合成関数の微分法

$z = f(x, y)$ と、$x = x(u, v), y = y(u, v)$ なる関数を合成した関数を $z = f^*(u, v)$ とする．つまり、$z = f^*(u, v) = f(x(u, v), y(u, v))$ である．このとき、以下が成り立つ．

$$\frac{\partial f^*}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f^*}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

これを行列の形にしたものは

$$\begin{pmatrix} \frac{\partial f^*}{\partial u} & \frac{\partial f^*}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

となる．この右の行列はヤコビ行列である．(この転置をヤコビ行列ということもある．)

宿題-5-1

$$x(r,\theta,\phi) = r\sin\theta\cos\phi,$$
$$y(r,\theta,\phi) = r\sin\theta\sin\phi,$$
$$z(r,\theta,\phi) = r\cos\theta$$

と $f(x,y,z)$ との合成関数を $f^*(r,\theta,\phi)$ とおくとき、次の等式が成り立つことを示せ．

❶ $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f^*}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f^*}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f^*}{\partial \phi}\right)^2$

❷ $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f^*}{\partial r^2} + \frac{2}{r}\frac{\partial f^*}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial f^*}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f^*}{\partial \phi^2}\right)$

❸ **上の変換によるヤコビアン $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ を求めよ．**

$$\left(\frac{\partial f^*}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f^*}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial f^*}{\partial \phi}\right)^2$$

(1)

$$\begin{aligned}\frac{\partial f^*}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} \\ &= \frac{\partial f}{\partial x}\sin\theta\cos\phi + \frac{\partial f}{\partial y}\sin\theta\sin\phi + \frac{\partial f}{\partial z}\cos\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial f^*}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} \\ &= \frac{\partial f}{\partial x}r\cos\theta\cos\phi + \frac{\partial f}{\partial y}r\cos\theta\sin\phi - \frac{\partial f}{\partial z}r\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial f^*}{\partial \phi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi} \\ &= -\frac{\partial f}{\partial x}r\sin\theta\sin\phi + \frac{\partial f}{\partial y}r\sin\theta\cos\phi\end{aligned}$$

$$\left(\frac{\partial f^*}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f^*}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2 \theta}\left(\frac{\partial f^*}{\partial \phi}\right)^2$$

$$\frac{\partial f^*}{\partial r} = \frac{\partial f}{\partial x}\sin\theta\cos\phi + \frac{\partial f}{\partial y}\sin\theta\sin\phi + \frac{\partial f}{\partial z}\cos\theta$$

$$\frac{1}{r}\frac{\partial f^*}{\partial \theta} = \frac{\partial f}{\partial x}\cos\theta\cos\phi + \frac{\partial f}{\partial y}\cos\theta\sin\phi - \frac{\partial f}{\partial z}\sin\theta$$

$$\frac{1}{r\sin\theta}\frac{\partial f^*}{\partial \phi} = -\frac{\partial f}{\partial x}\sin\phi + \frac{\partial f}{\partial y}\cos\phi$$

$$\frac{\partial^2 f^*}{\partial r^2},\ \frac{\partial^2 f^*}{\partial \theta^2},\ \frac{\partial^2 f^*}{\partial \phi^2}$$

(2)

$$\frac{\partial f^*}{\partial r} = \frac{\partial f}{\partial x}\sin\theta\cos\phi + \frac{\partial f}{\partial y}\sin\theta\sin\phi + \frac{\partial f}{\partial z}\cos\theta$$

$$\frac{\partial f^*}{\partial \theta} = \frac{\partial f}{\partial x}r\cos\theta\cos\phi + \frac{\partial f}{\partial y}r\cos\theta\sin\phi - \frac{\partial f}{\partial z}r\sin\theta$$

$$\frac{\partial f^*}{\partial \phi} = -\frac{\partial f}{\partial x}r\sin\theta\sin\phi + \frac{\partial f}{\partial y}r\sin\theta\cos\phi$$

この微分を求める.

$\dfrac{\partial^2 f^*}{\partial r^2}$

$$
\begin{aligned}
\frac{\partial}{\partial r}\frac{\partial f^*}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\sin\theta\cos\phi + \frac{\partial f}{\partial y}\sin\theta\sin\phi + \frac{\partial f}{\partial z}\cos\theta\right) \\
&= \left(\frac{\partial}{\partial r}\frac{\partial f}{\partial x}\right)\sin\theta\cos\phi + \left(\frac{\partial}{\partial r}\frac{\partial f}{\partial y}\right)\sin\theta\sin\phi + \left(\frac{\partial}{\partial r}\frac{\partial f}{\partial z}\right)\cos\theta \\
&= \left(\frac{\partial^2 f}{\partial x^2}\sin\theta\cos\phi + \frac{\partial^2 f}{\partial y\partial x}\sin\theta\sin\phi + \frac{\partial^2 f}{\partial z\partial x}\cos\theta\right)\sin\theta\cos\phi \\
&\quad + \left(\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\phi + \frac{\partial^2 f}{\partial y^2}\sin\theta\sin\phi + \frac{\partial^2 f}{\partial z\partial y}\cos\theta\right) r\sin\theta\sin\phi \\
&\quad + \left(\frac{\partial^2 f}{\partial x\partial z}\sin\theta\cos\phi + \frac{\partial^2 f}{\partial y\partial z}\sin\theta\sin\phi + \frac{\partial^2 f}{\partial z^2}\cos\theta\right) r\cos\theta
\end{aligned}
$$

$$\frac{\partial^2 f^*}{\partial \theta^2}$$

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\frac{\partial f^*}{\partial\theta} &= \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x} r\cos\theta\cos\phi + \frac{\partial f}{\partial y} r\cos\theta\sin\phi - \frac{\partial f}{\partial z} r\sin\theta\right) \\
= & \left(\frac{\partial}{\partial\theta}\frac{\partial f}{\partial x}\right) r\cos\theta\cos\phi + \left(\frac{\partial}{\partial\theta}\frac{\partial f}{\partial y}\right) r\cos\theta\sin\phi - \left(\frac{\partial}{\partial\theta}\frac{\partial f}{\partial z}\right) r\sin\theta \\
+ & \frac{\partial f}{\partial x}(-r\sin\theta\cos\phi) + \frac{\partial f}{\partial y}(-r\sin\theta\sin\phi) - \frac{\partial f}{\partial z} r\cos\theta \\
= & \left(\frac{\partial^2 f}{\partial x^2} r\cos\theta\cos\phi + \frac{\partial^2 f}{\partial y\partial x} r\cos\theta\sin\phi - \frac{\partial^2 f}{\partial z\partial x} r\sin\theta\right) r\cos\theta\cos\phi \\
+ & \left(\frac{\partial^2 f}{\partial x\partial y} r\cos\theta\cos\phi + \frac{\partial^2 f}{\partial y^2} r\cos\theta\sin\phi - \frac{\partial^2 f}{\partial z\partial y} r\sin\theta\right) r\cos\theta\sin\phi \\
- & \left(\frac{\partial^2 f}{\partial x\partial z} r\cos\theta\cos\phi + \frac{\partial^2 f}{\partial y\partial z} r\cos\theta\sin\phi - \frac{\partial^2 f}{\partial z^2} r\sin\theta\right) r\sin\theta \\
+ & \frac{\partial f}{\partial x}(-r\sin\theta\cos\phi) + \frac{\partial f}{\partial y}(-r\sin\theta\sin\phi) - \frac{\partial f}{\partial z} r\cos\theta
\end{aligned}
$$

$$\frac{\partial}{\partial \phi}\frac{\partial f^*}{\partial \phi} = \frac{\partial}{\partial \phi}\left(-\frac{\partial f}{\partial x} r\sin\theta\sin\phi + \frac{\partial f}{\partial y} r\sin\theta\cos\phi\right)$$

$$\begin{aligned}
= \quad & -\left(\frac{\partial}{\partial \phi}\frac{\partial f}{\partial x}\right) r\sin\theta\sin\phi + \left(\frac{\partial}{\partial \phi}\frac{\partial f}{\partial y}\right) r\sin\theta\cos\phi \\
& + \frac{\partial f}{\partial x} r\sin\theta\cos\phi + \frac{\partial f}{\partial y}(-r\sin\theta\sin\phi) \\
= \quad & -\left(-\frac{\partial^2 f}{\partial x^2} r\sin\theta\sin\phi + \frac{\partial^2 f}{\partial y\partial x} r\sin\theta\cos\phi\right) r\sin\theta\sin\phi \\
& + \left(-\frac{\partial^2 f}{\partial x\partial y} r\sin\theta\sin\phi + \frac{\partial^2 f}{\partial y^2} r\sin\theta\cos\phi\right) r\sin\theta\cos\phi \\
& + \frac{\partial f}{\partial x} r\sin\theta\cos\phi + \frac{\partial f}{\partial y}(-r\sin\theta\sin\phi)
\end{aligned}$$

(3)

$$\det\begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= r^2\det\begin{pmatrix} \sin\theta\cos\phi & \cos\theta\sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin^2\theta & 0 \end{pmatrix}$$

ここで、列の基本変形をすると、

$$= r^2\det\begin{pmatrix} \sin\theta\cos\phi & 0 & -\sin\phi \\ \sin\theta\sin\phi & 0 & \cos\phi \\ \cos\theta & -1 & 0 \end{pmatrix}$$

2 **列目に沿って展開をして、**

$$= r^2\det\begin{pmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{pmatrix} = r^2\sin\theta\det\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = r^2\sin\theta$$

宿題-5-2

つぎの証明のどこが不十分か答えなさい．十分な証明のためにはどのような議論を補うとよいか．

宿題-4-2.[全微分可能性]

$f(x,y) = xy$ **が点** (a,b) **で全微分可能であることを定義に基づいて示せ．**

（解答） $h = x - a, k = y - b$ **とおく．**

$$f(x,y) - f(a,b) - bh - ak = xy - ab - bh - ak = hk$$

となる．ここで、 $h = k$ **とおくことで、**

$$\lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = \lim_{h\to 0} \frac{h^2}{\sqrt{2h^2}} = \lim_{h\to 0} \frac{h}{\sqrt{2}} = 0$$

これは $hk = o(\sqrt{h^2+k^2})$ **を意味する．よって、任意の** (a,b) **に対して**

$$f(x,y) = f(a,b) + \alpha(x-a) + \beta(y-b) + o(\sqrt{(x-a)^2+(y-b)^2})$$

を満たす $\alpha = b, \beta = a$ **が存在するので、** $f(x,y)$ **は** (a,b) **で全微分可能．**
(証明終了)

（これは実際あった解答です.）

不十分な箇所

（解答） $h = x - a, k = y - b$ **とおく．**

$$f(x, y) - f(a, b) - bh - ak = xy - ab - bh - ak = hk$$

となる．ここで、 $h = k$ **とおくことで、**

$$\lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2 + k^2}} = \lim_{h\to 0} \frac{h^2}{\sqrt{2h^2}} = \lim_{h\to 0} \frac{h}{\sqrt{2}} = 0$$

これは $hk = o(\sqrt{h^2 + k^2})$ **を意味する．よって、任意の** (a, b) **に対して**

$$f(x, y) = f(a, b) + \alpha(x - a) + \beta(y - b) + o(\sqrt{(x - a)^2 + (y - b)^2})$$

を満たす $\alpha = b, \beta = a$ **が存在するので、** $f(x, y)$ **は** (a, b) **で全微分可能．** **(証明終了)**

補完する

任意の原点に近づく点列 (h_n, k_n) を $h_n = r_n \cos\theta_n, k_n = r_n \sin\theta_n$ とおく．ただし、$r_n \to 0 (n \to \infty)$ である．このとき、

$$\left|\frac{h_n k_n}{\sqrt{h_n^2 + k_n^2}}\right| = \left|\frac{r_n^2 \cos\theta_n \sin\theta_n}{r_n}\right| = r_n |\cos\theta_n \sin\theta_n| \leq r_n$$

ゆえに、$n \to \infty$ となるとき、$\frac{h_n k_n}{\sqrt{h_n^2+k_n^2}} \to 0$ が成り立つ．よって、任意の点列によって収束が示されたので、

$$\lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = 0$$

がいえる．
を補えば十分．

宿題-5-3

省略.