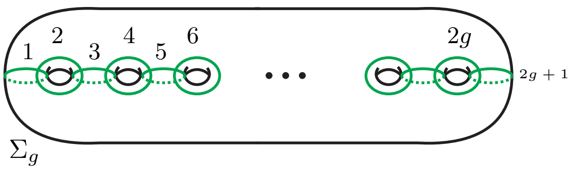
$(t_1 t_2 \cdots t_{2g} t_{2g+1} t_{2g+1} t_{2g} \ldots t_2 t_1)^2 = 1$

: the *hyperelliptic relation*, or the *Birman-Hilden relation*.

$X_{LF} = \mathbb{C}P^2 \# (4g+5)\overline{\mathbb{C}P^2}$.

$(\kappa_c, \kappa_l) = (g, 3g-4)$.

3.2 $(2g+1)$-chain relation



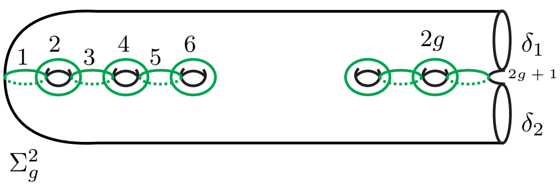
$(t_1 t_2 \cdots t_{2g+1})^{2g+2} = 1$

: the *chain relation with length* $2g+1$,

or simply, the $(2g+1)$-*chain relation*.

$X_{LF} =$ unknown?

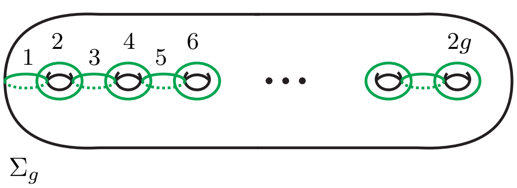
$(\kappa_c, \kappa_l) = (g(g+1)/2, (g+1)(3g-4)/2)$.



$(t_1 t_2 \cdots t_{2g+1})^{2g+2} = t_{\delta_1} t_{\delta_2}$.

$(\kappa_c, \kappa_l) = (g(g+1)/2, g(3g-1)/2)$.

3.3 $2g$-chain relation



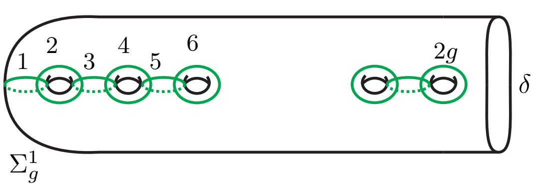
$(t_1 t_2 \cdots t_{2g})^{4g+2} = 1$

: the *chain relation with length* $2g$,

or simply, the $2g$-*chain relation*.

$X_{LF} =$ unknown?

$(\kappa_c, \kappa_l) = (g^2, g(3g-4))$.

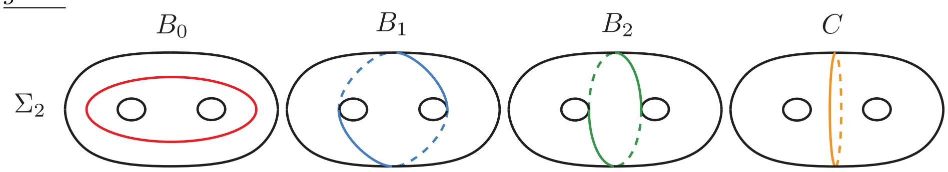


$(t_1 t_2 \cdots t_{2g})^{4g+2} = t_\delta$.

$(\kappa_c, \kappa_l) = (g^2, (g-1)(3g-1))$.

3.4 The Matsumoto-Cadavid-Korkmaz Lefschetz fibration

$\underline{g = 2}$: Matsumoto's Lefschetz fibration:

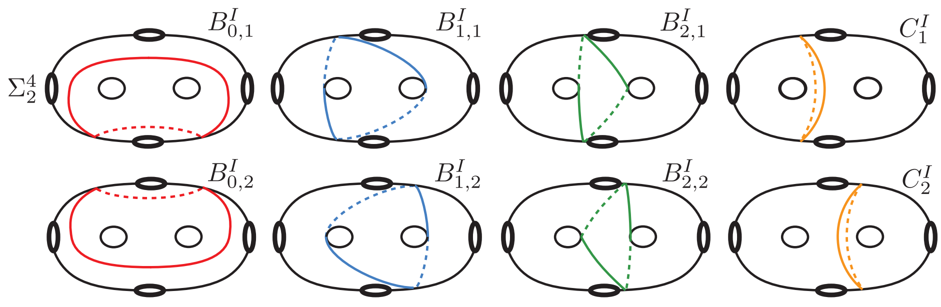


$(t_{B_0} t_{B_1} t_{B_2} t_C)^2 = 1.$

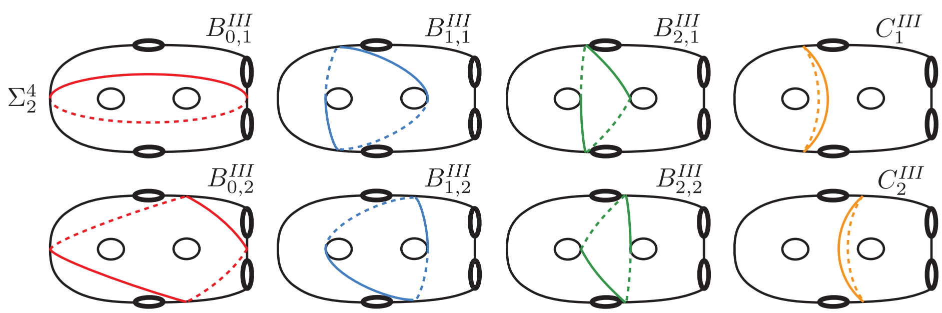
$X_{LF} = T^2 \times S^2 \# 4\overline{\mathbb{C}P^2}.$

$(\kappa_c, \kappa_l) = (1, 3).$

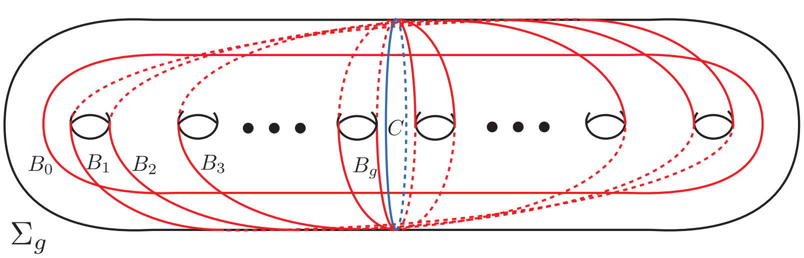
Type I lift : $t_{B^I_{0,1}} t_{B^I_{1,1}} t_{B^I_{2,1}} t_{C^I_1} \cdot t_{B^I_{0,2}} t_{B^I_{1,2}} t_{B^I_{2,2}} t_{C^I_2} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}.$ $(\kappa_c, \kappa_l) = (1, 7).$



Type III lift : $t_{B^{III}_{0,1}} t_{B^{III}_{1,1}} t_{B^{III}_{2,1}} t_{C^{III}_1} \cdot t_{B^{III}_{0,2}} t_{B^{III}_{1,2}} t_{B^{III}_{2,2}} t_{C^{III}_2} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}.$ $(\kappa_c, \kappa_l) = (1, 7).$



$\underline{g \text{ : even case}}$

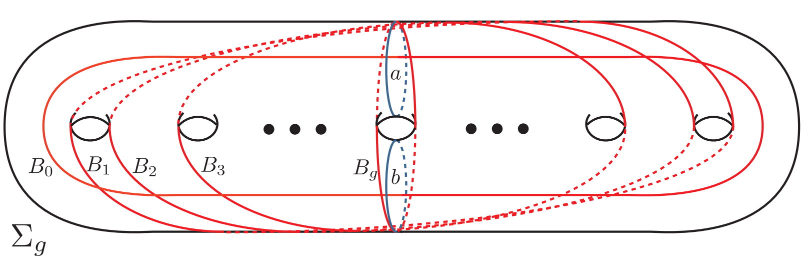


$(t_{B_0} t_{B_1} \cdots t_{B_g} t_C)^2 = 1.$

$X_{LF} = \Sigma_{g/2} \times S^2 \# 4\overline{\mathbb{C}P^2}.$

$(\kappa_c, \kappa_l) = (g/2, (7g - 8)/2).$

$\underline{g \text{ : odd case}}$



$(t_{B_0} t_{B_1} \cdots t_{B_g} t_a^2 t_b^2)^2 = 1.$

$X_{LF} = \Sigma_{(g-1)/2} \times S^2 \# 8\overline{\mathbb{C}P^2}.$

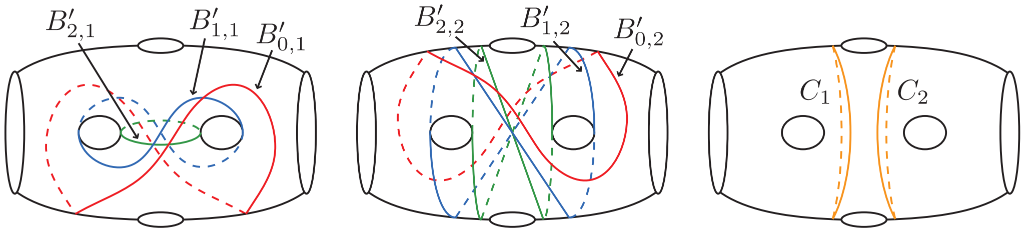
$(\kappa_c, \kappa_l) = ((g + 1)/2, (7g - 9)/2).$

3.5 Lefshetz pencils on the four-torus T^4

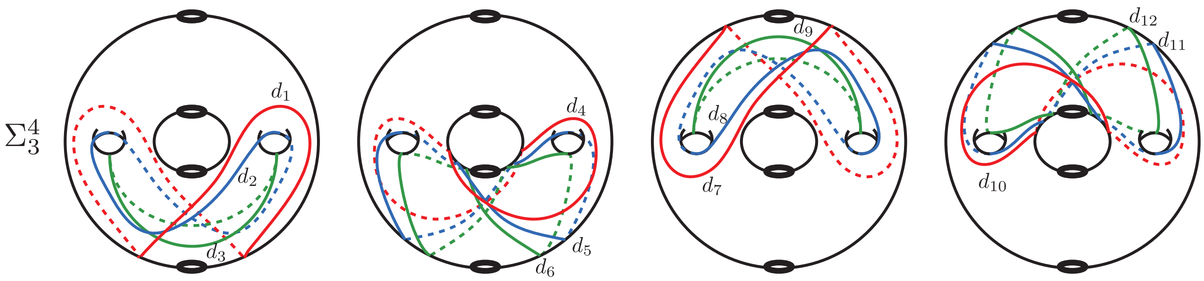
Modified Type I lift of Matsumoto's relation :

$$t_{B_{0,1}^{\prime\prime}} t_{B_{1,1}'} t_{B_{2,1}'} t_{B_{0,2}'} t_{B_{1,2}'} t_{B_{2,2}'} \cdot t_{C_1} t_{C_2} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}.$$

$(\kappa_c, \kappa_l) = (1, 7)$.



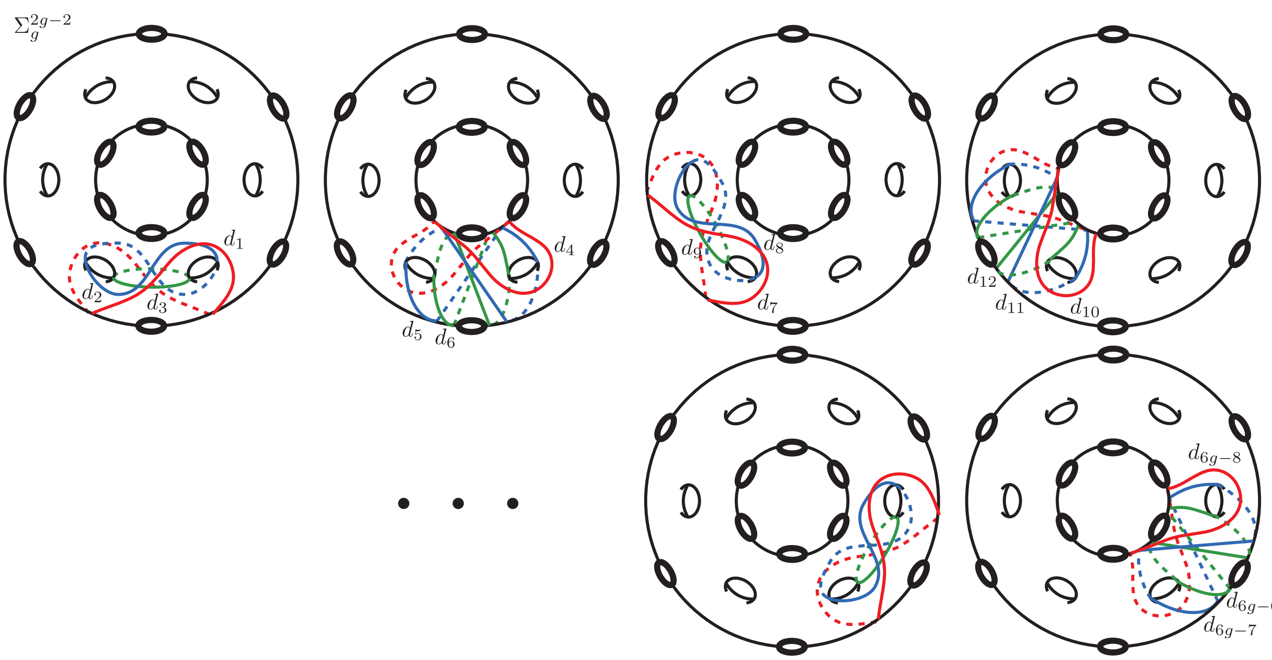
3.5.1 Smith's genus-3 holomorphic Lefshetz pencil



$$t_{d_1} t_{d_2} t_{d_3} \cdot t_{d_4} t_{d_5} t_{d_6} \cdot t_{d_7} t_{d_8} t_{d_9} \cdot t_{d_{10}} t_{d_{11}} t_{d_{12}} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}.$$

$X_{LP} = T^4$. $(\kappa_c, \kappa_l) = (2, 14)$.

3.5.2 A generalization of Smith's LP to higher genera ($g \geq 3$)



$$t_{d_1} t_{d_2} t_{d_3} \cdot t_{d_4} t_{d_5} t_{d_6} \cdots \cdot t_{d_{6g-8}} t_{d_{6g-7}} t_{d_{6g-6}} = t_{\delta_1} t_{\delta_2} \cdots t_{\delta_{2g-2}}.$$

The total space of the pencil is *homeomorphic* to T^4. $(\kappa_c, \kappa_l) = (2(g-1), 14(g-1))$.