Third terms of lens surgery polynomial

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$K \subset S^3$ lens space knot (surgery slope)

iff $\exists\, p > 0$ s.t. $S^3_p(K) = L(p, q)$

$\Delta(t)$ is lens surgery polynomial

iff $\exists$ lens space knot K. s.t. $\Delta_K(t) = \Delta(t)$

$\boxed{\text{Question 1 When is a polynomial } \Delta(t) \text{ a lens surgery polynomial?}}$

Here we write the coeff of $\Delta_K(t)$ $(d = g)$

$$\Delta_K = a_g t^g + a_{g-1} t^{g-1} + \cdots + a_{-g+1} t^{-g+1} + a_{-g} t^{-g}$$

where $a_i = a_{-i}$, $\Delta_K(1) = 1$

Fact 2 (Ozsváth-Szabó). $K \subset S^3$ is a lens space knot.

Then $\Delta_K(t)$ is the following form.

$$\Delta_K = (-1)^m + \sum_{j=1}^{m} (-1)^{j-1} (t^{n_j} + t^{-n_j})$$

$$d = n_1 > n_2 > \cdots > n_{m-1} > 0$$

$$\begin{cases} \text{flat} : f(t) \ \forall \text{coeff } a_i \quad |a_i| \leq 1 \\ \text{alternating}: a_0\, a_1 \cdots a_l : \text{non zero seq in order} \\ \qquad a_i = (-1)^i \end{cases}$$

Fact 3 (Ozsváth-Szabó) $K \subset S^3$ is a lens space knot

p: the surgery slope.

Then, $2g - 1 \leq p$.

Fact 4 (T. Hedden-Watson) $K \subset S^3$ is a l. sp knot

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$$\Delta_K(t) = t^g - t^{g-1} + \cdots$$

(c.f HW proved for any L-space knot the same statement.)

Question 5 (Teragaito) If a surg poly $\Delta(t)$ is $\Delta(t) = t^g - t^{g-1} + t^{g-2} - \cdots$

then $\Delta(t) = t^g - t^{g-1} + t^{g-2} - \cdots - t^{-g+1} + t^{-g}$

$= \Delta_{T(2,2g+1)}(t)$?

Then,

Δ : lens surgery polynomial

not $\Delta = t^g - t^{g-1} + t^{g-2} - \cdots - t^{-g}$.

then $a_{g-2} = 0$

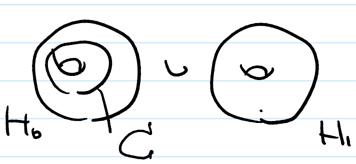
Main theorem 6 $K \subset S^3$ is a lens sp. knot

then Teragaito's question is true

To prove the main thm. we prepare the following infinite matrix. (A_{ij})

$Y : \mathbb{Z}HS^3$. $K \subset Y$ lens sp. knot.

$Y_p(K) = L(p,q) \supset \tilde{K}$ $\quad [\tilde{K}] \in H_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$

$C \subset L(p,q)$  $= \langle [c] \rangle$

$[\tilde{K}] = k[C]$ $\quad (p,k)$. lens surgery parameter lens surgery

φ parameter

dual class.

$[\alpha]_p$: the least absolute remainder when dividing by p.

$$-\frac{p}{2} < [\alpha]_p \leq \frac{p}{2}$$

$$k_2 := \left| [k^{-1}]_p \right|$$

<u>Constant</u>

- $e \equiv k k_2 \mod p$; $e = \pm 1$
- $c = \dfrac{(k-1)(k+1-p)}{2}$
- $m = \dfrac{k k_2 - e}{p}$
- $I_\alpha = \begin{cases} \{1, 2, \cdots \alpha\} & \alpha > 0 \\ \{\alpha+1, \alpha+2, \cdots -1, 0\} & \alpha < 0 \end{cases}$
- $q = [k^2]_p$, $q_2 = [k_2^2]_p$

Prop 7 K : lens sp. knot in S^3.
$Y_p(K) = L(p, q)$

$$\alpha_i = -em + e \cdot \#\{ j \in I_k \mid [q_2(j + ki + c)]_p \in I_{ek_2} \}$$

then

$$a_i = \alpha_i \qquad |i| \leq g.$$

Actually $a_i = \alpha_i \qquad |i| \leq p/2$.

$\bar{a}_i = a_{[i]_p}$ (periodic extension)

$$A_{i,j} = \overline{a}_{k_2(i-c)+j}$$

$$= \overline{a}_{k_2(x-c)}$$

$$= A(x) \qquad m = \frac{kk_2 - e}{p}$$

Example 8 $\quad x=0 \quad (i=j=0) \qquad \underset{m'}{\underset{\parallel}{me}} = \frac{ekk_2-1}{p}$

$$A(0) = \overline{a}_{-k_2c} = \overline{a}_{-ek_2c}$$

$$= -e(m - \#\{j \in I_k \mid [g_2(j + k(-ek_2c)+c)]_p \in I_{ek_2}\}$$

$$= -e(m - \#\{j \in I_k \mid [g_2 j]_p \in I_{ek_2}\}$$

$$\Big(= -e(m - (k - \#\{j \in I_k \mid [g_2 j]_p \in I_{k_2}\} \quad (e=-1)$$

$$= -e(m-k + \#\{j \in I_k \mid [g_2 j]_p \in I_{p-k_2}\}$$

$$= -m' + \#\{j \in I_k \mid [g_2 j]_p \in I_{p-k_2}\} \Big)$$

where $[\alpha]_p$ the remainder among $\{1, 2, \ldots, p$

$$m' = \begin{cases} \dfrac{kk_2-1}{p} & e=1 \\ \dfrac{k(p-k_2)-1}{p} & e=-1 \end{cases} = \frac{kk'-1}{p}$$

$$k' = [\overline{k}]_p$$

$$\therefore A(0) = -m' + \#\{j \in I_k \mid [g_2 j]_p \in I_{k'}\}$$

$$= -m' + \Phi(k) \quad \text{Saito's } \Phi.$$

$$A(-1) = \overline{a}_{ek_2(-1-c)} = -m' + \#\{j \in I_k \mid [g_2(j-1)]_p \in I_{k'}\}$$

$$= -m' + \#\{j \in I_{k-1} \mid [g_2 j]_p \in I_{k'}\}$$

$$= -m' + \#\{j \in I_k \mid [g_2 j]_p \in I_{k'}\}$$

$$= -m' + \Phi(0) - 1$$

$$= A(0) - 1$$

by O-S condition. (Fact 1)

$$(A(0), A(-1)) = (1,0), (0,-1)$$

$$\therefore -m' + \Phi(k) = 1 \text{ or } 0$$

multiplying by p

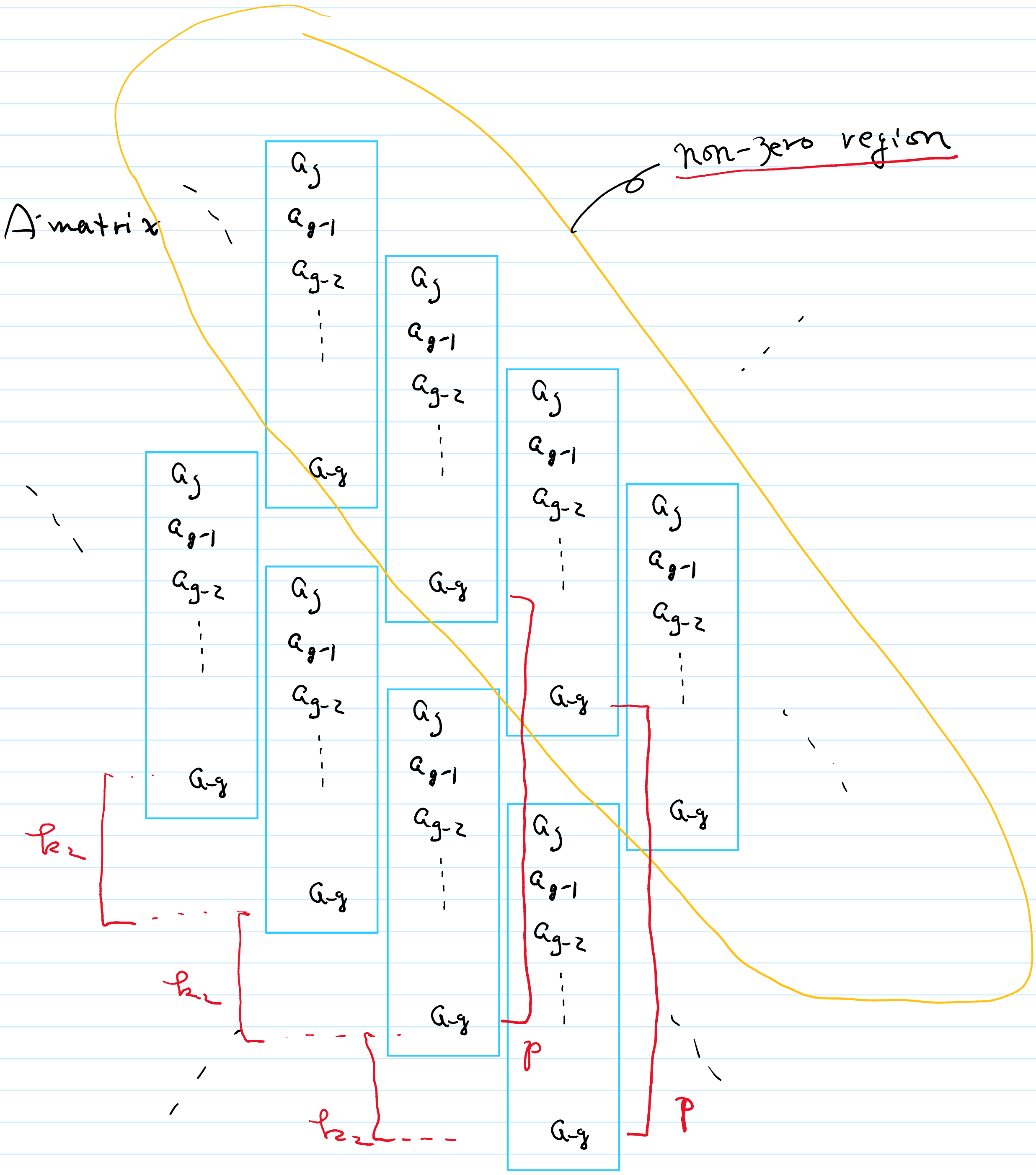
$$1 - kk' + p\Phi(k) = p \text{ or } 0$$

multiplying by p

$$1 - kb' + p\,\Phi(k) = p \quad \text{or} \quad 0$$

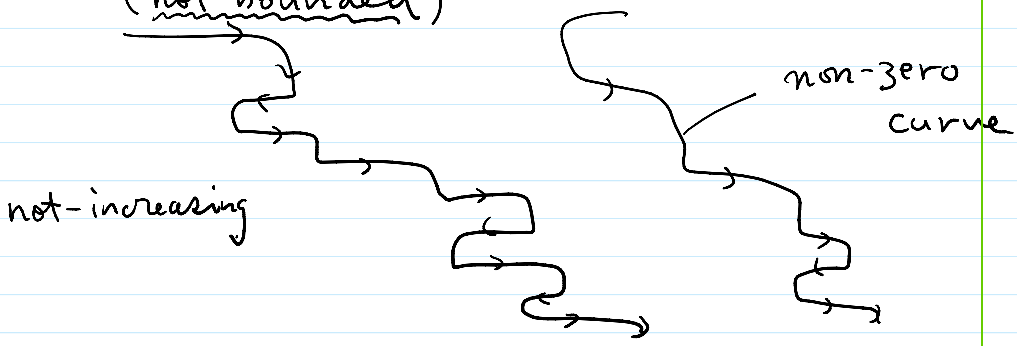
$$-p\,\Phi(k) + kb' = -p+1 \quad \text{or} \quad -1$$

(Saito's condition)



Prop 9 $\forall$ A-matrix

$\exists$ an oriented curve in $\mathbb{R}^2$
(not bounded)



s.t. $\rightarrow \cdots \rightarrow \;=\; 1\ 1 \cdots 1\ 1$

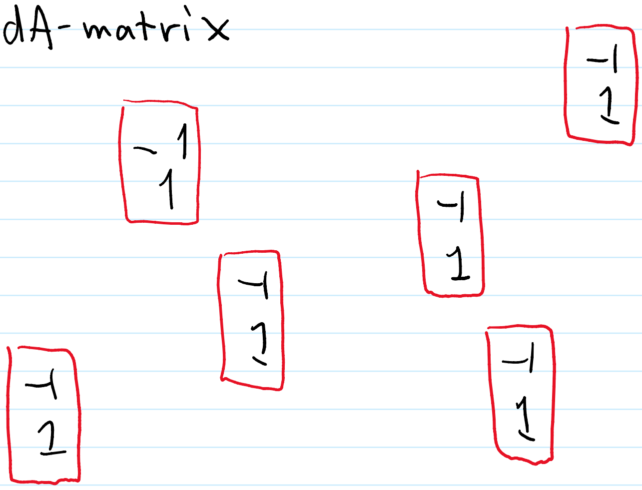
$\leftarrow \quad \leftarrow \;=\; -1\ -1 \cdots -1\ -1$

$\begin{array}{c} \cdots \rightarrow \\ \cdots \leftarrow \end{array} \;=\; \begin{array}{c} \cdots 1\ \ 1 \\ \cdots -1\ -1 \end{array}$

$\phi \;=\; \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{array}$

$dA_{i,j} = A_{i,j} - A_{i-1,j}$

dA-matrix

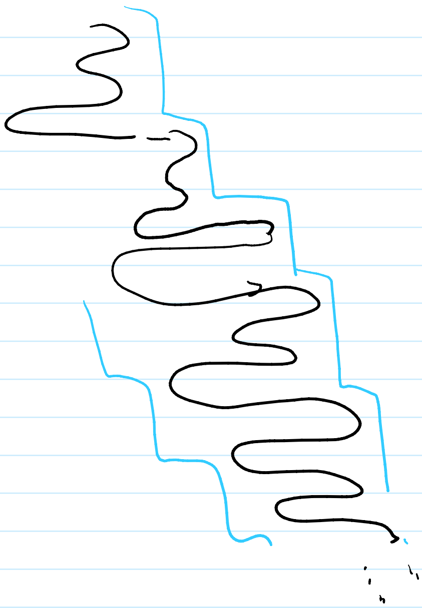


Lem 10 A-matrix of a lens sp. knot.
There exists $m \geq 0$, s.t.

$$dA. \quad \begin{array}{l} \boxed{\begin{matrix} -1 \\ 1 \end{matrix}} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} m \text{ or } m+1 \\ \boxed{\begin{matrix} -1 \\ 1 \end{matrix}} \end{array}$$

Prop 11 In each non-zero region for any A-matrix.

there is one comp non-zero curve only.



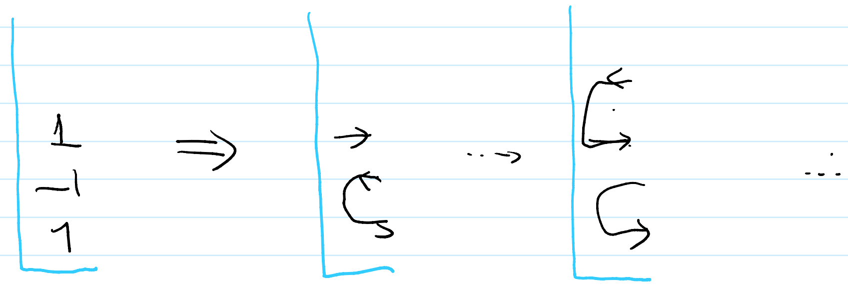
Proof of Theorem 6

$K \subset S^3$ lens sp. knot

Suppose $\Delta_K(t) = t^g - t^{g-1} + t^{g-2} - \cdots$

K : c.d. lens sp. knot

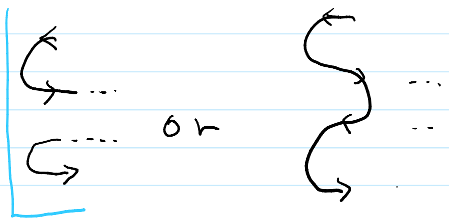
Suppose $\Delta_K(t) = t^g - t^{g-1} + t^{g-2} - \cdots$



$\therefore$

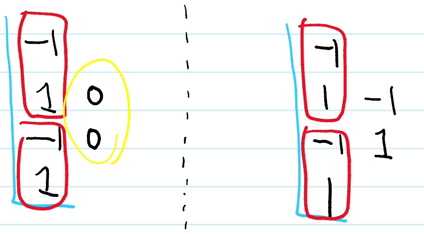
$$\Delta_K = t^g - t^{g-1} + t^{g-2} - t^{g-3} + \cdots$$

A $\qquad\qquad$ A



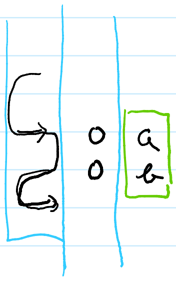
Case I $\qquad\qquad$ Case II

dA $\qquad\qquad$ dA

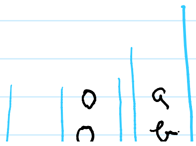


Contradiction to Lemma. 10 ☆

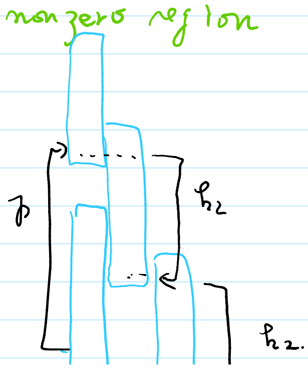
☆ If $ab = 0$, then

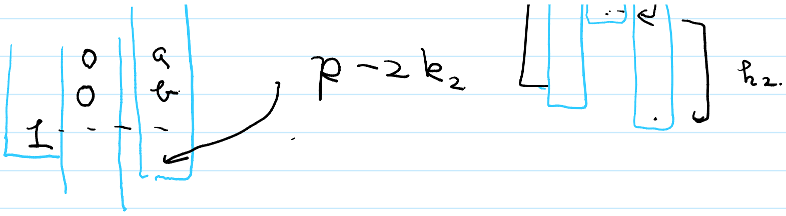


in the right nonzero region



, $p - 2k_2$





$\therefore\ p - 2k_2 \leq 2.$

by $\quad 2k_2 < p$

$(k_2, 2) = 1$

$\therefore\ p - 2k_2 = 2$ or $1.$

$k_2 = 2K+1$

$\therefore\ p = \underbrace{2k_2+2}$ or $\underbrace{2k_2+1}$

$4K+3 = p$

$\downarrow \qquad\qquad \hookrightarrow k = 2.$

$(k_2, 2) = 1$

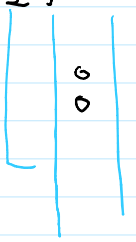
$k_2^2 - 1 = \frac{k_2-1}{2}(2k_2+2) \equiv 0 \ (p)$

$\therefore\ q_2 = 1 \qquad$ a surgery yielding $L(p.1)$

Here we use this classification $\rightsquigarrow k_2 = 1 \quad$ KMOS.

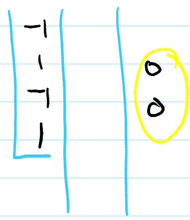
$\boxed{\begin{array}{l} \underline{\text{Thm 11 (T)}}\ K \subset Y : \text{a lens sp. knot} \\ \qquad Y_p(K) = L(p.q) \quad \text{with } (p\ k) \\ \qquad \text{the following conditions are equivalent} \\ \qquad \left[\begin{array}{ll} 1) & (p\ k) \text{ is realized by } (2.2g+1) \\ 2) & k = 2. \\ 3) & k_2 = 2g+1 \text{ or } 2g \\ 4) & \Delta_K(t) = \Delta_{T(2,2g+1)}(t) \end{array}\right. \end{array}}$

If $\quad A \quad$ satisfies



then

dA



contradiction
to Lemma 10

Therefore such a surgery is realized by
$T(2.2g+1)$

In particular, $\Delta_K = \Delta_{T(2,2g+1)}$ //

Cor 12 $K \subset S^3$ a lens sp. knot with $\Delta_K = \Delta_{T(2,2g+1)}$

then any lens surg. parameter (p, k) of K. (i.e. $S^3_p(K) = L(p, q)$ para. (p, k))

is realized by $T(2, 2g+1)$ for some integer g.

(p, k) is realized by a knot $K \subset Y$

$Y_p(K) = L(p, q)$ has the parameter (p, k)

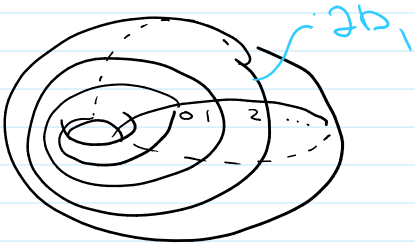
§2. $K_{p,k} \subset Y_{p,k}$

(p, k). relatively prime

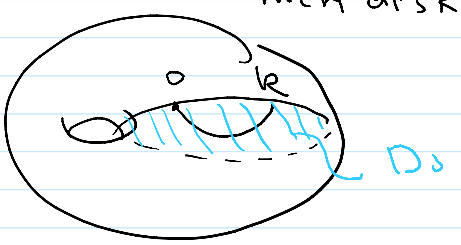
$Y_{p,k}$ $\mathbb{Z}HS^3$ $K_{p,k} \subset Y_{p,k}$

$$(Y_{p,k})_p(K_{p,k}) = \underset{\substack{\| \\ H_0 \cup_{T^2} H_1}}{L(p, k^2)} \supset \tilde{K}_{p,k}$$

$H_i \supset D_i$ meri disk.



H_0



H_0

Question 13 Any $K_{p,k}$ is $\Delta_{K_{p,k}} = t^g - t^{g-1} + t^{g-2} \cdots$

then

$K_{p,k} = T(2, 2g+1)$