Dehn surgery on knots : survey

Kimihiko Motegi (Nihon University)

Differential Topology 22

19 March 2022

目標

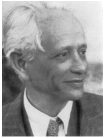
デーン手術の地図を描く！

1904 : Poincaré described a remarkable 3–manifold which has a *trivial homology*, but whose *fundamental group is nontrivial*. This 3–manifold is now called the Poincaré homology 3–sphere, and it clarifies the difference between "homology" and "homotopy".

Poincaré conjecture

A closed 3–manifold with trivial fundamental group is homeomorphic to S^3

1907 : Dehn tried to construct a homology 3–sphere which is not S^3 by identifying boundaries of two knot exteriors suitably.

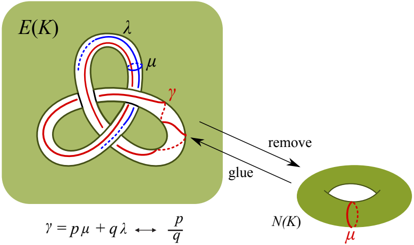


However, he was not able to show the resultant 3–manifold is not S^3.

1910 : Dehn introduced a new idea what we call "*Dehn surgery*".

Dehn surgery

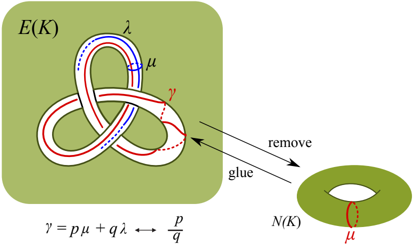
K: a knot in the 3-sphere S^3



$K(\frac{p}{q})$: a 3-manifold obtained from S^3 by $\frac{p}{q}$-Dehn surgery on K

Dehn surgery

K: a knot in the 3-sphere S^3

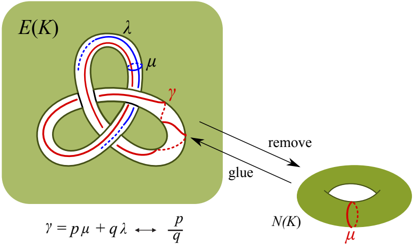


$K(\frac{p}{q})$: a 3-manifold obtained from S^3 by $\frac{p}{q}$-Dehn surgery on K

The result of Dehn surgery is uniquely determined by the image of the meridian of the glued solid torus (independent of the image of the longitude)!

Dehn surgery

K: a knot in the 3-sphere S^3



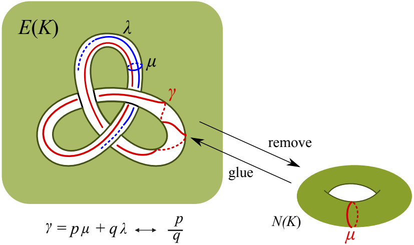
$K(\frac{p}{q})$: a 3-manifold obtained from S^3 by $\frac{p}{q}$-Dehn surgery on K

The result of Dehn surgery is uniquely determined by the image of the meridian of the glued solid torus (independent of the image of the longitude)!

We call $\frac{p}{q}$ a surgery slope.
The gluing solid torus is often called $\frac{p}{q}$-Dehn filling.

Dehn surgery

K: a knot in the 3-sphere S^3



$\gamma = p\mu + q\lambda \longleftrightarrow \frac{p}{q}$

$K(\frac{p}{q})$: a 3-manifold obtained from S^3 by $\frac{p}{q}$-Dehn surgery on K

The result of Dehn surgery is uniquely determined by the image of the meridian of the glued solid torus (independent of the image of the longitude)!

We call $\frac{p}{q}$ a surgery slope.
The gluing solid torus is often called $\frac{p}{q}$-Dehn filling.

In the following, we consider nontrivial surgery $\frac{p}{q} \neq \frac{1}{0}$

$$G(K) = \pi_1(E(K))$$

$$\pi_1(K(p/q)) = G(K) \big/ \langle\!\langle \mu^p \lambda^q \rangle\!\rangle$$

$$\Big\downarrow$$

$$G(K) \big/ \langle\!\langle \mu \rangle\!\rangle \cong \pi_1(K(1/0)) = \pi_1(S^3) = \{1\}$$

$$\Big\downarrow$$

$$G(K) = \langle\!\langle \mu \rangle\!\rangle \overset{\varphi}{\twoheadrightarrow} \pi_1(K(p/q)) = \langle\!\langle \varphi(\mu) \rangle\!\rangle$$

weight 1

$$\Big\downarrow$$

$$H_1(E(K)) = \mathbf{Z} = \langle \mu \rangle \qquad\qquad H_1(K(p/q)) = \mathbf{Z} \big/ p\mathbf{Z}$$

Examples

Lens space

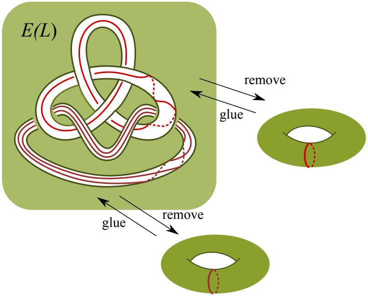
Let K be the trivial knot O. Then $O(p/q)$ is a lens space $L(p, q)$.
In particular, $O(1/n)$ is the 3–sphere for any integer n.

Homology 3–spheres

For any knot K, $K(1/n)$ is a homology 3–sphere for any integer n.

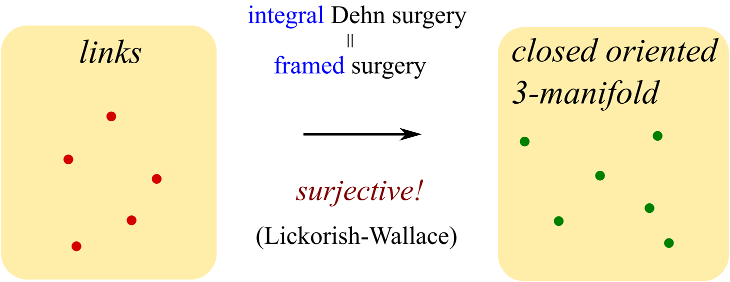
If K is a trefoil knot $T_{3,2}$, then $T_{3,2}(1)$ is the Poincaré homology 3–sphere.

One may easily generalize Dehn surgery on "knots" to Dehn surgery on "links".

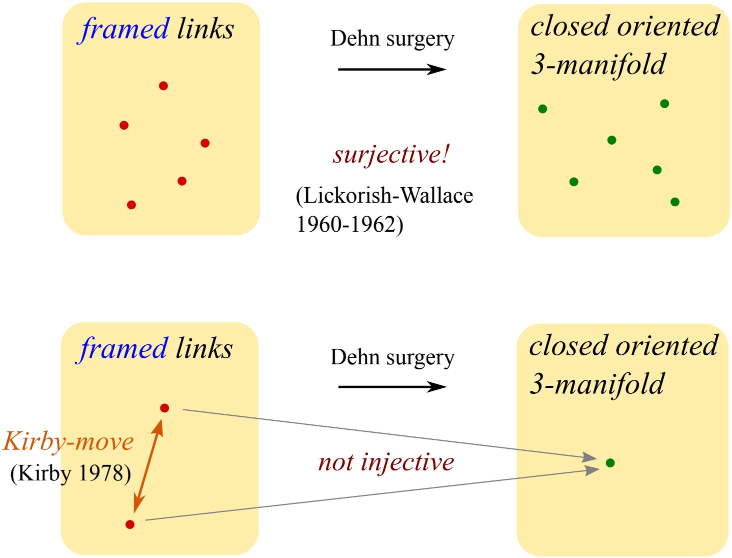


Theorem (Lickorish 1962, Wallace 1960)

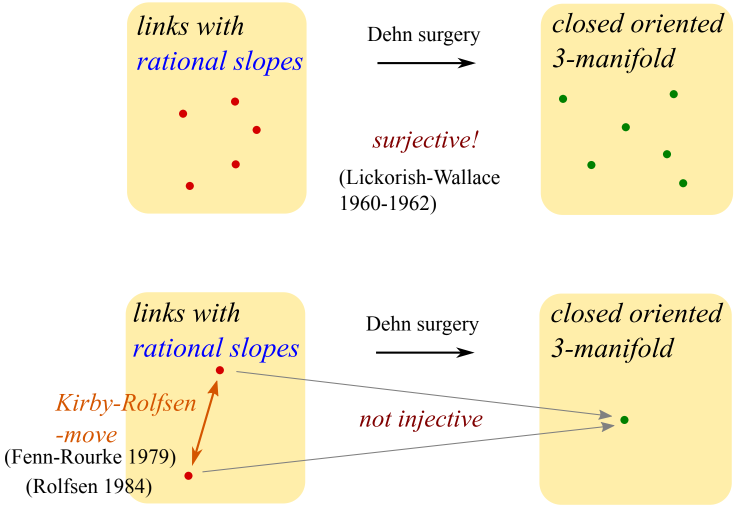
Any orientable closed 3–manifold can be obtained from S^3 by Dehn surgery on a "link".


links
integral Dehn surgery
||
framed surgery
surjective!
(Lickorish-Wallace)
closed oriented 3-manifold

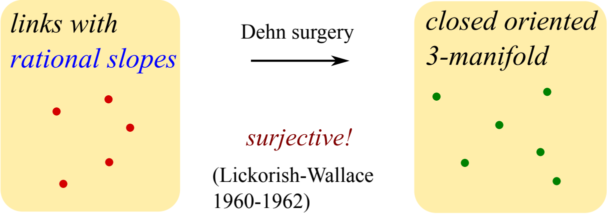
Lickorish-Wallace + Kirby

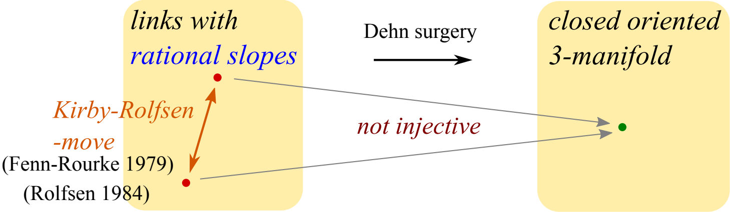


Lickorish-Wallace + Kirby-Fenn-Rourke-Rolfsen



Lickorish-Wallace + Kirby-Fenn-Rourke-Rolfsen

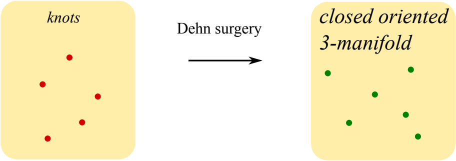




We will focus on Dehn surgery on knots in S^3*.*

Problems in study of Dehn surgery

Given a knot $K \subset S^3$ and $r \in \mathbb{Q}$, we obtain a closed oriented 3–manifold $K(r)$.



More explicitly we may regard Dehn surgery as a map:

$$\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$

$$\mathcal{D}(K, r) = K(r)$$

Exceptional surgery problem

$$\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$
$$\mathcal{D}(K, r) = K(r)$$

Consider a class

$$\mathcal{C}_* \subset \{\text{oriented closed 3–manifolds}\}$$

with "property $*$".

Describe K and/or r such that $\mathcal{D}(K, r) \in \mathcal{C}_*$.

In general, we take a "property $*$" as an exceptional one which describe a kind of "degeneration" happens, so we call a problem of this type an exceptional surgery problem.

Surjectivity (realization) problem

$$\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$
$$\mathcal{D}(K, r) = K(r)$$

Recall first that any 3–manifold obtained by a Dehn surgery on a knot in S^3 has the fundamental group with weight one.

So we focus on

$$\mathcal{W}_1 = \{\text{closed, orientable 3–manifolds with weight one fundamental group}\}$$

and ask

Is $\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \mathcal{W}_1$ surjective?

Injectivity problem

$$\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$
$$\mathcal{D}(K, r) = K(r)$$

We may ask:

Is the map $\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$ injective.

As we will discuss later, the injectivity problem may be divided into two more specific problems according as we fix a knot K or a surgery slope r.

Property P conjecture

In the exceptional surgery problem, let us consider

$$\mathcal{C}_* = \{\text{simply connected, closed 3–manifolds}\}$$

We imagine that there were some
"trials to find counterexamples to the Poincare Conjecture"
via Dehn surgery on knots.

Later Bing and Martin propose:

Property P conjecture (Bing-Martin 1960)

Let K be a nontrivial knot.

$$\pi_1(K(r)) = \{1\} \Leftrightarrow r = \infty$$

Take $\mathcal{C}_* = \{S^3\}$, Gordon and Luecke prove:

Theorem (Gordon-Luecke 1989)

Let K be a nontrivial knot.

$$K(r) = S^3 \Leftrightarrow r = \infty$$

Corollary (Knot complement conjecture : Tietze 1908 – Gordon-Luecke 1989)

Let K_1 and K_2 be knots in S^3.

$$S^3 - K_1 \cong S^3 - K_2 \Leftrightarrow K_1 \cong K_2$$

Proof ($\Leftarrow$) is obvious.

($\Rightarrow$) Assume that we have an orientation preserving homeomorphism $S^3 - K_1 \to S^3 - K_2$.
By concentricity theorem we have an orientation preserving homeomorphism $E(K_1) \to E(K_2)$.

• If K_1 is itrivial, then $E(K_2) \cong E(K_1) \cong S^1 \times D^2$, and hence K_2 is also trivial.

• Assume K_1 and K_2 are nontrivial. If h sends μ_1 to μ_2, then extending h to an orientation preserving homeomorphism $\overline{h}\colon S^3 \to S^3$ such that $\overline{h}(K_1) = K_2$.

Suppose for a contradiction that $h(\mu_1) = \gamma_2 \neq \mu_2$.
Then extend h to obtain $S^3 \cong K_1(\mu_1) \to K_2(\gamma_2)$.
Since $\gamma_2 \neq \mu_2$, Gordon-Luecke Theorem says $K_2(\gamma_2) \not\cong S^3$,
a contradiction!

Two branches of Gordon-Luecke's S^3–Surgery Theorem

Reformulation of Gordon-Luecke theorem.

Theorem (reformulation 1)

Let K be a nontrivial knot.
$K(r) \cong K(\infty) \Leftrightarrow r = \infty$

$\Rightarrow$ cosmetic surgery slopes (discuss later)

Theorem (reformulation 2)

Let K be a knot and O a trivial knot.
$K(1/n) \cong O(1/n) = S^3 \ (n \neq 0) \Leftrightarrow K \cong O$

$\Rightarrow$ characterizing slopes (discuss later)

Cyclic Surgery Theorem

Theorem (Culler-Gordon-Luecke-Shalen 1987)

Let K be a knot in S^3 other than a torus knot.
Assume that both $K(p_1/q_1)$ and $K(p_2/q_2)$ have cyclic fundamental groups.
Then the distance $\Delta(p_1/q_1, p_2/q_2) = |p_1q_2 - p_2q_1| \leq 1$.
Hence, there are at most three such slopes.

Since $K(1/0) = S^3$ has the trivial, and hence cyclic fundamental group, as a consequence of this theorem we have the following:

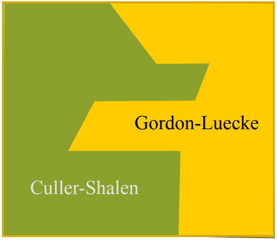
• If K is not a torus knot, then $K(p/q)$ has a cyclic fundamental group only when $|q| = 1$, i.e. cyclic surgery is integral.

For torus knots the Property P conjecture is known to be true. Thus noting $H_1(K(p/q)) \cong \mathbb{Z}_{|p|}$, the Cyclic Surgery Theorem implies that

• $\pi_1(K(r)) = 1 \Rightarrow r = \pm 1$.

The proof of the Cyclic Surgery Theorem requires

"Culler-Shalen theory" + "Analysis of intersection graphs"
(Culler-Shalen) (Gordon-Luecke)



Culler-Shalen theory (Varieties of group representations and splittings of 3–manifolds 1983)
$\Rightarrow$ Kabaya's talk

Analysis of intersection graphs
$\Rightarrow$ Litherland, Gordon-Litherland, Gordon-Luecke

Property P Conjecture and essential lamination

Essential lamination is a generalization of

"incompressible surface" and "taut foliation".

$^{\exists}3$–manifolds that contain essential laminations but do *not* contain incompressible surfaces. (Gabai)

Gabai-Oertel 1989

If a compact orientable 3–manifold contains an essential lamination, then its universal cover is homeomorphic to $\mathbb{R}^3$.

Application to Dehn surgery.

- If $K(r)$ has an essential lamination, then $|\pi_1(K(r))| = \infty$ (strong Property P).

This leads to study of persistent lamination of knot exteriors (1990's). (Brittenham, Delman, Roberts, Wu....)

$\Rightarrow$ Ito's talk

Property P "Theorem"

Theorem (Kronheimer-Mrowka 2004)

Let K be a nontrivial knot.

$$\pi_1(K(r)) = \{1\} \Leftrightarrow r = \infty.$$

Theorem (reformulation)

Let K be a nontrivial knot and $r \in \mathbb{Q} \cup \{\infty\}$ a slope.

$$\langle\langle r \rangle\rangle = G(K) = \langle\langle \infty \rangle\rangle \ \Leftrightarrow r = \infty$$

Theorem (Ito-M-Teragaito 2021)

Let K be a nontrivial knot and $r, r' \in \mathbb{Q} \cup \{\infty\}$ slopes.

$$\langle\langle r \rangle\rangle = \langle\langle r' \rangle\rangle \ \Leftrightarrow r = r'$$

Classification of 3*–manifolds with* $\partial M = \emptyset$ *or tori*

$|\pi_1(M)| < \infty$ $|\pi_1(M)| = \infty$

Some terminologies

A 3–manifold M is

reducible if it contains a 2–sphere which does not bound a 3–ball in M; otherwise it is irreducible.

$|\pi_1(M)| < \infty$

$|\pi_1(M)| = \infty$

irreducible

reducible

$|\pi_1(M)| < \infty$

$|\pi_1(M)| = \infty$

irreducible

reducible

$$M = \underbrace{M_1 \# \quad \# M_n}_{\text{prime decomposition}}$$

M_i : irreducible

or

$S^2 \times S^1$

Some terminologies

A 3–manifold M is

reducible if it contains a 2–sphere which does not bound a 3–ball in M; otherwise it is irreducible.

toroidal if it contains a torus T such that $i_* \colon \pi_1(T) \to \pi_1(M)$ is injective. otherwise it is atoroidal.

$|\pi_1(M)| < \infty$

$|\pi_1(M)| = \infty$

irreducible

reducible

atoroidal

toroidal

$$M = \underbrace{M_1 \# \quad \# M_n}_{\text{prime decomposition}}$$

M_i : irreducible

or

$S^2 \times S^1$

Some terminologies

A 3–manifold M is

reducible if it contains a 2–sphere which does not bound a 3–ball in M; otherwise it is irreducible.

toroidal if it contains a torus T such that $i_* \colon \pi_1(T) \to \pi_1(M)$ is injective. otherwise it is atoroidal.

Seifert fibered if it consists of pairwise disjoint circles (fibers) in which each fiber has a fibered solid torus neighborhood.

$|\pi_1(M)| < \infty$

- Seifert
- ?

$|\pi_1(M)| = \infty$

- irreducible
  - atoroidal
    - Seifert
    - ?
  - toroidal
    - $M = \cup N_i$
    - N_i : atoroidal or Seifert
- reducible
  - $M = \underbrace{M_1 \# \quad \# M_n}_{\text{prime decomposition}}$
  - M_i : irreducible or $S^2 \times S^1$

Hyperbolization theorem

A 3–manifold M is

hyperbolic if it admits a complete hyperbolic metric of finite volume.

Theorem (Thurston-Perelman)

Let M be an irreducible 3–manifold with $\partial M = \emptyset$ or tori.
If M is atoroidal, then it is either Seifert fibered or hyperbolic.

Classification of 3–manifolds

$|\pi_1(M)| < \infty$

$|\pi_1(M)| = \infty$

Seifert

irreducible

reducible

atoroidal

toroidal

$M = M_1 \# \quad \# M_n$

prime decomposition

Seifert

hyperbolic

$M = \cup N_i$

N_i : hyperbolic

or

Seifert

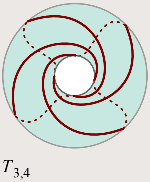
M_i : irreducible

or

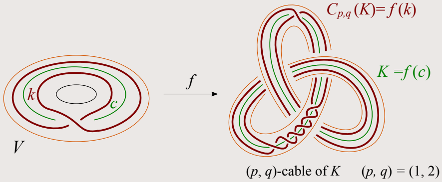
$S^2 \times S^1$

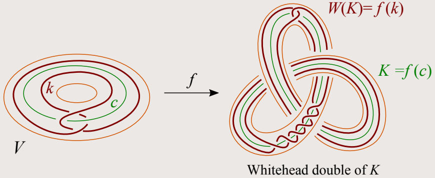
Classification of knots

torus knots



satellite knots





and **others**

$E(K)$

irreducible

~~reducible~~

Alexander - Schönflies

atoroidal

toroidal

Seifert

hyperbolic

satellite knot

torus knot

hyperbolic knot

Thurston's hyperbolic Dehn surgery

Let K be a hyperbolic knot in S^3,

i.e. $S^3 - K$ admits a complete hyperbolic metric of finite volume.

$\mathbb{Q} \cup \{\infty\} = \mathcal{H}_K \cup \mathcal{E}_K$

Thurston's hyperbolic Dehn surgery

Let K be a hyperbolic knot in S^3,

i.e. $S^3 - K$ admits a complete hyperbolic metric of finite volume.

$\mathbb{Q} \cup \{\infty\} = \mathcal{H}_K \cup \mathcal{E}_K$

$\mathcal{H}_K = \{r \mid K(r)$ is hyperbolic$\}$, the set of hyperbolic surgery slopes
$\mathcal{E}_K = \{r \mid K(r)$ is not hyperbolic$\}$, the set of "exceptional "surgery slopes

Thurston's hyperbolic Dehn surgery

Let K be a hyperbolic knot in S^3,

i.e. $S^3 - K$ admits a complete hyperbolic metric of finite volume.

$\mathbb{Q} \cup \{\infty\} = \mathcal{H}_K \cup \mathcal{E}_K$

$\mathcal{H}_K = \{r \mid K(r)$ is hyperbolic$\}$, the set of hyperbolic surgery slopes
$\mathcal{E}_K = \{r \mid K(r)$ is not hyperbolic$\}$, the set of "exceptional "surgery slopes

Theorem (Thurston's hyperbolic Dehn surgery)

For any hyperbolic knot K, $|\mathcal{E}_K| < \infty$.

K : figure-eight knot

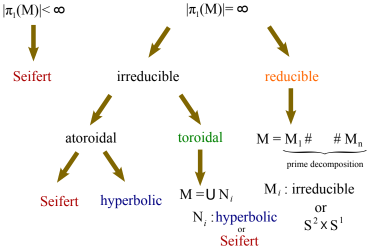
$\mathcal{E}_K = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, \infty\}$ [Thurston]

Two directions of studies of "exceptional" surgeries on hyperbolic knots

♠ Refine the hyperbolic Dehn surgery theory.

⇒ Ichihara's talk

♠ Take a closer look at each exceptional surgery.

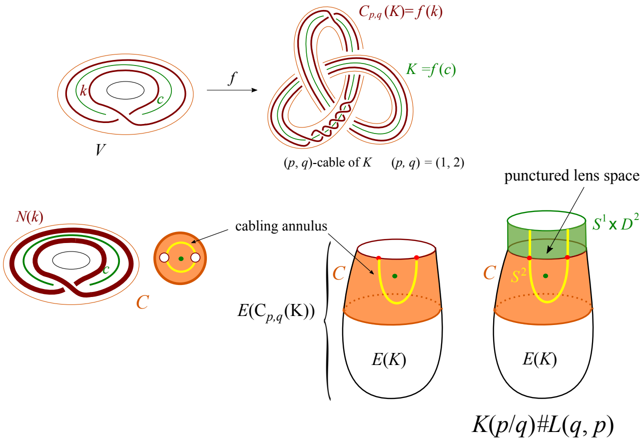


- Reducible surgery
- Seifert surgery
- Toroidal surgery,

and put some restrictions.

Cabling Conjecture (González-Acuña-Short 1986)

If $K(r)$ is reducible, then K is a (p,q)–cable of a knot and $r = pq$



Conjecture (Gordon et al.)

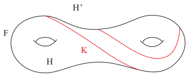
If $K(r)$ is a Seifert fiber space,
then $r \in \mathbb{Z}$, or K is a torus knot or a cable of a torus knot.

Conjecture (Gordon-Wu 2008)

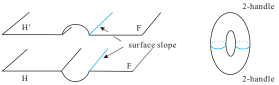
A hyperbolic knot K has at most three toroidal surgeries.

$\{$Lens space surgeries$\} \subset \{$Seifert fibered surgery$\}$

Let K be a knot contained in a genus 2 Heegaard surface F which splits S^3 into two genus 2 handlebodies H and H', i.e. $S^3 = H \cup_F H'$.



Performing Dehn surgery on K along the surface slope m,

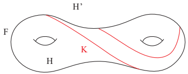


we obtain a 3-manifold

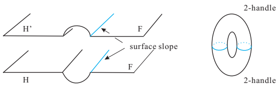
$$K(m) = H[K] \cup H'[K]$$

$\{$Lens space surgeries$\} \subset \{$Seifert fibered surgery$\}$

Let K be a knot contained in a genus 2 Heegaard surface F which splits S^3 into two genus 2 handlebodies H and H', i.e. $S^3 = H \cup_F H'$.



Performing Dehn surgery on K along the surface slope m,



we obtain a 3-manifold

$$K(m) = H[K] \cup H'[K] = (S^1 \times D^2) \cup (S^1 \times D^2),$$

which is a lens spaces.
This construction is called Berge's primitive/primitive construction.

Berge introduced "Berge's lens space surgeries"
$= 12$ classes of primitive/primitive knots (positions)

Berge's Conjecture

Every lens space surgery is a Berge's lens space surgery.

(i) Every lens space surgery arises from primitive/primitive construction,

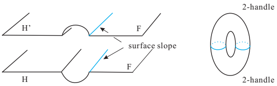
(ii) all primitive/primitive knots belong to Berge's list. (Greene 2010)

$\Rightarrow$ Teragaito's talk & Tange's talk & Yamada's talk

Then by performing Dehn surgery on K along the surface slope m, we obtain a 3-manifold

$$K(m) = H[K] \cup H'[K] = (S^1 \times D^2) \cup (S^1 \times D^2),$$

which is a lens spaces.



This construction is called Berge's primitive/primitive construction.

Berge introduced "Berge's lens space surgeries" $= 12$ classes of primitive/primitive knots (positions)

Berge's Conjecture

Every lens space surgery is a Berge's lens space surgery.

(i) Every lens space surgery arises from primitive/primitive construction,
(ii) all primitive/primitive knots belong to Berge's list. (Greene 2010)

$\Rightarrow$ Teragaito's talk & Tange's talk & Yamada's talk

What comes next?

Reducible surgeries, toroidal surgeries and Seifert surgeries are "exceptional" with respect to *Thurston' hyperbolic Dehn surgery*.

L-space surgeries, generalization of lens space surgeries, are "exceptional" with respect to in *Heegaard Floer homology theory*.

L–spaces

M : rational homology 3–sphere

M : rational homology 3–sphere

$\widehat{HF}(M)$: Heegaard Floer homology with coefficients in $\mathbb{Z}_2$

M : rational homology 3–sphere

$\widehat{HF}(M)$: Heegaard Floer homology with coefficients in $\mathbb{Z}_2$

$$\dim_{\mathbb{Z}_2}\widehat{HF}(M) \geq |H_1(M;\mathbb{Z})|$$

L–spaces

M : rational homology 3–sphere

$\widehat{HF}(M)$: Heegaard Floer homology with coefficients in $\mathbb{Z}_2$

$$\dim_{\mathbb{Z}_2} \widehat{HF}(M) \geq |H_1(M;\mathbb{Z})|$$

M is an L–space if "$=$" holds, i.e. $\dim_{\mathbb{Z}_2} \widehat{HF}(M) = |H_1(M;\mathbb{Z})|$.

M : rational homology 3–sphere

$\widehat{HF}(M)$: Heegaard Floer homology with coefficients in $\mathbb{Z}_2$

$$\dim_{\mathbb{Z}_2} \widehat{HF}(M) \geq |H_1(M;\mathbb{Z})|$$

M is an L–space if "$=$" holds, i.e. $\dim_{\mathbb{Z}_2} \widehat{HF}(M) = |H_1(M;\mathbb{Z})|$.

Example

Lens spaces ($\neq S^2 \times S^1$), more generally, 3–manifolds with S^3–geometry are L–spaces.

L–space knots

A knot K is called an L–space knot if it admits a nontrivial, L–space surgery.

L–space knots

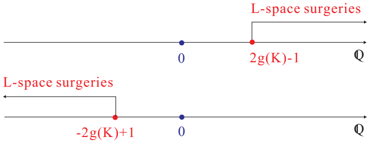
A knot K is called an L–space knot if it admits a nontrivial, L–space surgery.

- K: a nontrivial L–space knot

L–space knots

A knot K is called an L–space knot if it admits a nontrivial, L–space surgery.

• K: a nontrivial L–space knot
$K(r)$ is an L–space $\Leftrightarrow r \geq 2g(K) - 1$ or $r \leq -2g(K) + 1$.
(Ozsváth-Szabó 2011)

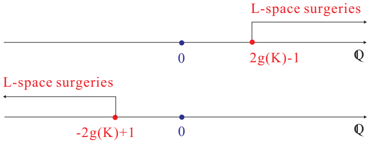


Knot Floer homology, surgery formula $\Rightarrow$ Sato's talk

L–space knots

A knot K is called an L–space knot if it admits a nontrivial, L–space surgery.

• K: a nontrivial L–space knot
$K(r)$ is an L–space $\Leftrightarrow r \geq 2g(K) - 1$ or $r \leq -2g(K) + 1$.
(Ozsváth-Szabó 2011)



Knot Floer homology, surgery formula $\Rightarrow$ Sato's talk

• If K is hyperbolic L-space knot, it produces infinitely many hyperbolic L–spaces by Dehn surgery. (Thurston's hyperbolic Dehn surgery theorem)

Constraints for L–space knots

• For any L–space knot K,
the non-zero coefficients of the Alexander polynomial of K are ± 1 and alternate in sign. (Ozsváth-Szabó 2005)

$\Rightarrow$ Teragaito's talk

Constraints for L–space knots

• For any L–space knot K,
the non-zero coefficients of the Alexander polynomial of K are ± 1 and alternate in sign. (Ozsváth-Szabó 2005)

$\Rightarrow$ Teragaito's talk

• An L-space knot is a fibered knot,
i.e. its exterior is a surface bundle over S^1. (Ni 2007)

Corollary (Ni)

A knot with lens space surgery, more generally finite surgery is fibered.

• L-space knot is prime (1-string prime). (Krcarovich 2015)

(*It should be compare with: Connected sum of L-spaces is an L-space!*)

• An L-space knot has no essential Conway sphere, i.e. 2-string prime. (Lidman-Moore-Zibrowius 2020)

[Hanselman-Rasmussen-Watson;
Bordered Floer homology for manifolds with torus boundary via immersed curves] (Ito's talk)

enables us to prove:

• Any satellite L-space knot has an
L-space knot companion and L-space pattern knot (Hom 2016)
and
braided pattern. (Baker-M 2019)

Surjectivity (realization) problem of Dehn surgery

Is $\mathcal{D} : \{\text{knots in } S^3\} \to \{\text{closed 3–manifolds with weight one } \pi_1\}$ surjective?

Boyer-Lines (1990) found (first) examples of weight one 3–manifolds which cannot be obtained by Dehn surgery on knots.
These 3–manifolds are Seifert fiber spaces.

Auckly (1993) gave an example of a hyperbolic homology 3–sphere M which is *not* obtained by Dehn surgery on a knot, but it is *not* known that $\pi_1(M)$ has weight one.

Hoffman and Walsh (2015) give infinite family of hyperbolic 3–manifolds with weight one fundamental group none of which cannot be obtained by surgery on a knot in S^3.

Hom, Karakurt and Lidman (2016) have given an infinite family of Seifert homology 3–spheres each of which has weight one, but not obtained by Dehn surgery on a knot in S^3.

Injectivity problem of Dehn surgery

$$\mathcal{D} : \{\text{knots in } S^3\} \times \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$
$$\mathcal{D}(K, r) = K(r)$$

Then fixing a knot $K \subset S^3$ or a surgery slope $r \in \mathbb{Q}$, we may consider two kinds of injectivity of $\mathcal{D}$.

- Fix a knot K in S^3. Then is the map

$$\mathcal{D}_K : \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$

injective?
In other words, if $K(r) = K(r')$, then $r = r'$?

- Fix a rational number $r \in \mathbb{Q}$. Then is the map

$$\mathcal{D}_r : \{\text{knots in } S^3\} \to \{\text{oriented closed 3–manifolds}\}$$

injective?
In other words, if $K(r) = K'(r)$, then $K = K'$?

Injectivity of the map $\mathcal{D}_K$

Is the map

$$\mathcal{D}_K : \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$

$$\mathcal{D}_K(r) = K(r)$$

injective?

Cosmetic surgery slope

For a given knot K we say that $r \in \mathbb{Q}$ is a cosmetic surgery slope for K if ${}^{\exists}r' \neq r$ s.t. $K(r') \cong K(r)$.

Since $H_1(K(p/q)) \cong \mathbb{Z}_p$ $(p \geq 0)$, if $K(p/q) \cong K(p'/q')$, then $p = p'$.

Let O be the trivial knot in S^3.
Then $O(1/n) \cong S^3$ for any integer n, and thus $\mathcal{D}_O$ is not injective.
$1/n$ is a cosmetic surgery slope for "trivial knot".
(Actually there is an orientation preserving homeomorphism of $E(O) = S^1 \times D^2$ sending $1/n$ to $1/0$.)

Theorem (Gordon-Luecke: reformulation 1)

Let K be a nontrivial knot.
∞ is not a cosmetic surgery slope for K, i.e. $K(r) \cong K(\infty) \Leftrightarrow r = \infty$.

Cosmetic surgery conjecture

Any nontrivial knot in S^3 does not have a cosmetic surgery slope, i.e.

$$\mathcal{D}_K : \mathbb{Q} \to \{\text{oriented closed 3–manifolds}\}$$

$$\mathcal{D}_K(r) = K(r)$$

is injective.

If $M = K(r_1) \cong K(r_2)$, then some nontrivial surgery on $K^*_{r_1} \subset M = K(r_1)$ converts $M = K(r_1)$ into $M = K(r_2) = K(r_1)$.
This is the original meaning of "cosmetic" surgery.

$\mathcal{D}_K : \mathbb{Q} \to \{\text{unorineted closed 3–manifolds}\}$ is not injective.

Indeed, Mathieu observed $T_{3,2}(9) \cong -T_{3,2}(9/2)$, where $-M$ denotes M with the opposite orientation.

Theorem (Ni-Wu 2015)

Assume that K is nontrivial and $K(p/q) \cong K(p/q')$ for distinct slopes $p/q, p/q' \in \mathbb{Q} \cup \{\infty\}$.
Then $q = -q'$ and $q^2 \equiv -1 \pmod{p}$.

Theorem (Hanselman 2019)

Assume that $K(r) \cong K(r')$ for distinct slopes $r, r' \in \mathbb{Q}$. Then

1. $\{r, r'\} = \{\pm 2\}$ *or* $\{\pm 1/q\}$
2. *If $\{r, r'\} = \{\pm 1/q\}$, then there are at most finitely many q; the number of q is depending on q and decidable.*

$\Rightarrow$ Ito's talk

Is the map

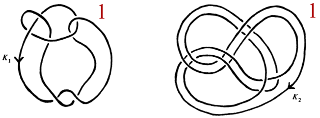
$$\mathcal{D}_r : \mathbb{K} \to \{\text{oriented closed 3–manifolds}\}$$

$$\mathcal{D}_r(K) = K(r)$$

injective?

Example

Lickorish provided the first example of distinct knots K_1 and K_2 which satisfy $K_1(1) \cong K_2(1)$, answering Kirby's question (1976).



Characterizing slope

A slope $r \in \mathbb{Q}$ is a characterizing slope for a knot K if whenever $K'(r) \cong K(r)$, then K' is isotopic to K.

Theorem (Gordon-Luecke: reformulation 2)

Let K be a knot and O a trivial knot.

$$K(1/n) \cong O(1/n)\ (n \neq 0) \Leftrightarrow K \cong O$$

This means that

$1/n\ (n \neq 0)$ is a characterizing slope for the trivial knot.

- Gordon had conjectured that

every $r \in \mathbb{Q}$ is a characterizing slope for the trivial knot.

about ten years before he proved knot complement conjecture with Luecke.

Theorem (Kronheimer-Mrowka-Ozsváth-Szabó 2007)

For the trivial knot every $r \in \mathbb{Q}$ is a characterizing slope.

This answers Gordon's conjecture affirmatively.

Apply this theorem to prove "projective space conjecture".

Theorem (projective space conjecture)

If $K(r)$ is the projective space $\mathbb{R}P^3$, then K is the trivial knot.

Proof. Assume that $K(m/n) \cong \mathbb{R}P^3$. Then $m = 2$ for homological reason.
If K is a nontrivial torus knot $T_{p,q}$, then we must have $|pqn - 2| = 1$, which is impossible because $|pq| \geq 6$.
So K is not a torus knot, and Cyclic Surgery Theorem shows that $|n| = 1$.
$K(2) \cong \mathbb{R}P^3 \cong O(2)$ implies $K \cong O$ by the above theorem. □

Theorem (Kronheimer-Mrowka-Ozsváth-Szabó 2007)

For the trivial knot every $r \in \mathbb{Q}$ is a characterizing slope.

This theorem is generalized to

Theorem (Ozsváth-Szabó 2019)

Let K be a trefoil knot or the figure-eight knot.
Then every $r \in \mathbb{Q}$ is a characterizing slope for K.

Theorem (Ni-Zhang 2014)

Let K be a torus knot $T_{p,q}$ $(p > q \geq 2)$.
If $r > 30(p^2 - 1)(q^2 - 1)/67$, then r is a characterizing slope for K.

Question (Ni-Zhang 2014) Let K be a hyperbolic knot.
Then is p/q a characterizing slope for K if $|p| + |q|$ is sufficiently large?

On the other hand,

- For given integer m,

$$^{\exists}\text{hyperbolic knots } K,\ K' \text{ s.t. } K(m) \cong K'(m).$$

(Kawauchi)

Non-characterizing slopes

Theorem (Baker-M 2018)

1. *There exists a hyperbolic knot for which every integer is a non-characterizing slope.*
2. *For a given nontrivial knot k, there exists a satellite knot with companion knot k for which every integer is a non-characterizing slope.*

This is based on Gompf-Miyazaki's "surgery dual" (1995).

Existence of characterizing slopes

Theorem (Lackenby 2019, McCoy 2019)

1. *Every hyperbolic knot has a characterizing slope.* *(Lackenby)*
2. *Any torus knot $T_{r,s}$ with $r, s > 1$ has only finitely many non-characterizing slopes which are not negative integers.* *(McCoy)*
3. *Any hyperbolic knot can have only finitely many non-characterizing slopes with $q \geq 3$* *(McCoy)*

Conjecture (Universal characterizing slope)

There exist a universal characterizing slope r, i,e, the map

$$\mathcal{D}_r : \mathbb{K} \to \{\text{oriented closed 3–manifolds}\}$$

$$\mathcal{D}_r(K) = K(r)$$

injective. In other words, r is a characterizing slope for all knots.

Let $m \in \mathbb{Z}$ be a non-characterizing slope for a knot K.
Then by definition, there is a knot K' which is not isotopic to K and $K'(m) \cong K(m)$.

It is interesting to ask if there are infinitely knots K_i $(i = 1, 2, \dots)$ such that

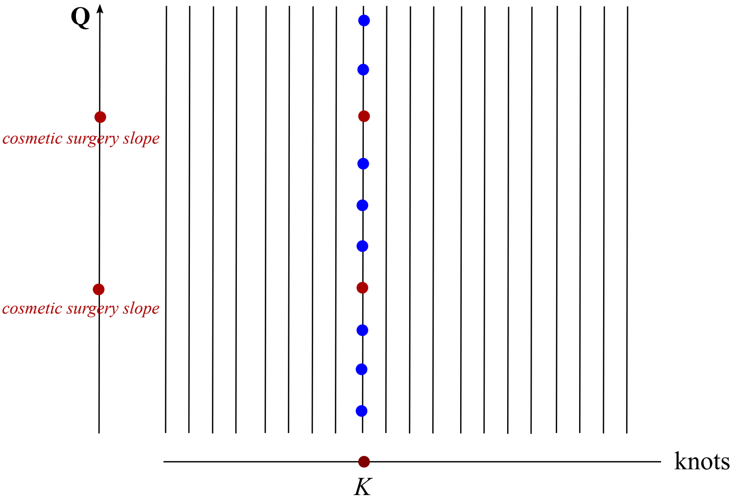
$$K_1(m) \cong K_2(m) \cong \cdots \cong K_n(m) \cong \cdots$$

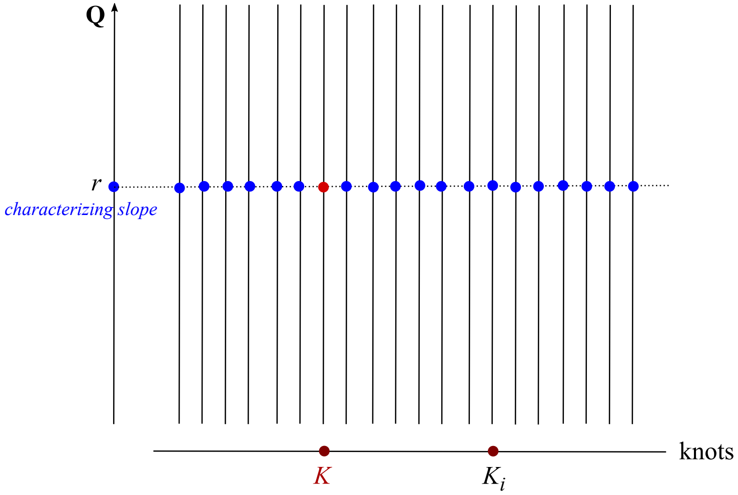
(Kirby)

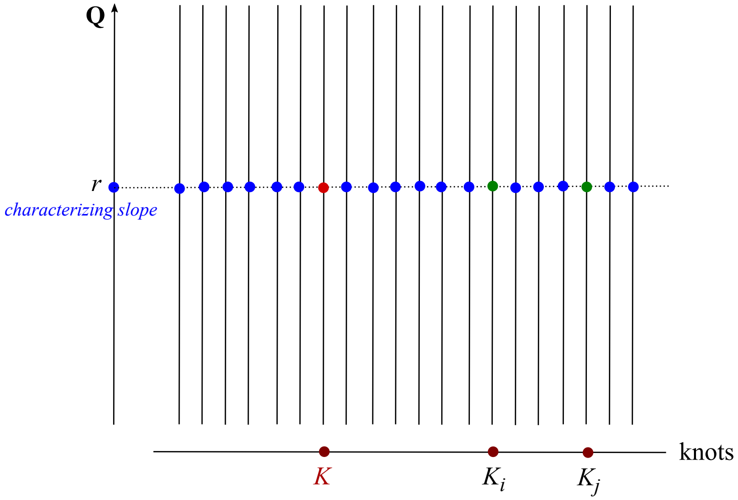
Osoinach answered this question in the positive when $m = 0$.
Some extensions were given by Teragaito, Kouno.

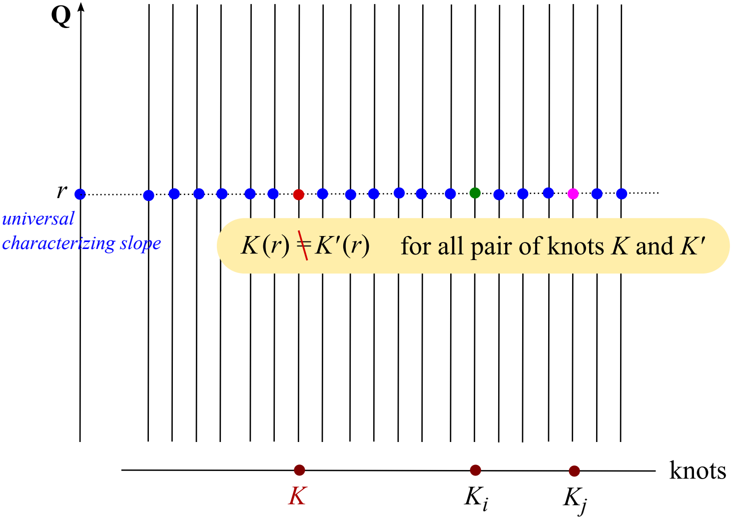
Theorem (Abe, Jong, Luecke and Osoinach 2015)

For every integer n, there exist infinitely many knots $K_1, K_2 \dots$ such that $K_1(n) \cong K_2(n) \cong \cdots$.


Q
cosmetic surgery slope
cosmetic surgery slope
knots
K


Q
r
characterizing slope
knots
K
K_i


Q
r
characterizing slope
knots
K
K_i
K_j


Q
r
universal
characterizing slope
$K(r) \neq K'(r)$ for all pair of knots K and K'
knots
K
K_i
K_j

L-space surgery and left-orderable surgery

Left orderable

A nontrivial group G is said to be left-orderable
if there exists a strict total ordering $<$ that is invariant under multiplication from the left, i.e. if $g < h$, then $fg < fh$ for any $f, g, h \in G$.

- A left-orderable group has no torsion.

Let G be an left-orderable group and g a nontrivial element. Assume that $1 < g$.
Then $1 < g = g \cdot 1 < g \cdot g = g^2 < g^3 < \cdots < g^n$ for any $n \geq 2$.

So if $K(r)$ is a lens space, $\pi_1(K(r))$ is not left-orderable.

L-space conjecture (Boyer, Gordon and Watson 2013)

M: irreducible, rational homology 3–sphere
M is an L-space
$\Leftrightarrow$
$\pi_1(M)$ is not left-orderable.

A 3–manifold with geometric structure other than hyperbolic structure satisfies the L-space conjecture.
Furthermore, many hyperbolic 3–manifolds are known to satisfy the conjecture. (Boyer, Gordon and Watson 2013)

For any knot $K \subset S^3$, since $G(K) = \pi_1(E(K))$ admits an epimorphism onto the infinite cyclic group $\mathbb{Z}$ via its abelianization, $G(K)$ is left-orderable. (Howie-Short 1985, Boyer-Rolfsen-Wiest 2005)

However, the fundamental group of $K(r)$ may not be left-orderable.

$$S_L(K) = \{r \in \mathbb{Q} \mid K(r) \text{ is an L-space}\}$$
$$S_{LO}(K) = \{r \in \mathbb{Q} \mid \pi_1(K(r)) \text{ is left-orderable}\}$$

Conjecture

Let K be a knot in S^3. Assume that K is not a cable of a nontrivial knot. Then $S_L(K) \cup S_{LO}(K) = \mathbb{Q}$ and $S_L(K) \cap S_{LO}(K) = \emptyset$.

Torus knot

For (p,q)–torus knot $T_{p,q}$ $(p > q \geq 2)$,
$\mathcal{S}_{LO}(T_{p,q}) = (-\infty,\ pq-p-q) \cap \mathbb{Q}$, and
$\mathcal{S}_{L}(T_{p,q}) = [pq-p-q, \infty) \cap \mathbb{Q}$.

• There exist infinitely many hyperbolic knots K such that
$\mathcal{S}_{LO}(K) = \mathbb{Q}$ and $\mathcal{S}_{L}(K) = \emptyset$. (M-Teragaito 2014)