微積分演習S [FBA1722,FBA1732]

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宿題解答編

宿題 7-1 (1) $\int_0^1 \frac{x^4}{1+x^2}dx = \int_0^1 \frac{x^4-1+1}{1+x^2}dx = \int_0^1 \left(x^2-1+\frac{1}{1+x^2}\right)dx = \left[\frac{x^3}{3} - x + \mathrm{Arctan}\, x\right]_0^1 = \frac{1}{3} - 1 + \frac{\pi}{4} = -\frac{2}{3} + \frac{\pi}{4}$

(2) $\tan\frac{x}{2} = t$ とおくとき、$\cos^2\frac{x}{2} = \frac{1}{1+\tan^2\frac{x}{2}} = \frac{1}{1+t^2}$ であることから、$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2}$ と $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$ であるから、$dt = \frac{1}{2\cos^2\frac{x}{2}}dx = \frac{t^2+1}{2}dx$ となる。$\tan^2\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2$ であり、$\tan\frac{\pi}{8} > 0$ であるから $\tan\frac{\pi}{8} = \sqrt{2}-1$ である。よって、

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1}{\sin x + \cos x}dx &= \int_0^{\sqrt{2}-1} \frac{1}{\frac{2t+1-t^2}{1+t^2}} \frac{2dt}{1+t^2} = \int_0^{\sqrt{2}-1} \frac{2dt}{1+2t-t^2} \\
&= -2\int_0^{\sqrt{2}-1} \frac{dt}{(t-1-\sqrt{2})(t-1+\sqrt{2})} = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}-1}\left(\frac{1}{t-1+\sqrt{2}} - \frac{1}{t-1-\sqrt{2}}\right)dt \\
&= \frac{1}{\sqrt{2}}\left[\log\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right|\right]_0^{\sqrt{2}-1} = \frac{1}{\sqrt{2}}\left(\log\frac{2\sqrt{2}-2}{2} - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \\
&= \frac{\log(\sqrt{2}+1)}{\sqrt{2}}
\end{aligned}$$

宿題 7-2

(1) $\int \frac{x+1}{2x^2+2x+1}dx = \int \frac{\frac{1}{4}(4x+2)+\frac{1}{2}}{2x^2+2x+1}dx = \frac{1}{4}\log(2x^2+2x+1) + \frac{1}{2}\int\frac{1}{2x^2+2x+1}dx = \frac{1}{4}\log(2x^2+2x+1) + \int\frac{dx}{(2x+1)^2+1} = \frac{1}{4}\log(2x^2+2x+1) + \frac{1}{2}\mathrm{Arctan}\,(2x+1) + C$

(2) $\int \frac{x+1}{\sqrt{1-x^2}}dx = \frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx + \int\frac{1}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + \mathrm{Arcsin}\, x$

宿題 7-3

(1) $2+\sqrt{x-4} = t$ とおくと、$t^2 = 4+4\sqrt{x-4}+x-4 = x+4\sqrt{x-4}$

$$\begin{aligned}
\int_5^7 \frac{1}{x+4\sqrt{x-4}}dx &= \int_3^{2+\sqrt{3}} \frac{2t-4}{t^2}dt = \left[2\log t + \frac{4}{t}\right]_3^{2+\sqrt{3}} \\
&= 2\log\frac{2+\sqrt{3}}{3} + \frac{4}{2+\sqrt{3}} - \frac{4}{3} = 2\log\frac{2+\sqrt{3}}{3} - 4\sqrt{3} + \frac{20}{3}
\end{aligned}$$

(2) $\frac{2x+1}{\sqrt{3}} = t$ とおき、$\frac{2}{\sqrt{3}}dx = dt$ を使うと、

$\int_{-\frac{1}{2}}^0 \sqrt{1+x+x^2}dx = \frac{\sqrt{3}}{2}\int_{-\frac{1}{2}}^0 \sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}dx = \frac{3}{4}\int_0^{\frac{1}{\sqrt{3}}}\sqrt{t^2+1}dt$ となる。さらに、$t = \sinh r$ と

おく。また、$\mathrm{Arsinh}\,\dfrac{1}{\sqrt{3}} = \log\sqrt{3}$ であるから、

$$\begin{aligned}
\frac{3}{4}\int_0^{\frac{1}{\sqrt{3}}} \sqrt{t^2+1}\,dt &= \frac{3}{4}\int_0^{\log\sqrt{3}} \cosh^2 r dr \\
&= \frac{3}{4}\int_0^{\log\sqrt{3}} \frac{1+\cosh 2r}{2} dr = \frac{3}{4}\left[\frac{2r+\sinh 2r}{4}\right]_0^{\log\sqrt{3}} \\
&= \frac{3(\log 3 + \sinh(\log 3))}{16} = \frac{3(\log 3 + \frac{e^{\log 3} - e^{-\log 3}}{2})}{8} \\
&= \frac{3\log 3 + \frac{3}{2}(3-\frac{1}{3})}{16} = \frac{3\log 3 + 4}{16}
\end{aligned}$$

(コメント) 最後の積分に関して、$\displaystyle\int \sqrt{t^2+1}\,dt$ を計算す際、$t = \tan\theta$ と置換すると $\displaystyle\int \frac{1}{\cos^3\theta} d\theta$ を計算する必要が生じる。ここでは $t = \sinh r$ と置換する方が良い。