§10 The technique of Fintushel-Stern

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Thm 10.1

M: closed, connected, oriented, positive definite topological 4-mfd

$H_1(M:\mathbb{Z}) = 0$

$$
(*)\left(\begin{array}{l}
\exists \alpha \in H^2(M:\mathbb{Z}) \text{ s.t.} \\
\quad \alpha^2 = \omega(\alpha,\alpha) = 2 \\
\not\exists \beta, \gamma \in H^2(M:\mathbb{Z}) \quad \beta^2 = \gamma^2 = 1, \ \alpha = \beta + \gamma
\end{array}\right.
$$

$\Longrightarrow$ M is not smoothable

Donaldson's thm $\Rightarrow$ Thm 10.1

SO(3)-instanton moduli space を用いる.

collar thm が必要ない.

モジュライ空間のコンパクト性がよくなる.

(境界付4次元多様体の交差形式 $\Leftarrow$ SO(3) instanton を使う)

<u>Example</u> nE_8 sastisfies ⊛

<u>Example</u> $\underbrace{E_8 \otimes \cdots \otimes E_8}_{s}$ は⊛を満たさない。 $s \geq 2$

$$\alpha^2 \geq 2^s$$

$\square$

<u>SO(3) - instanton</u>

M : closed, oriented smooth 4-mfd

g : M の Riemann metric on M

<u>Fact 10.2</u>

(1) $\{\xi \xrightarrow{SO(3)} M\} / \text{iso}$

$$\xleftrightarrow{1:1} \left\{ (w_2, p_1) \in H^2(M : \mathbb{Z}_2) \times H^4(M : \mathbb{Z}) \;\middle|\; w_2^2 \equiv p_1 \mod 4 \right\}$$

$\xi \mapsto (w_2(\xi), p_1(\xi))$ (Dold-Whitney)

(2) $\tilde{\xi}$: $U(2)$-bundle lift of ξ $(\tilde{\xi}/S^1 = \xi)$

$(U(2)/S^1 = SO(3))$

$\Leftrightarrow$ $\exists c_1 \in H^2(M;\mathbb{Z})$ $c_1 \equiv w_2(\xi) \mod 2$

$$\left(\begin{array}{l} c_1(\tilde{\xi}) = c_1 \ (\equiv w_2(\xi) \mod 2), \\ 2c_1(\tilde{\xi})^2 - 4c_2(\tilde{\xi}) = p_1(\xi) \end{array} \right)$$

(3) $$p_1(\xi) = -\frac{1}{2\pi^2}\int_M \mathrm{Tr}(F_D^2)$$

(D : conn on ξ)

$$= -\frac{1}{2\pi^2}\left(\|F_D^-\|^2 - \|F_D^+\|^2\right)$$

(Chern-Weil)

$\square$

Lemma 10.3

(1) $H_1(M;\mathbb{Z}) = 0 \Rightarrow \exists \tilde{\xi}$: $U(2)$-lift of ξ.

(2) $M = S^4 \Rightarrow p_1(\xi) \equiv 0 \mod 4$

(3) If $\exists D$: Self-Dual instanton on ξ

$\Rightarrow p_1(\xi) \geqq 0$

$=$ holds $\Leftrightarrow$ D is flat

$\because$ (1) Bockstein exact seq $(\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2)$

$$H^2(M:\mathbb{Z}) \xrightarrow{\times 2} H^2(M:\mathbb{Z}) \to H^2(M:\mathbb{Z}_2) \to H^3(M:\mathbb{Z})$$

$\uparrow$ surjective

$H^3(M:\mathbb{Z}) \cong H_1(M:\mathbb{Z}) = 0$ (P.D)

$\Rightarrow \exists \tilde{\xi}$: $U(2)$-lift.

(2) $p_1(\xi) = c_1^2(\tilde{\xi}) - 4c_2(\tilde{\xi}) = -4c_2(\tilde{\xi}) \equiv 0\ (4)$

$c_1(\tilde{\xi}) \in H^2(S^4;\mathbb{Z}) = 0$

(3) D: SD on ξ $\Rightarrow$ $p_1(\xi) = \frac{1}{2\pi^2}\|F_D^+\|^2 \geqq 0$

$\square$

Def 10.4 $\xi \xrightarrow{SO(3)} M$

D : connection on ξ

D is reducible $\Longleftrightarrow$

$\xi \cong \lambda \oplus \varepsilon$ $\lambda : SO(2)$-bundle

ε: trivial real line bundle

$$D = \underset{\substack{\uparrow \\ \text{connection on } \lambda}}{d_1} \oplus \underset{\substack{\uparrow \\ \text{trivial connection on } \varepsilon.}}{d}$$

$\mathcal{G} = \mathrm{Aut}(\xi)$: gauge group of ξ.

$\mathcal{G}_D = \{ g \in \mathcal{G} \mid g^* D = D \}$
(the stabilizer of D)

$\boxed{\begin{array}{l} \underline{\text{Prop 10.5}} \quad D: \text{connection on } \xi, \text{ not flat} \\ \quad \text{The following are equivalent} \\ \text{(a) } \mathcal{G}_D = SO(2) \quad (\cong U(1)) \\ \text{(a) } \ker\left(D: \Omega^0(\mathrm{ad}\,\xi) \to \Omega^1(\mathrm{ad}\,\xi)\right) \neq 0 \\ \text{(c) } D \text{ is reducible} \\ \text{(d) } \mathcal{G}_D \neq 1 \end{array}}$

$$\mathcal{M} := \{ D \text{ connection on } \xi \mid F_D^- = 0 \} / \mathcal{G}$$

moduli space of SD instanton on ξ.

$$\dim \mathcal{M} = 2p_1(\xi) - 3(1 - b_1(M) + b^-(M))$$

↑ index thm

$\boxed{\underline{\text{Prop 10.6}} \quad p_1(\xi) \leqq 3 \Rightarrow \mathcal{M} \text{ is compact}}$

SU(2)と違うところ.

∵ $[D_j] \in \mathcal{M}$ $\quad (j = 1, 2, \cdots)$

It is sufficient to show that

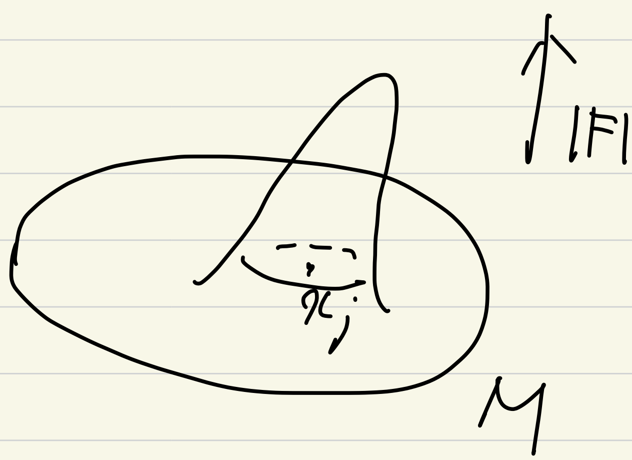
$$\max_{x \in M} |F_{D_j}(x)| < C \quad \text{(independent of } j\text{)}$$

4-mfd

If not, ∃ subseq s.t. $|F_{D_j}(x_j)| \to \infty$

$$e^*_{\lambda_j, x_j}(D_j) \longrightarrow I$$

x_j の近くでのスケール変換



I : not flat SD instanton on S^4

$$3 \geq p_1(\xi) = \frac{1}{2\pi^2} \int_M |F^+_{D_j}|^2 d\mu$$

$$\geq \frac{1}{2\pi^2} \int_{x_j \text{の近傍}} |F^+_{D_j}|^2 d\mu$$

$$\xrightarrow[j\to\infty]{} \frac{1}{2\pi^2}\int_{S^4} |F_\xi^+|^2 \, d\mu$$

$$= p_1(\xi_{S^4}) \geq 4$$

contradiction

$\Rightarrow \mathcal{M}$ compact. //

Prop 10.7 M: positive definite 4-mfd

$H_1(M;\mathbb{Z}) = 0$

$\xi \xrightarrow{SO(3)} M$, $p_1(\xi) = 2$,

gauge eq

$D_\alpha \sim D_{-\alpha}$

$\Rightarrow$ (1) $\dim \mathcal{M} = 1$

(2) $\mathcal{M}$: compact.

(3) $m := \frac{1}{2} \# \left\{ \alpha \in H^2(M;\mathbb{Z}) \,\middle|\, \begin{array}{l} \alpha \equiv w_2(\xi) \bmod 2 \\ \alpha^2 = 2 \end{array} \right\}$

$\mathcal{M}$ has m reducible SD instantons

$[D_1], \cdots, [D_m]$

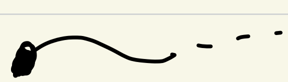
and $\mathcal{M} \setminus \{[D_1], \cdots, [D_m]\}$

is smooth for generic g

(4) $[D_j] \in \mathcal{M}$ reducible

nbd of $[D_j]$ in $\mathcal{M} \cong \mathbb{C}/S^1$

$(\cong [0,1))$

D_j 

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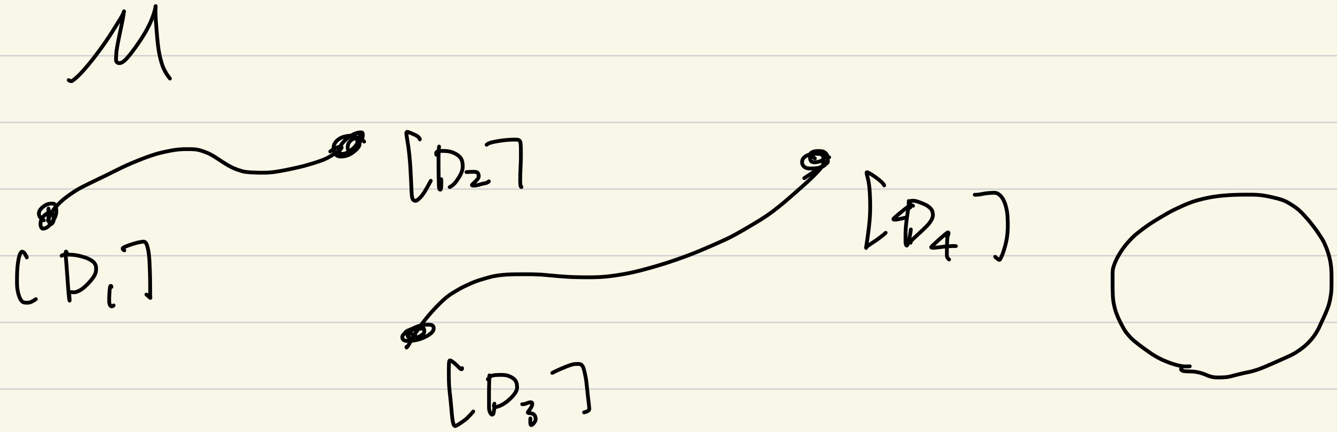

Thm 10.1 の証明

M: positive def 4-mfd $\quad H_1(M:\mathbb{Z})=0$

(※) $\exists \alpha \in H^2(M:\mathbb{Z})$ s.t.

$\alpha^2 = 2$

$\nexists \beta, \gamma \in H^2(M:\mathbb{Z}), \quad \beta^2 = \gamma^2 = 1, \quad \alpha = \beta + \gamma.$

$\Rightarrow$ M is not smoothable

$\because$ Assume M is smoothable

$\xi \xrightarrow{SO(3)} M$

$p_1(\xi) = 2, \quad w_2(\xi) \equiv \alpha \mod 2$

$\mathcal{M}$: moduli space of SD instantons on ξ.

$\mathcal{M}$ is cpt 1-mfd.

$$\# \partial \mathcal{M} = \frac{1}{2} \# \underbrace{\{ e \in H^2(M;\mathbb{Z}) \mid e^2 = 2,\ e \equiv w_2(\xi) \bmod 2 \}}_{\substack{\alpha,\ -\alpha}}$$

$=: m$

We will prove that $m = 1$. (contradiction)

$e \in H^2(M;\mathbb{Z})$, $e^2 = 2$, $e \equiv w_2(\xi) \bmod 2$

$e = \alpha + 2\beta \qquad \exists \beta \in H^2(M;\mathbb{Z})$

$$2 = e^2 = \underset{\substack{\shortparallel \\ 2}}{\alpha^2} + 4\alpha\beta + 4\beta^2$$

$\therefore\ \alpha\beta + \beta^2 = 0 \qquad \therefore\ \beta^2 = -\alpha\beta.$

$$0 \leqq (\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 = 2 - \beta^2$$

$\uparrow$

M: positive def

$$0 \leqq \beta^2 \leqq 2$$

- $\beta^2 = 0 \Rightarrow \beta = 0 \Rightarrow e = \alpha.$

$\uparrow$

M: positive def

O.K.

- $\beta^2 = 2 \Rightarrow (\alpha+\beta)^2 = 0 \Rightarrow \alpha + \beta = 0$

$\Rightarrow \beta = -\alpha \Rightarrow e = -\alpha$ O.K

- $\beta^2 = 1 \Rightarrow (\alpha+\beta)^2 = 1$

$\alpha = (\alpha+\beta) + (-\beta)$

$(\alpha+\beta)^2 = 1, \quad \beta^2 = 1$

これは (*) に反する.

$\beta^2 = 1$ は起こらない.

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