

# L-SPACE KNOT の 形式的半群について

寺垣内 政一

(広島大学大学院人間社会科学研究科)

微分トポロジー '22 (2022年3月20日)

## 1 Formal semigroup

## 2 Doubly primitive knot

- Type IX:  $p = \frac{1}{11}(2k^2 + k + 1)$ ,  $k = 11j + 2$
- Type X:  $p = \frac{1}{11}(2k^2 + k + 1)$ ,  $k = 11j + 3$

# FORMAL SEMIGROUP

# L-SPACE KNOT の ALEXANDER 多項式

L-space knot の Alexander 多項式は, flat かつ alternating

[Ozsváth-Szabó]

$$\Delta(t) = 1 - t^{n_1} + t^{n_2} - \dots - t^{n_{k-1}} + t^{n_k}$$

$$(0 < n_1 < n_2 < \dots < n_{k-1} < n_k = 2g)$$

# L-SPACE KNOT の ALEXANDER 多項式

L-space knot の Alexander 多項式は, flat かつ alternating

[Ozsváth-Szabó]

$$\Delta(t) = 1 - t^{n_1} + t^{n_2} - \dots - t^{n_{k-1}} + t^{n_k}$$

$$(0 < n_1 < n_2 < \dots < n_{k-1} < n_k = 2g)$$

## 例

■ trefoil:  $1 - t + t^2$

■  $(2, n)$ -torus knot:

$$\frac{(t^{2n} - 1)(t - 1)}{(t^2 - 1)(t^n - 1)} = 1 - t + t^2 + \dots + t^{n-3} - t^{n-2} + t^{n-1}$$

■  $(-2, 3, 7)$ -pretzel knot:  $1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}$

- [Shida Wang](#),  
*Semigroups of L-space knots and nonalgebraic iterated torus knots*, *Math. Res. Lett.* **25** (2018), no. 1, 335–346.
- [Maciej Borodzik and Charles Livingston](#),  
*Semigroups, d-invariants and deformations of cuspidal singular points of plane curves*, *J. London Math. Soc.* **93** (2016), no. 2, 439–463.
- [Maciej Borodzik and Charles Livingston](#),  
*Heegaard Floer homology and rational cuspidal curves*, *Forum Math. Sigma* **2** (2014), e28, 23 pages.
- [David Kratovich](#), *A restriction on the Alexander polynomials of L-space knots*, *Pacific J. Math.* **297** (2018), no. 1, 117–129.

# 形式的半群 (FORMAL SEMIGROUP)

L-space knot に対して, Alexander 関数 (Milnor torsion or Turaev torsion) の形式的べき級数展開によって, 形式的半群  $S$  を定義する.

定義

$$\frac{\Delta(t)}{1-t} = \sum_{s \in S} t^s \in \mathbb{Z}[[t]]$$

# TREFOIL

Trefoil :  $\Delta(t) = 1 - t + t^2$



Trefoil :  $\Delta(t) = 1 - t + t^2$

$$\begin{aligned}\frac{\Delta(t)}{1-t} &= \frac{1-t}{1-t} + \frac{t^2}{1-t} \\ &= 1 + t^2 \cdot \frac{1}{1-t} \\ &= 1 + t^2(1 + t + t^2 + \dots) \in \mathbb{Z}[[t]]\end{aligned}$$

Trefoil :  $\Delta(t) = 1 - t + t^2$

$$\begin{aligned} \frac{\Delta(t)}{1-t} &= \frac{1-t}{1-t} + \frac{t^2}{1-t} \\ &= 1 + t^2 \cdot \frac{1}{1-t} \\ &= 1 + t^2(1 + t + t^2 + \dots) \in \mathbb{Z}[[t]] \end{aligned}$$

こうして,

$$\mathcal{S} = \{0, 2, 3, \dots\} = \{2a + 3b \mid a, b \geq 0\} = \langle 2, 3 \rangle$$

# TORUS KNOT

$$T(p, q): \Delta(t) = \frac{(1 - t^{pq})(1 - t)}{(1 - t^p)(1 - t^q)}$$

$$\begin{aligned} \frac{\Delta(t)}{1 - t} &= \frac{1 - t^{pq}}{(1 - t^p)(1 - t^q)} \\ &= \frac{1}{1 - t^p} (1 + t^q + t^{2q} + \dots + t^{(p-1)q}) \\ &= (1 + t^p + t^{2p} + t^{3p} + \dots) \\ &\quad + t^q (1 + t^p + t^{2p} + t^{3p} + \dots) \\ &\quad + t^{2q} (1 + t^p + t^{2p} + t^{3p} + \dots) \\ &\quad \dots \\ &\quad + t^{(p-1)q} (1 + t^p + t^{2p} + t^{3p} + \dots) \end{aligned}$$

$$S = \{ap + bq \mid a, b \geq 0\} = \langle p, q \rangle : \text{semigroup}$$

# CABLING FORMULA

$K$ : nontrivial L-space knot

$K_{p,q}$ :  $(p, q)$ -cable of  $K$  ( $p$ : longitude 方向)

S. Wang

$p \geq 2$  かつ  $q \geq p(2g(K) - 1)$  ならば

$$\mathcal{S}_{K_{p,q}} = p\mathcal{S}_K + q\mathbb{Z}_{\geq 0} = \{pa + qb \mid a \in \mathcal{S}_K, b \geq 0\}$$

\*[Hedden], [Hom]

$K_{p,q}$  が L-space knot  $\iff p \geq 2, q \geq p(2g(K) - 1)$

# 形式的半群に関する予想

- $(p, q)$ -torus knot に対して, 形式的半群は  $\langle p, q \rangle$
- cable knot に対する cabling formula

## Wang's Question

iterated torus knot ではない L-space knot で, 形式的半群が半群になるものはあるのか.

# 形式的半群に関する予想

- $(p, q)$ -torus knot に対して, 形式的半群は  $\langle p, q \rangle$
- cable knot に対する cabling formula

## Wang's Question

iterated torus knot ではない L-space knot で, 形式的半群が半群になるものはあるのか.

## 強気な予想

hyperbolic L-space knot に対して, 形式的半群は半群ではない.

## 弱気な予想

doubly primitive, hyperbolic knot に対して, 形式的半群は半群ではない.

## 定理

hyperbolic Montesinos L-space knot に対して，予想は正しい。  
すなわち，形式的半群は半群ではない。

## 定理

hyperbolic Montesinos L-space knot に対して, 予想は正しい.  
すなわち, 形式的半群は半群ではない.

- $(-2, 3, q)$ -pretzel knot ( $q \geq 7, \text{ odd}$ ) のみが, hyperbolic Montesinos L-space knot [Lidman-Moore], [Baker-Moore]
- E. Hironaka によれば

$$\begin{aligned}\Delta_q &= \frac{t^{q+4} + 1}{t + 1} - t^2(t - 1) \frac{t^{q-1} - 1}{t + 1} \\ &= ((t^{q+3} - t^{q+2}) + (t^{q+1} - t^q) + \dots + (t^2 - t) + 1) \\ &\quad - (t - 1)((t^q - t^{q-1}) + (t^{q-2} - t^{q-3}) + \dots + (t^3 - t^2))\end{aligned}$$



# PRETZEL KNOT $(-2, 3, q)$ , $q \geq 7$

$$\begin{aligned}\frac{\Delta_q}{1-t} &= -(t^{q+2} + t^q + \dots + t^3 + t) + \frac{1}{1-t} \\ &\quad + ((t^q - t^{q-1}) + (t^{q-2} - t^{q-3}) + \dots + (t^3 - t^2)) \\ &= \frac{1}{1-t} - (t^{q+2} + t) - (t^{q-1} + t^{q-3} + \dots + t^4 + t^2)\end{aligned}$$

# PRETZEL KNOT $(-2, 3, q)$ , $q \geq 7$

$$\begin{aligned}\frac{\Delta_q}{1-t} &= -(t^{q+2} + t^q + \cdots + t^3 + t) + \frac{1}{1-t} \\ &\quad + ((t^q - t^{q-1}) + (t^{q-2} - t^{q-3}) + \cdots + (t^3 - t^2)) \\ &= \frac{1}{1-t} - (t^{q+2} + t) - (t^{q-1} + t^{q-3} + \cdots + t^4 + t^2)\end{aligned}$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \in \mathbb{Z}[[t]]$$

を代入して

# PRETZEL KNOT $(-2, 3, q)$ , $q \geq 7$

$$\begin{aligned}\frac{\Delta_q}{1-t} &= -(t^{q+2} + t^q + \dots + t^3 + t) + \frac{1}{1-t} \\ &\quad + ((t^q - t^{q-1}) + (t^{q-2} - t^{q-3}) + \dots + (t^3 - t^2)) \\ &= \frac{1}{1-t} - (t^{q+2} + t) - (t^{q-1} + t^{q-3} + \dots + t^4 + t^2)\end{aligned}$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots \in \mathbb{Z}[[t]]$$

を代入して

$$\frac{\Delta_q}{1-t} = 1 + (t^3 + t^5 + t^7 + \dots + t^{q-2}) + t^q + t^{q+1} + (t^{q+3} + \dots)$$

$S = \{0, 3, 5, 7, \dots, q-2, q, q+1\} \cup \mathbb{Z}_{\geq q+3}$  となって, 半群ではない. ( $3 \in S$ ,  $6 \notin S$ )

# DUNFIELD'S LIST OF L-SPACE KNOTS

Dunfield Identified 630 L-space knots among 1267 knot complements in  $S^3$  with at most 9 ideal tetrahedra.

```
import pandas
exfil = pandas.read_csv("exceptional_fillings.csv")
qhst = pandas.read_csv("QHSolidTori.csv")
S3knot = exfil.loc[(exfil['kind'] == 'S3)]['cusped'].to_list();
knownFloerSimple = qhst.loc[(qhst['floer_simple'] == 1)]['name'].to_list();
unknownFloerSimple = qhst.loc[(qhst['floer_simple'] == 0)]['name'].to_list();
Lspaceknot = list(set(S3knot) & set(knownFloerSimple));
Lspaceknot.sort(); Lspaceknot.sort(key=len); # for ordering
maybeLspaceknot = list(set(S3knot) & set(unknownFloerSimple));
D = Lspaceknot + maybeLspaceknot
```

\* C. Anderson, K. Baker, X. Gao, et al.

*L-space knots with tunnel number > 1 by experiment,*

arXiv:1909.00790, Jan 18, 2021.

## 9 ASYMMETRIC L-SPACE KNOTS

Knot	$\mathcal{S}$	Defect
$K8_{290}$	0, 4, 7, 9, 12, 13, 15, 17, 18, 20, 21, 22, 24, ...	$4 \in \mathcal{S}, 8 \notin \mathcal{S}$
$K8_{296}$	0, 5, 9, 12, 15, 17, 20, 21, 24, 25, 27, 29, 30, 32, 33, 35, 36, 37, 39 – 42, 44, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$
$K9_{620}$	0, 6, 9, 12, 15, 17, 19, 21, 23, 24, 26, 27, 29, 30, 32 – 36, 38, ...	$9 \in \mathcal{S}, 18 \notin \mathcal{S}$
$K9_{624}$	0, 6, 9, 13, 15, 18, 21, 22, 24, 27, 28, 30, 31, 33, 35, 36, 37, 39, 40, 42 – 46, 48, ...	$6 \in \mathcal{S}, 12 \notin \mathcal{S}$

# 9 ASYMMETRIC L-SPACE KNOTS (CONTINUED)

Knot	$\mathcal{S}$	Defect
$K9_{674}$	0, 7, 12, 15, 19, 21, 24, 27, 28, 31, 33, 35, 36, 39, 40, 42, 43, 45, 47, 48, 49, 51, 52, 54 – 57, 59 – 64, 66, ...	$7 \in \mathcal{S}, 14 \notin \mathcal{S}$
$K9_{679}$	0, 5, 8, 11, 13, 15, 17, 18, 20, 21, 23 – 26, 28, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$
$K9_{680}$	0, 6, 9, 12, 15, 17, 19, 21, 23, 25, 27, 28, 30, 31, 33, 34, 36 – 40, 42, ...	$9 \in \mathcal{S}, 18 \notin \mathcal{S}$
$K9_{685}$	0, 5, 9, 11, 14, 15, 18, 19, 21, 23 – 25, 27 – 30, 32, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$
$K9_{723}$	0, 5, 8, 11, 13, 15, 17, 19, 21, 22, 24, 25, 27 – 30, 32, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$

# 13 TUNNEL NUMBER TWO L-SPACE KNOTS

Knot	$\mathcal{S}$	Defect
$K8_{186}$	0, 4, 6, 9, 11, 12, 14, ...	$4 \in \mathcal{S}, 8 \notin \mathcal{S}$
$K8_{189}$	0, 6, 10, 14, 16, ...	$6 \in \mathcal{S}, 12 \notin \mathcal{S}$
$K8_{195}$	0, 5, 8, 12, 13, 16, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$
$K8_{220}$	0, 4, 7, 10, 12, 14, 15, ...	$4 \in \mathcal{S}, 8 \notin \mathcal{S}$
$K9_{412}$	0, 8, 14, 18, 22, 26, 28, ...	$8 \in \mathcal{S}, 16 \notin \mathcal{S}$
$K9_{429}$	0, 7, 10, 14, 20, 22, 24, ...	$7 \in \mathcal{S}, 21 \notin \mathcal{S}$
$K9_{481}$	0, 7, 10, 15, 17, 20, 23, ...	$7 \in \mathcal{S}, 14 \notin \mathcal{S}$
$K9_{497}$	0, 5, 9, 11, 14, 16, 19, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$
$K9_{511}$	0, 5, 10, 14, 16, 19, 21, 24, ...	$5 \in \mathcal{S}, 15 \notin \mathcal{S}$
$K9_{513}$	0, 7, 12, 16, 19, 23, 25, ...	$7 \in \mathcal{S}, 14 \notin \mathcal{S}$
$K9_{524}$	0, 4, 7, 10, 13, 15, 17, 18, ...	$4 \in \mathcal{S}, 8 \notin \mathcal{S}$
$K9_{565}$	0, 10, 14, 20, 24, 28, 31, 34, ...	$10 \in \mathcal{S}, 30 \notin \mathcal{S}$
$K9_{656}$	0, 5, 8, 12, 14, 16, ...	$5 \in \mathcal{S}, 10 \notin \mathcal{S}$

# MORE EXAMPLES WITH AT MOST 8 IDEAL TETRAHEDRA

Knot	$\mathcal{S}$	Defect
$m016$	$0, 3, 5, 7, 8, 10, \dots$	$3 \in \mathcal{S}, 6 \notin \mathcal{S}$
$m071$	$0, 5, 7, 10, 12, 13, 15, 17 - 20, 22, \dots$	$7 \in \mathcal{S}, 14 \notin \mathcal{S}$
$m082$	$0, 3, 6, 8, 10, 11, 13, 14, 16, \dots$	$3 \in \mathcal{S}, 9 \notin \mathcal{S}$
$m103$	$0, 5, 8, 10, 13, 15, 17, 18, 20, 22, 23,$ $25 - 28, 30, \dots$	$8 \in \mathcal{S}, 16 \notin \mathcal{S}$
$m118$	$0, 4, 7, 9, 11, 13, 14, 16 - 18, 20, \dots$	$4 \in \mathcal{S}, 8 \notin \mathcal{S}$
$m144$	$0, 3, 6, 9, 11, 13, 14, 16, 17, 19, 20, 22, \dots$	$3 \in \mathcal{S}, 12 \notin \mathcal{S}$
$m194$	$0, 5, 8, 10, 13, 14, 16, 18, 19, 21 - 24,$ $26, \dots$	$5 \in \mathcal{S}, 15 \notin \mathcal{S}$
$m198$	$0, 5, 7, 10, 12, 14, 16, 17, 19, 21 - 24,$ $26, \dots$	$5 \in \mathcal{S}, 20 \notin \mathcal{S}$



# SNAPPY IN SAGE

teragaito — IPython: Dropbox/dlist — python3 - python3 /private/var/tr

```
sage: for mfld in D:
.....:     M=snappy.Manifold(mfld)
.....:     A=M.alexander_polynomial()
.....:     R.<a>=PowerSeriesRing(QQ)
.....:     S=A/(1-a)
.....:     print(mfld,S)
.....:

m016 1 + a^3 + a^5 + a^7 + a^8 + a^10 + a^11 + a^12 + a^13 + a^14 + a^15 + a^16 + a^17 + a^18 + a^19 +
m071 1 + a^5 + a^7 + a^10 + a^12 + a^13 + a^15 + a^17 + a^18 + a^19 + O(a^20)
m082 1 + a^3 + a^6 + a^8 + a^10 + a^11 + a^13 + a^14 + a^16 + a^17 + a^18 + a^19 + O(a^20)
m103 1 + a^5 + a^8 + a^10 + a^13 + a^15 + a^17 + a^18 + O(a^20)
m118 1 + a^4 + a^7 + a^9 + a^11 + a^13 + a^14 + a^16 + a^17 + a^18 + O(a^20)
m144 1 + a^3 + a^6 + a^9 + a^11 + a^13 + a^14 + a^16 + a^17 + a^19 + O(a^20)
m194 1 + a^5 + a^8 + a^10 + a^13 + a^14 + a^16 + a^18 + a^19 + O(a^20)
m198 1 + a^5 + a^7 + a^10 + a^12 + a^14 + a^16 + a^17 + a^19 + O(a^20)
m211 1 + a^4 + a^6 + a^9 + a^10 + a^12 + a^14 + a^15 + a^16 + a^18 + a^19 + O(a^20)
m223 1 + a^3 + a^5 + a^7 + a^9 + a^10 + a^12 + a^13 + a^14 + a^15 + a^16 + a^17 + a^18 + a^19 + O(a^20)
m239 1 + a^4 + a^7 + a^9 + a^11 + a^13 + a^15 + a^16 + a^18 + a^19 + O(a^20)
m240 1 + a^4 + a^7 + a^10 + a^11 + a^14 + a^15 + a^17 + a^18 + O(a^20)
m270 1 + a^5 + a^8 + a^11 + a^13 + a^16 + a^17 + a^19 + O(a^20)
m276 1 + a^5 + a^9 + a^11 + a^14 + a^16 + a^18 + O(a^20)
m281 1 + a^4 + a^8 + a^11 + a^13 + a^15 + a^17 + a^19 + O(a^20)
s042 1 + a^5 + a^10 + a^12 + a^15 + a^17 + O(a^20)
s068 1 + a^5 + a^10 + a^13 + a^15 + a^18 + O(a^20)
s086 1 + a^3 + a^6 + a^9 + a^12 + a^14 + a^16 + a^17 + a^19 + O(a^20)
s104 1 + a^7 + a^10 + a^14 + a^17 + a^19 + O(a^20)
s114 1 + a^7 + a^11 + a^14 + a^18 + O(a^20)
s294 1 + a^5 + a^9 + a^11 + a^14 + a^16 + a^18 + O(a^20)
s291 1 + a^5 + a^10 + a^13 + a^15 + a^18 + O(a^20)
16 17 + O(a^20)
```

## 結果

Dunfield のリストのうち、 $K8\_201(t09847)$  と  $K9\_449(o9\_30634)$  以外の 628 個について、形式的半群は半群ではない。

## 結果

Dunfield のリストのうち、 $K8\_201(t09847)$  と  $K9\_449(o9\_30634)$  以外の 628 個について、形式的半群は半群ではない。

\*ideal tetrahedra 9 個以下の 1-cusped hyperbolic manifold に対する Dunfield の計算では、 $o9\_30150$  と  $o9\_31440$  が L-space knot かどうか不明だった。 → Baker-Kegel-McCoy が肯定的に解決。

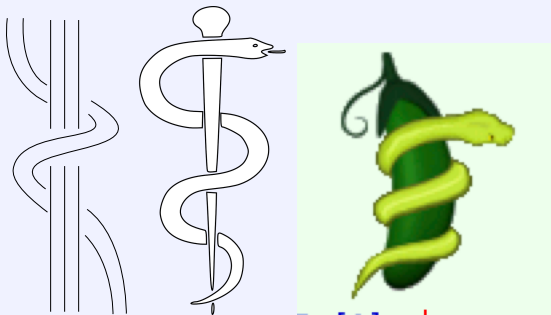
なお、これらの形式的半群は半群ではない。

# COUNTEREXAMPLE: $K8\_201(t09847)$

$$K8\_201 = (\sigma_1\sigma_2\sigma_3\sigma_4)^8(\sigma_1^{-1}\sigma_2^{-1})^7$$

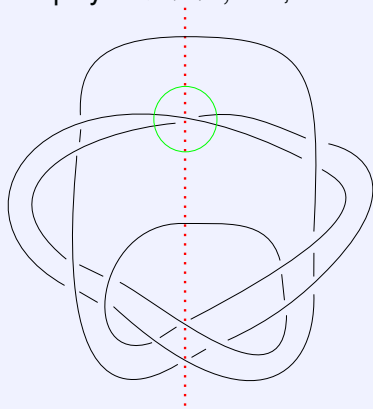
$$\Delta(t) = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{10} - t^{13} + t^{14}$$

$$\mathcal{S} = \{0, 4, 7, 8, 10, 11, 12\} \cup \mathbb{Z}_{\geq 14} = \langle 4, 7, 10 \rangle$$



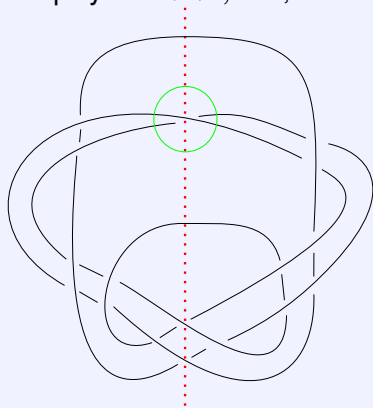
# COUNTEREXAMPLE: $K8\_201(t09847)$

SnapPy によれば, 18, 19-surgery が例外的.



# COUNTEREXAMPLE: $K8_{-201}(t09847)$

SnapPy によれば, 18, 19-surgery が例外的.



Montesinos trick から

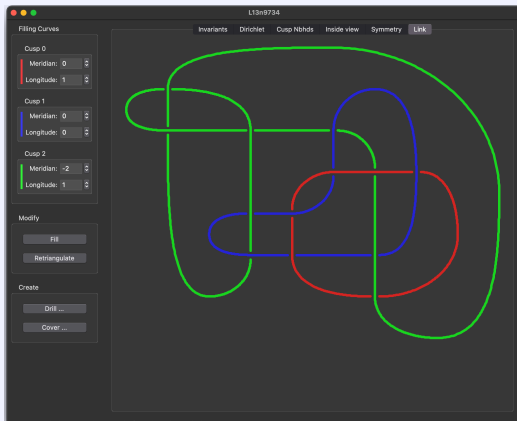
- 18-surgery:  $M(-1/2, 2/7, 3/8)$   
→ L-space
- 19-surgery: Graph manifold  
 $D^2(2, 3) \cup D^2(2, 3)$

# COUNTEREXAMPLE: $K9\_449(o9\_30634)$

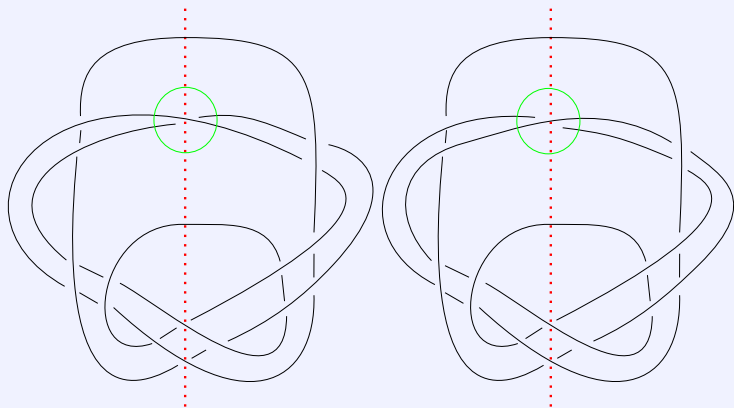
$$K9\_449 = L13n9734(0, 1)(0, 0)(-2, 1)$$

$$\Delta(t) = 1 - t + t^4 - t^5 + t^6 - t^7 + t^8 - t^{11} + t^{12}$$

$$S = \{0, 4, 6, 8, 9, 10\} \cup \mathbb{Z}_{\geq 12} = \langle 4, 6, 9 \rangle$$



## TWO COUNTEREXAMPLES: $K8\_201$ , $K9\_449$

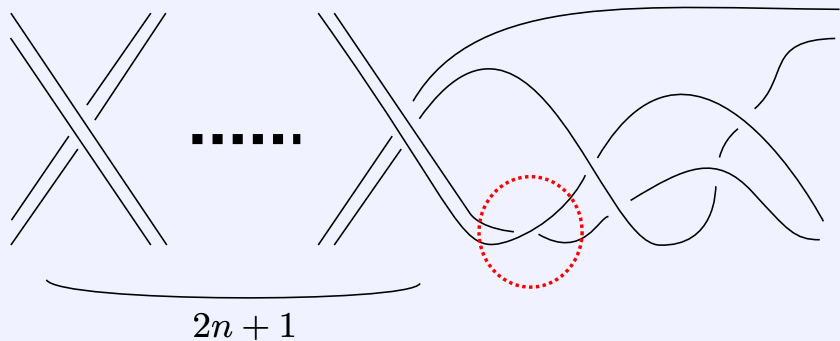


\*  $K9\_449$ : Montesinos trick より, 14-surgery  $\rightarrow$   
 $M(-1/2, 3/8, 2/9)$  で L-space surgery



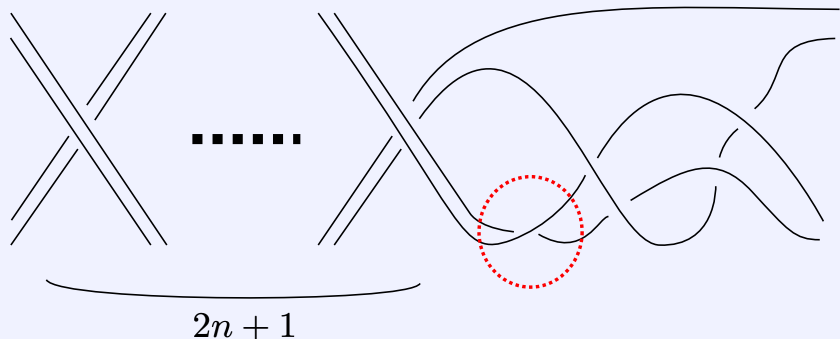
# BAKER-KEGEL の KNOT

$K_n$  ( $n \geq 1$ ): hyperbolic L-space knot  $\rightarrow K_1 = K9\_449$



# BAKER-KEGEL の KNOT

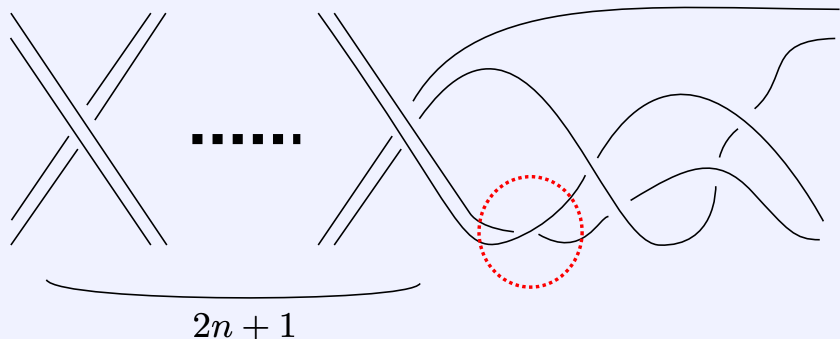
$K_n$  ( $n \geq 1$ ): hyperbolic L-space knot  $\rightarrow K_1 = K9\_449$



$$S = \langle 4, 4n + 2, 4n + 5 \rangle$$

# BAKER-KEGEL の KNOT

$K_n$  ( $n \geq 1$ ): hyperbolic L-space knot  $\rightarrow K_1 = K9\_449$



$$S = \langle 4, 4n + 2, 4n + 5 \rangle$$

\*  $S = \langle 6, 6n + 4, 6n + 8, 12n + 11, 12n + 15 \rangle$  の実現

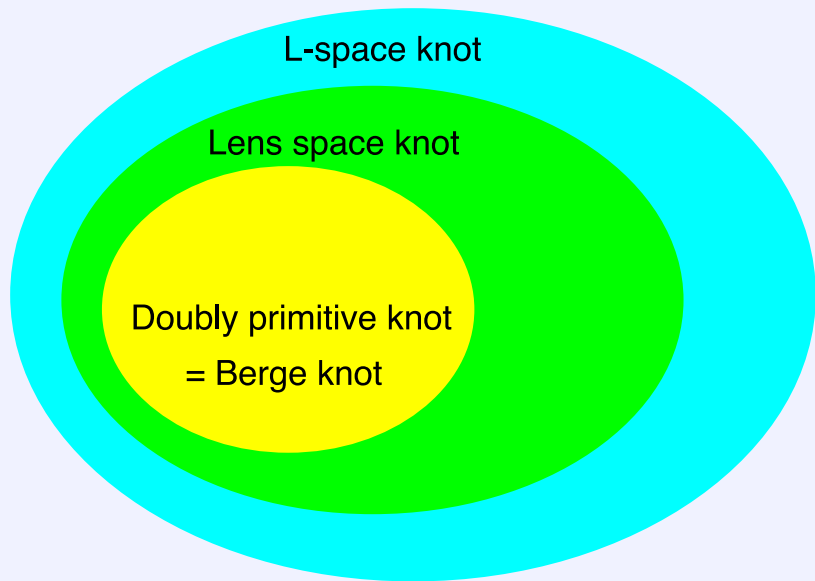
# 根本的な問題

## 疑問

L-space knot に対して，その形式的半群が半群になるかどうかの分水嶺は何なのか？

# DOUBLY PRIMITIVE KNOT

# DOUBLY PRIMITIVE KNOT



# DOUBLY PRIMITIVE KNOT

$H \subset S^3$ : standard genus two Heegaard surface  
 $S^3 = H_1 \cup_H H_2$

## 定義

knot  $K \subset H$  が,  $\pi_1(H_1)$  と  $\pi_1(H_2)$  の自由生成元であるとき,  
 $K$  を **doubly primitive** という.

# DOUBLY PRIMITIVE KNOT

$H \subset S^3$ : standard genus two Heegaard surface  
 $S^3 = H_1 \cup_H H_2$

## 定義

knot  $K \subset H$  が,  $\pi_1(H_1)$  と  $\pi_1(H_2)$  の自由生成元であるとき,  
 $K$  を **doubly primitive** という.

最大の特徴: surface slope で surgery すると, lens space を生じる.

\* doubly primitive position は一意とは限らない.



doubly primitive knot  $K$  から, lens space  $L(p, q)$  が得られるとき,  $L(p, q)$  内の surgery core  $K^*$  を, (associated) **dual knot** という.

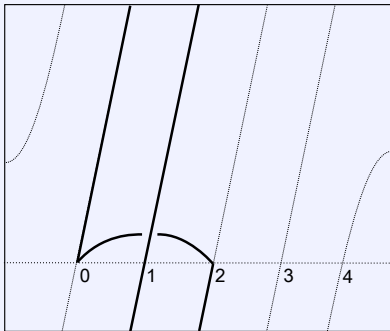
## Fact

dual knot  $K^*$  は,  $L(p, q)$  内の simple  $(1, 1)$ -knot になる.  
→ integer triplet  $(p, q, k)$  で表示可能.

\*simple  $(1, 1)$ -knot: 2つの defining arc がそれぞれ meridian disk になる.

# DUAL KNOT (5, 1, 2)

$L(5, 1)$  の Heegaard 分解  $V_1 \cup V_2$  に対して,  
 $K \cap V_i = t_i \subset D_i$



\*  $(5, 1, 2) \sim (5, 1, 3)$

# ALEXANDER 多項式

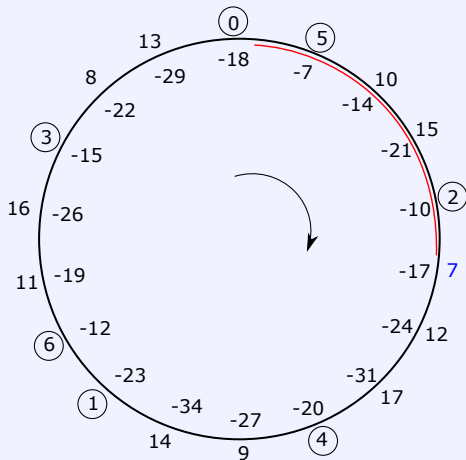
$(p, q, k) = (18, 5, 7)$  の基本列  $\{nq \pmod{p}\}$

⑤, 10, 15, ②, 7, 12, 17, ④, 9, 14, ①, ⑥, 11, 16, ③, 8, 13, ⑦.

$i$	0	1	2	3	4	5	6
$\Psi(i)$	18	11	4	15	8	1	12
$\Phi(i)$	6	3	1	5	2	0	4
$p\Phi(i) - k\Psi(i)$	-18	-23	-10	-15	-20	-7	-12
底上げ $f(i)$	5	0	13	8	3	16	11
剰余類 $\pmod{k}$	5	0	6	1	3	2	4
重複度 $m(i)$	0	0	1	1	0	2	1

\*  $\Psi(k) = 5, \Phi(k) = 2 \rightarrow p\Phi(k) - k\Psi(k) = 1$  [Saito's condition]

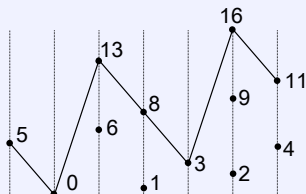
# 巡回基本列



\*常に  $k$  ずつ減少, 丸付き数へは増加  $p - k$

# 折れ線グラフ

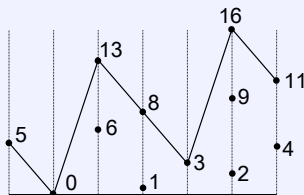
$i$	0	1	2	3	4	5	6
底上げ $f(i)$	5	0	13	8	3	16	11
剰余類 (mod $k$ )	5	0	6	1	3	2	4
重複度 $m(i)$	0	0	1	1	0	2	1



減少量  $\Psi(k) = 5$   
 増加量  $\rho - \Psi(k) = 13$

$$\begin{aligned} \Delta(t) &= 1 + (t-1)(t^6 + t^1 + t^9 + t^2 + t^4) \\ &= 1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10} \end{aligned}$$

# GAP SEQUENCE



折れ線グラフより下にある数列 (1, 2, 4, 6, 9) を,  
[Borodzik-Livingston] は **gap sequence** とよんだ。

$$S = \mathbb{Z}_{\geq 0} - \{\text{gap sequence}\} \longrightarrow \mathcal{S} = \{0, 3, 5, 7, 8\} \cup \mathbb{Z}_{\geq 10}$$

\*(18, 5, 7) は (-2, 3, 7)-pretzel knot に対応

# DOUBLY PRIMITIVE KNOT の形式的半群

doubly primitive knot に対しては, gap sequence  $(a_1, a_2, \dots, a_g)$  によって

$$\Delta(t) = 1 + (t-1)(t^{a_1} + t^{a_2} + \dots + t^{a_g})$$

なので

$$\begin{aligned} \frac{\Delta(t)}{1-t} &= \frac{1}{1-t} - (t^{a_1} + t^{a_2} + \dots + t^{a_g}) \\ &= (1 + t + t^2 + t^3 + \dots) - (t^{a_1} + t^{a_2} + \dots + t^{a_g}) \end{aligned}$$

$\mathbb{Z}_{\geq 0}$  - {gap sequence} が形式的半群を与える.

# BERGE KNOTS

Rasmussen's table

$$q \equiv -k^2 \pmod{p}$$

Type	Order	Constraint
III(a)	$p \equiv \pm(2k - 1)d \pmod{k^2}$	$d k + 1, \frac{k+1}{d} : \text{odd}$
III(b)	$p \equiv \pm(2k + 1)d \pmod{k^2}$	$d k - 1, \frac{k-1}{d} : \text{odd}$
IV(a)	$p \equiv \pm(k - 1)d \pmod{k^2}$	$d 2k + 1$
IV(b)	$p \equiv \pm(k + 1)d \pmod{k^2}$	$d 2k - 1$
V(a)	$p \equiv \pm(k + 1)d \pmod{k^2}$	$d k + 1, d : \text{odd}$
V(b)	$p \equiv \pm(k - 1)d \pmod{k^2}$	$d k - 1, d : \text{odd}$
VII	$k^2 + k + 1 \equiv 0 \pmod{p}$	
VIII	$k^2 - k - 1 \equiv 0 \pmod{p}$	
IX	$p = \frac{1}{11}(2k^2 + k + 1)$	$k \equiv 2 \pmod{11}$
X	$p = \frac{1}{11}(2k^2 + k + 1)$	$k \equiv 3 \pmod{11}$

$d \geq 2$  for hyperbolicity

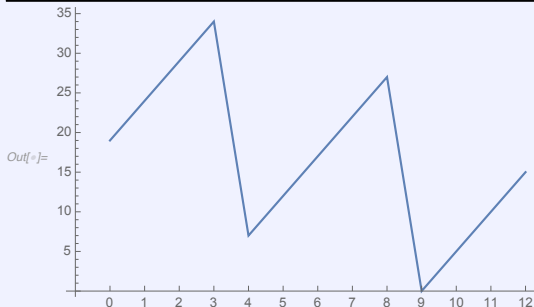


# DOUBLY PRIMITIVE KNOT

**TYPE IX:**  $p = \frac{1}{11}(2k^2 + k + 1)$ ,  $k = 11j + 2$

$$j = 1: (32, 23, 13)$$

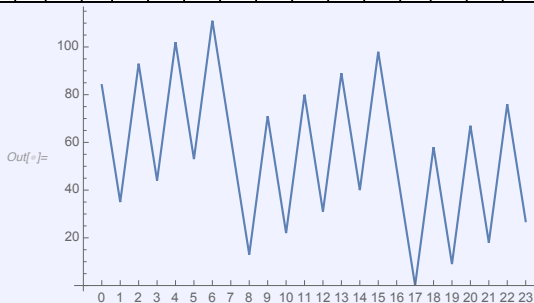
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	32	7	14	21	28	3	10	17	24	31	6	13	20	27
	12	2	5	8	10	0	3	6	9	11	1	4	7	10
Out[*]=	-32	-27	-22	-17	-44	-39	-34	-29	-24	-51	-46	-41	-36	-31
	19	24	29	34	7	12	17	22	27	0	5	10	15	20
	6	11	3	8	7	12	4	9	1	0	5	10	2	7
	1	1	2	2	0	0	1	1	2	0	0	0	1	1



$$f(3j) = 34, f(4j) = 7 \longrightarrow f(3j) \equiv 3f(4j) \pmod{k}$$

$$j = 2: (107, 66, 24)$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Out[-j]	107	60	13	73	26	86	39	99	52	5	65	18	78	31	91	44	104	57	10	70	23	83	36	96	49
	23	12	2	15	5	18	8	21	10	0	13	3	16	6	19	9	22	11	1	14	4	17	7	20	10
	-107	-156	-98	-147	-89	-138	-80	-129	-178	-120	-169	-111	-160	-102	-151	-93	-142	-191	-133	-182	-124	-173	-115	-164	-106
	84	35	93	44	102	53	111	62	13	71	22	80	31	89	40	98	49	0	58	9	67	18	76	27	85
	12	11	21	20	6	5	15	14	13	23	22	8	7	17	16	2	1	0	10	9	19	18	4	3	13
	3	1	3	1	4	2	4	2	0	2	0	3	1	3	1	4	2	0	2	0	2	0	3	1	3



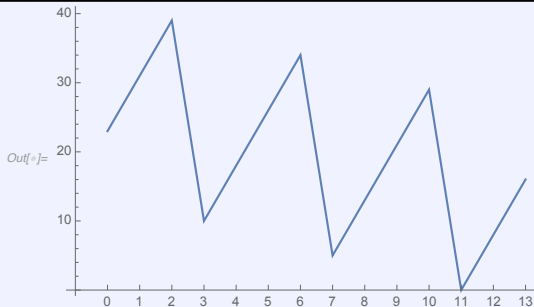
$$f(3j) = 111, f(4j) = 13 \longrightarrow f(3j) \equiv 3f(4j) \pmod{k}$$

# DOUBLY PRIMITIVE KNOT

**TYPE X:**  $p = \frac{1}{11}(2k^2 + k + 1)$ ,  $k = 11j + 3$

$$j = 1: (37, 26, 14)$$

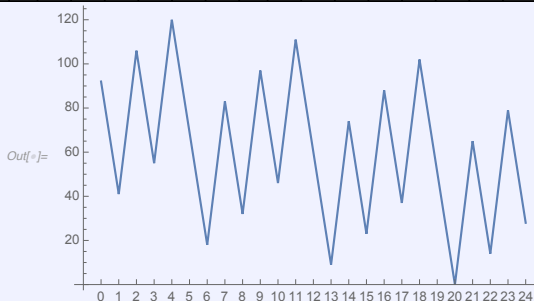
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	37	10	20	30	3	13	23	33	6	16	26	36	9	19	29
	13	3	7	10	0	4	8	11	1	5	9	12	2	6	10
$Out[*]=$	-37	-29	-21	-50	-42	-34	-26	-55	-47	-39	-31	-60	-52	-44	-36
	23	31	39	10	18	26	34	5	13	21	29	0	8	16	24
	9	3	11	10	4	12	6	5	13	7	1	0	8	2	10
	1	2	2	0	1	1	2	0	0	1	2	0	0	1	1



$$f(2j) = 39, f(6j + 1) = 5 \longrightarrow f(2j) \equiv 5f(6j + 1) \pmod{k}$$

$$j = 2: (116, 71, 25)$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	116	67	18	85	36	103	54	5	72	23	90	41	108	59	10	77	28	95	46	113	64	15	82	33	100	51
	24	13	3	17	7	21	10	0	14	4	18	8	22	11	1	15	5	19	9	23	12	2	16	6	20	10
Out <sub>[*]=</sub>	-116	-167	-102	-153	-88	-139	-190	-125	-176	-111	-162	-97	-148	-199	-134	-185	-120	-171	-106	-157	-208	-143	-194	-129	-180	-115
	17	16	6	5	20	19	18	8	7	22	21	11	10	9	24	23	13	12	2	1	0	15	14	4	3	18
	3	1	4	2	4	2	0	3	1	3	1	4	2	0	2	0	3	1	4	2	0	2	0	3	1	3



$$f(2j) = 120, f(6j + 1) = 9 \longrightarrow f(2j) \equiv 5f(6j + 1) \pmod{k}$$

2020 年度修了

- 豊福 共輝
- 山本 陽介

## 定理

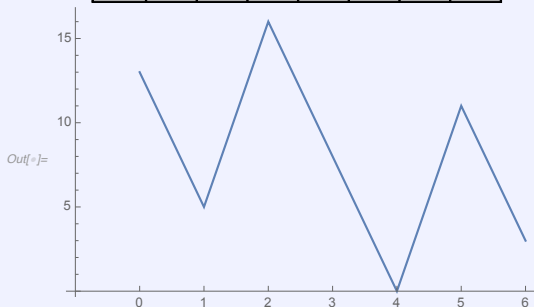
Berge knot の Type VII と VIII 以外については, hyperbolic の場合, 形式的半群が半群にならない.

\*  $p < 500$  でも確認済み

\*  $k \leq 10$  でも確認済み

# TYPE VII: (19, 8, 7)

	0	1	2	3	4	5	6	7
19	12	5	17	10	3	15	8	
6	3	1	5	2	0	4	2	
<i>Out[*]=</i>	-19	-27	-16	-24	-32	-21	-29	-18
13	5	16	8	0	11	3	14	
6	5	2	1	0	4	3	0	
1	0	2	1	0	1	0	2	

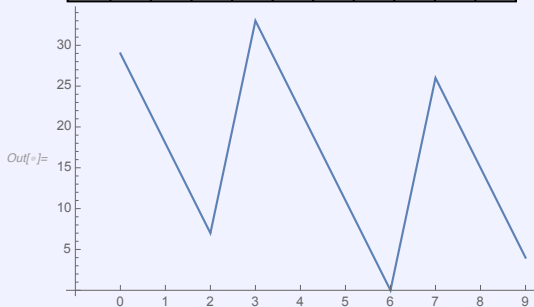


$$f(6) = 3 \longrightarrow f(2) \equiv 3f(6) \pmod{k}$$



# TYPE VII: (37, 11, 10)

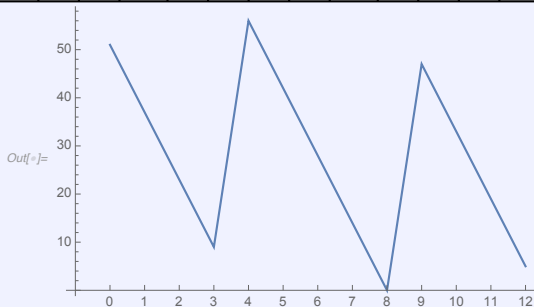
	0	1	2	3	4	5	6	7	8	9	10
	37	27	17	7	34	24	14	4	31	21	11
	9	6	3	1	8	5	2	0	7	4	2
$Out[+]=$	-37	-48	-59	-33	-44	-55	-66	-40	-51	-62	-36
	29	18	7	33	22	11	0	26	15	4	30
	9	8	7	3	2	1	0	6	5	4	0
	2	1	0	3	2	1	0	2	1	0	3



$$f(7) = 26, f(9) = 4 \longrightarrow f(7) \equiv 4f(9) \pmod{k}$$

# TYPE VII: (61, 14, 13)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	61	48	35	22	9	57	44	31	18	5	53	40	27	14
	12	9	6	3	1	11	8	5	2	0	10	7	4	2
$Out[*]=$	-61	-75	-89	-103	-56	-70	-84	-98	-112	-65	-79	-93	-107	-60
	51	37	23	9	56	42	28	14	0	47	33	19	5	52
	12	11	10	9	4	3	2	1	0	8	7	6	5	0
	3	2	1	0	4	3	2	1	0	3	2	1	0	4



$$f(4) = 56, f(12) = 5 \longrightarrow f(4) \equiv 6f(12) \pmod{k}$$