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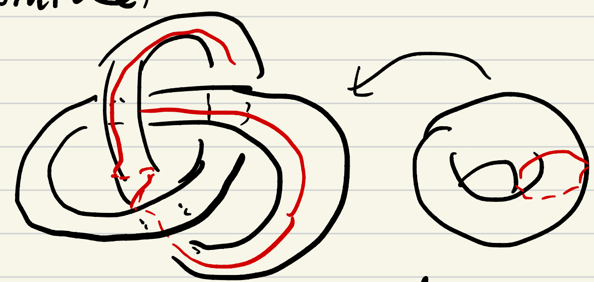
The third term in lens surgery polynomial.

§1. Introduction.

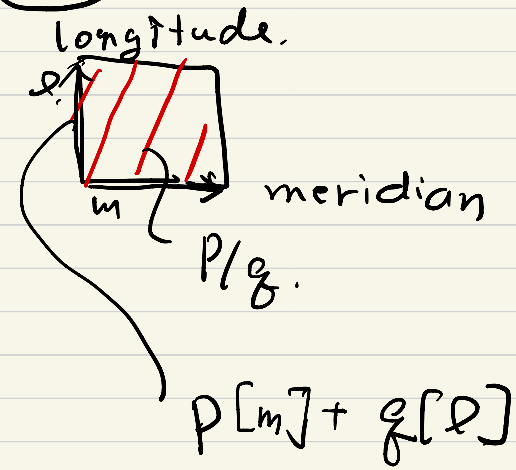
K



$K \subset S^3$: a knot

Def.

$S^3_{p/q}(K)$: p/q-surgery along K.
(p/q: slope)



It is called Dehn surgery

$S^3_p(K)$ is called an integral surgery.

Ex & Def $S^3_{p/q}(U) = L(p,q)$ (oriented)
(U: unknot)

②

Basic results

Fact $S^3_{p/q}(K) = S^3 \Rightarrow K = U$ (KMOS)

Fact $S^3_0(K) = S^2 \times S^1 \Rightarrow K = U$ (Gabai)

positive.

Def. A knot $K \subset S^3$ satisfies $S^3_p(K) = L(p, q)$ then K is called a lens space knot.

Def If a positive Dehn surgery along $K \subset S^3$ is an L-space, then K is called an L-space knot

$\widehat{HF}(S^3) = \widehat{HF}(\ \ , \mathfrak{s})$ any Spin^c str.

ex: lens space

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Conj: $S^3_p(K) = L(p,q) \Rightarrow K$ is a Berge's knot (Gordon - Berge)

(The cases with $p/q \notin \mathbb{Z}$ are classified.)

Fact (Greene) $S^3_p(K) = L(p,q) \Rightarrow \exists$ a Berge's knot $S^3_p(B) = L(p,q)$

Necessary conditions

$\boxed{\text{Fact 1}}$ (Ozsvath–Szabó)

$$\underset{\text{Seifert genus}}{} \quad S^3_p(K) = L(p,q)$$

Any lens space knot has $2g(K)+1 \leq p$

$\boxed{\text{Fact 2}}$ (Ni)

Any lens space knot is fibered.

$\boxed{\text{Fact 3}}$ (Ozsvath–Szabó)

Any L-space K

Alexander poly.

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{j-1}(t^{n_j} + t^{-n_j})$$

$$0 < n_1 < n_2 < \cdots < n_m = \deg(\Delta_K)$$

For example, $t - 1 + t^{-1}$, $t^5 - t^4 + t^2 - t + 1 - t^{-1} + t^{-2} - t^{-4} + t^{-5}$.

④

Let K be a lens space knot.

The Alexander polynomial $\Delta_K(t)$ is called a lens surgery polynomial.

$$(-1)^m + \sum_{j=1}^{m} (-1)^{j-1}\left(t^{n_j} + t^{-n_j}\right) \quad (0 < n_1 < n_2 < \cdots < n_m = d)$$

Main

$\boxed{\text{Problem: Characterize lens space polynomial.}}$

Fact (T., Hedden et al) For any lens surgery polynomial $n_{m-1} = n_m - 1$

i.e. $\Delta_K = t^d - t^{d-1} + \cdots$

$\neq t^d - t^{d-2}$

Question (Teragaito) For any lens space polynomial $n_{m-2} = n_m - 2$

then is $\Delta_K = t^d - t^{d-1} + t^{d-2} - \cdots - t^{-d+1} + t^{-d}$

$= \Delta_{T(2,2d+1)}$?

Here, $T(2,2d+1)$: $(2,2d+1)$-torus knot.

⑤

Thm. (T.) Teragaito's question is true.

Fact (Caudell) " is true with changemaker vectors.

(alternative proof)

<u>Cor 1.</u> If a lens space knot K has $\Delta_K \neq \Delta_{T(2,2d+1)}$ for some d.

, then, $\Delta_K = t^d - t^{d-1} + \underset{\sim}{0} \cdot t^{d-2} + \cdots$

the coefficient is zero!

<u>Cor 2.</u> If a lens space knot K has $\Delta_K = t^d - t^{d-1} + t^{d-2} \cdots$

then K is isotopic to $T(2,2d+1)$

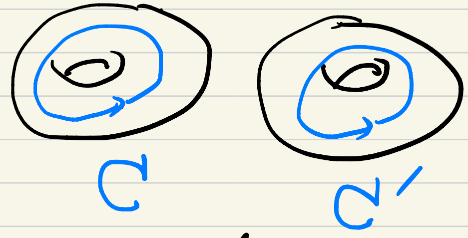
⑥

$$\Delta_K = (-1)^m + \sum_{j=1}^{m} (-1)^{j-1} (t^{n_j} + t^{-n_j}) = \sum_{j=-d}^{d} a_j t^j$$

$-\frac{p}{2} \le j \le \frac{p}{2}$.

$$\bar{a}_j = \begin{cases} a_j & -d < j \le d. \\ 0 & \text{o/w.} \end{cases}$$

$S^3_p(K) = L(p.q) \supset \tilde{K} \rightsquigarrow [\tilde{K}] = k[C]$ ← generator.



$$\begin{cases} k^2 \equiv q \ (p) \\ k_2 = |[k^{-1}]_p| \quad k k_2 \equiv e \ (p). \\ k k_2 = e + mp \quad m \in \mathbb{Z}. \quad e = \pm 1 \end{cases}$$

$j \equiv [j]_p \ (\mathrm{mod}\ p)$

$c = (k-1)(p+1-k)/2$

(p, k) : parameter (of Lens space surgery.)

<u>general i</u>

$\bar{a}_j = \bar{a}_{[j]_p} \qquad -\frac{p}{2} \le [j]_p \le \frac{p}{2}, \qquad j \equiv [j]_p \ (p)$

$$\underline{A_{ij}} = \bar{a}_{k_2(i+jek-c)} = \bar{a}_{k_2 i + j - k_2 c}$$

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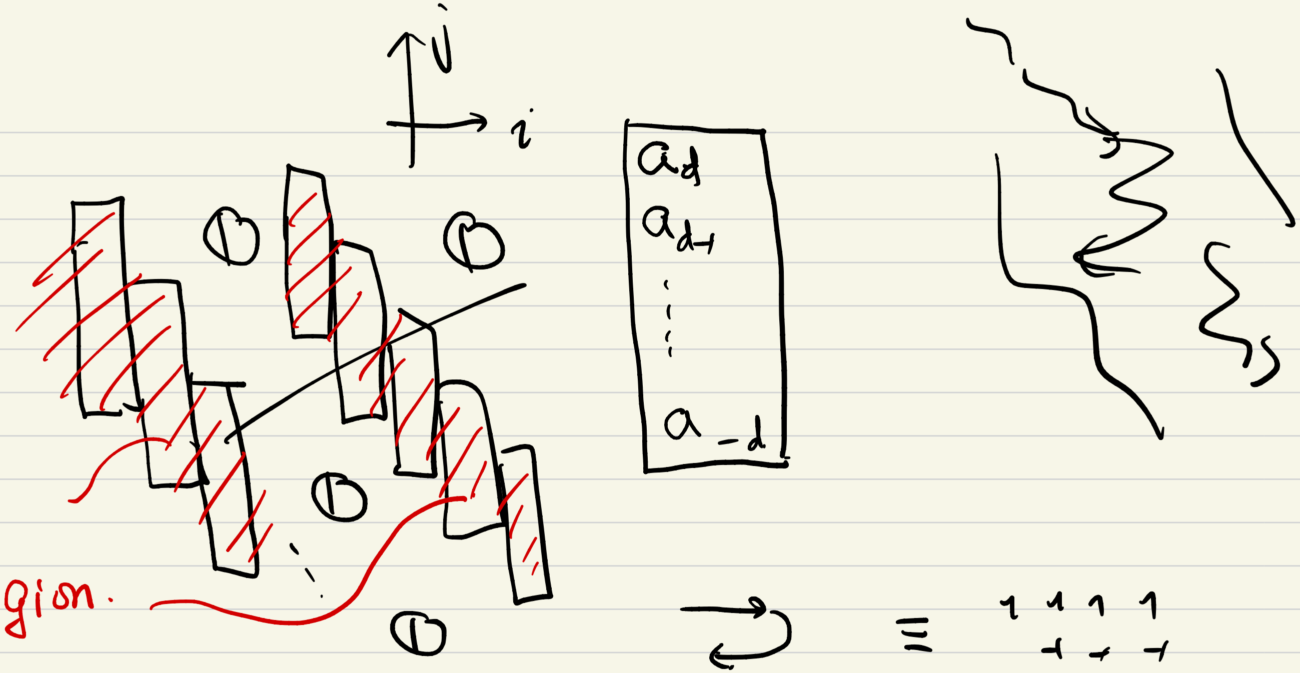
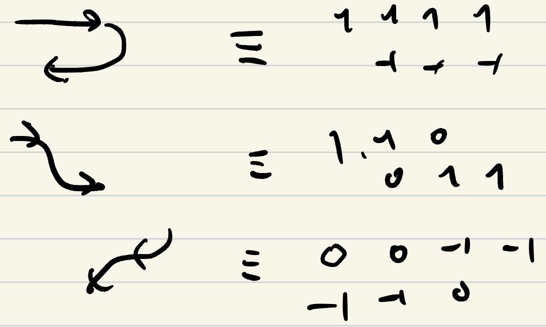
Fact A_{ij}

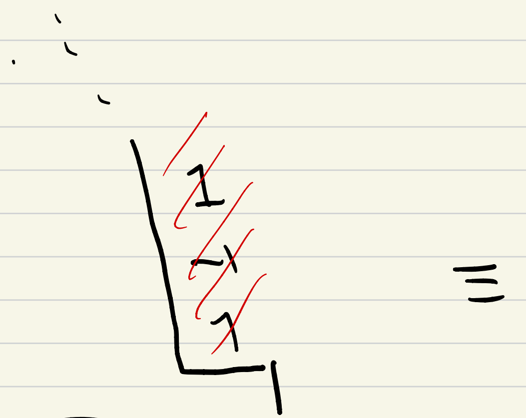


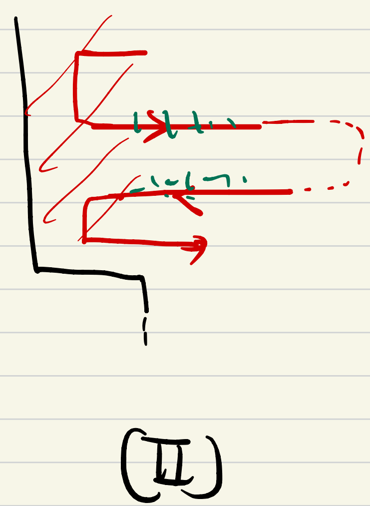


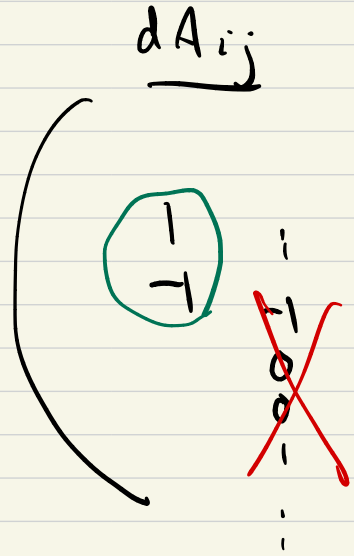
- non-zero curve passes all non-zero coefficients of A_{ij}
- Any nonzero curve is included in a non-zero region.
- one component is a region
- decreasing with respect to j-coord

Proof of Main thm

Assume that $\Delta_K = t^d - t^{d-1} + t^{d-2} - \cdots$





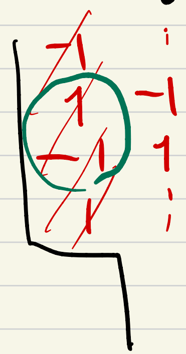


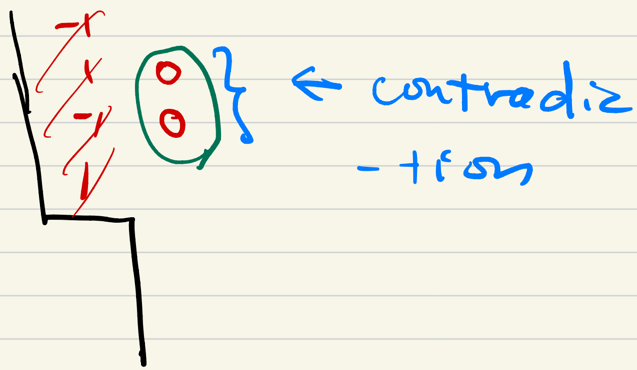
Def

$$\boxed{dA_{ij} = A_{ij} - A_{i-1\,j}}$$

Lemma.

If dA_{ij} has $\begin{matrix} 1 \\ -1 \end{matrix}$ in $\mathbb{R}^2$, then the # of vertical consecutive 0's is at least one





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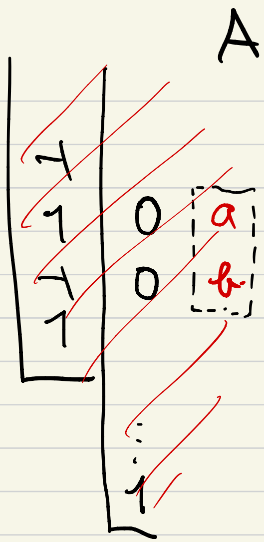
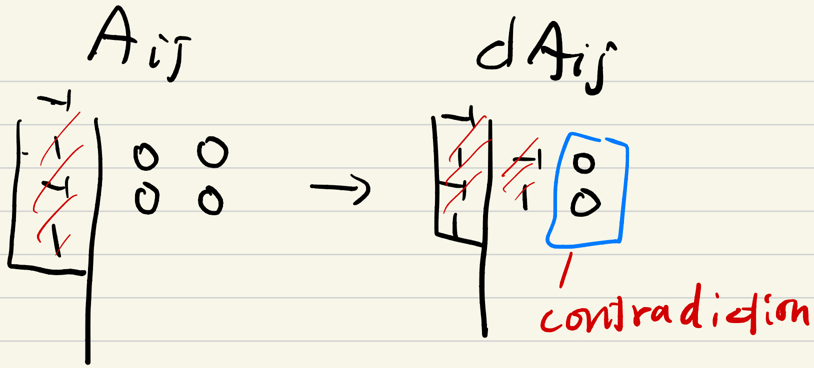
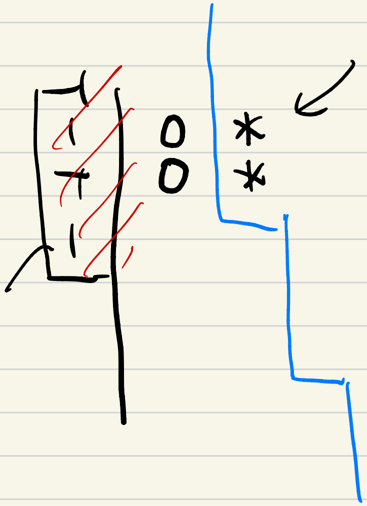


If $a^2+b^2=0$, then

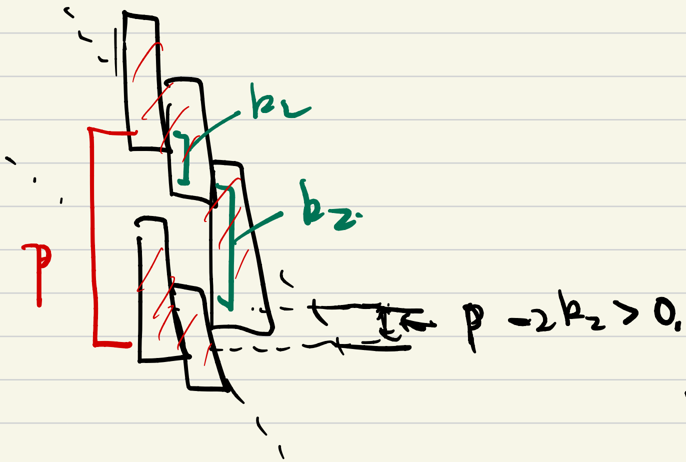


Hence $a^2+b^2>0$.

$$\boxed{A_{ij} = \bar{a}_{k_2 i + j - k_2 c}}$$



Included in the right next nonzero region.



$\therefore\ p = 2k_2+1$

or

$p = 2k_2+2$

$$k_2^2 - 1 = \frac{p}{2}(k_2-1) = p\,\frac{k_2-1}{2}$$
$$\equiv 0\ (p)$$

by KMOS $\Rightarrow k_2 = 1$

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$p = 2k_2 + 1 \Rightarrow 2k_2 \equiv -1 \ (p)$

$\therefore \ k = 2.$

$\boxed{\begin{array}{l} \text{Fact (T.)} \quad K: \text{lens space knot in } Y. \quad (\mathbb{Z}HS^3) \\ \qquad Y_p(K) = L(p,q) \quad (p,k) \\ \quad \text{The following conditions are equivalent.} \\ \text{(i)} \quad k = 2 \\ \text{(ii)} \quad k_2 = 2g+1, \ 2g \\ \text{(iii)} \quad \Delta_K = \Delta_{T(2,2g+1)} \\ \text{(iv)} \ (p,k) \ \text{is } \underline{\text{realized}} \text{ by } T(2,2g+1) \\ \qquad\qquad\qquad \leftarrow S^3_p(T(2,2g+1)) = L(p,q) \end{array}}$ with parameter (p,k)

$\therefore$ by using (i) $\Rightarrow$ (iii)

$\Delta_K = \Delta_{T(2,2g+1)}$ //.

(11)

<u>Cor 1</u> If a lens space knot K has $\Delta_K \neq \Delta_{T(2,2d+1)}$ for some d.

then, $\Delta_K = t^d - t^{d-1} + 0 \cdot t^{d-2} + \cdots$

<u>proof</u>. Equivalent to Thm.

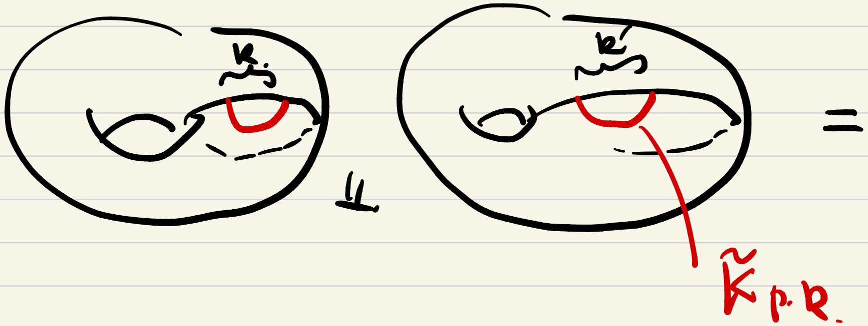
<u>Cor 2.</u> If a lens space knot K has $\Delta_K = t^d - t^{d-1} + t^{d-2} - \cdots$

then K is isol $T(2,2d+1)$

<u>proof.</u> If $S^3_p(K) = L(p,4)$, then K is isotopic to $T(2.2d+1)$. (K. Baker)

$K_{p,k}$: parameter (p,k)

↑ the dual of simple $(1,1)$-knot in $L(p,k^2)$



$= L(p,k^2) \overset{\text{surgery}}{\rightsquigarrow} Y_{p,k} \supset K_{p,k}$

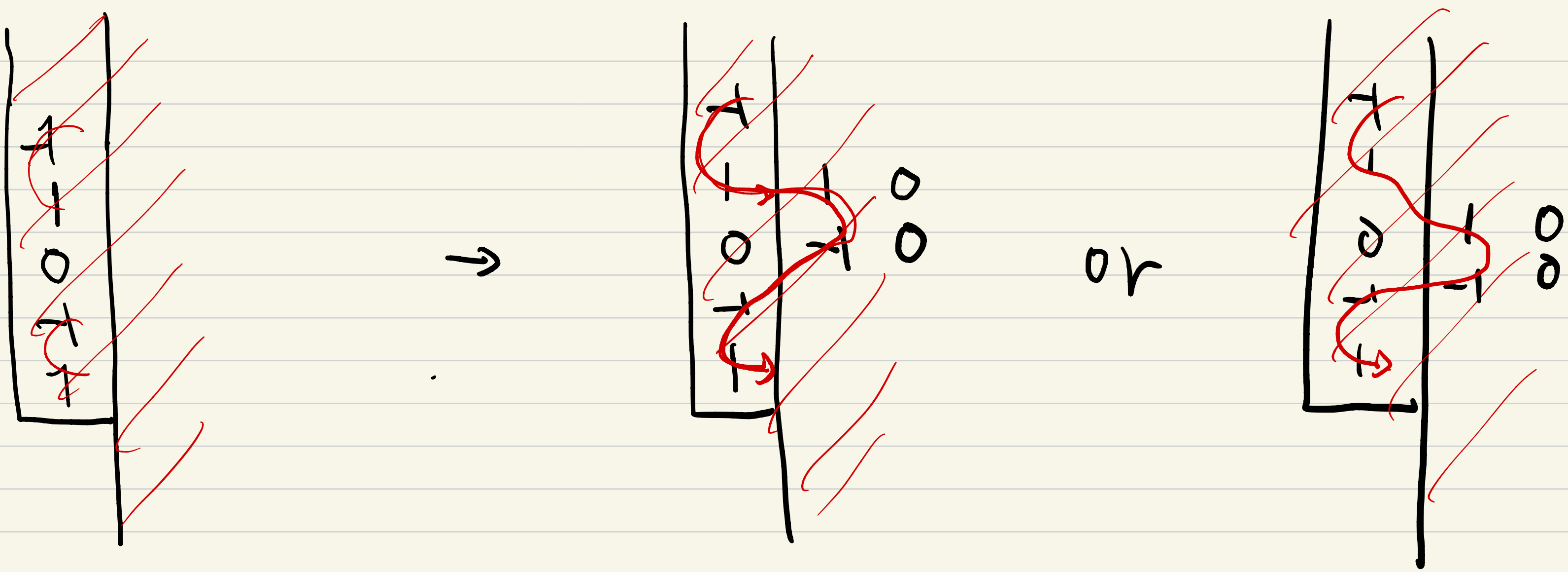
$\cup$ $\quad$ ZHS^3.

$\tilde{K}_{p,k}$

In the case of $Y_{p,k} = S^3$, $K_{p,k}$: Berge's knot (Berge, Greene)

<u>Question 1</u>: $K = K_{p,k}$. If $\Delta_K = t^d - t^{d-1} + t^{d-2} - \cdots$, then, $K = K_{p,2} = T(2, 2d+1)$?

<u>Problem</u> Classify lens space knot with lens surgery $\Delta = t^d - t^{d-1} + t^{d-3} - t^{d-4} + \cdots$



$$\Delta = t^{d} - t^{d-1} + t^{d-3} - t^{d-4} + \cdots - t^{d-k_2-1} + t^{d-k_2-2} + \cdots$$

$$\Rightarrow \quad 2k_2 < p < 4k_2$$

Prop $\quad \Delta_K = t^{d} - t^{d-1} + t^{d-3} - t^{d-4} + t^{d-5} - t^{d-6}$

$$\Rightarrow \quad \Delta_K = t^{d} - t^{d-1} + t^{d-3} - t^{d-4} + \cdots - t^{-d+4} + t^{-d+3} - t^{-d+1} - t^{-d}$$

$$1 \; -1 \; 0 \; 1 \; -1 \; \cdots\cdots \; -1 \; 1 \; 0 \; -1 \; 1$$

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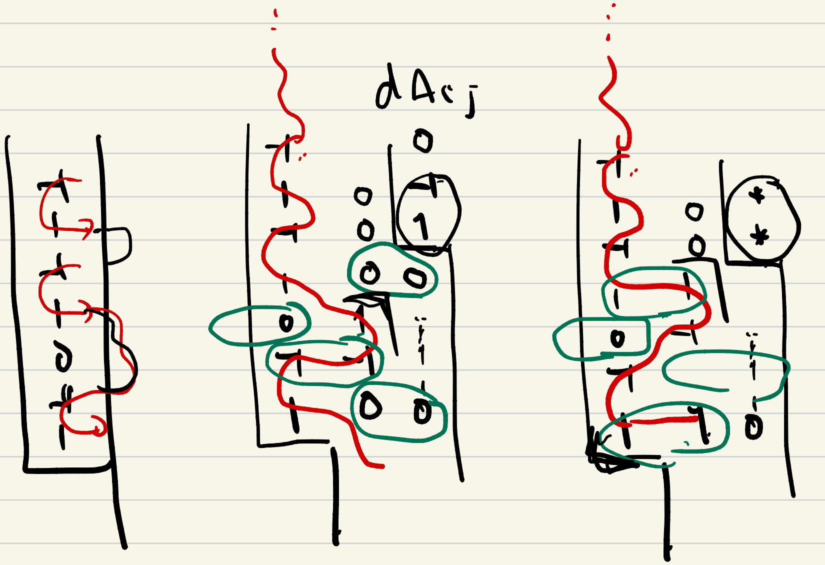
$$\therefore \quad \Delta_K = t^{d} - t^{d-1} + t^{d-2} - t^{d-3}$$



$p = 2b_2+4$ $\qquad$ $p = 2b_2+5$

then $\qquad$ $p = 2b_2+4$, or $p = 2b_2+5$

$$\exists \quad \Delta_K = t^{d} - t^{d-1} + t^{d-3} - t^{d-4} + \cdots - t^{-d+4} + t^{-d+3} - t^{-d+1} - t^{-d}$$

Particularly. $\qquad d = 5$,

$$\Delta_K = \Delta_{Pr(-2,3,7)}$$