

HOMOLOGY SPHERES WITH E_8 -FILLINGS AND ARBITRARILY LARGE CORRECTION TERMS

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ABSTRACT. In this paper we construct families of homology spheres which bound 4-manifolds with intersection forms isomorphic to $-E_8$. We show that these families have arbitrary large correction terms. This result says that among homology spheres, the difference of the maximal rank of minimal sub-lattice of definite filling and the maximal rank of even definite filling is arbitrarily large.

1. INTRODUCTION

1.1. Definite fillings and homology cobordism invariants. If a 3-manifold Y bounds X , then we call X *filling* of Y . If a filling of Y has a definite (even, or spin) intersection form, then the filling is called *definite filling* (*even filling* or *spin filling* respectively). Under the assumption that the homology of a filling has no 2-torsions, an even filling is equivalent to spin filling. If a definite filling has a positive (or negative) definite intersection, then we call the filling *positive-definite filling* (or *negative-definite filling* respectively).

Let Y be an integer homology sphere. Rohlin invariant $\mu(Y)$ is defined to be $\sigma(W)/8 \in \mathbb{Z}/2\mathbb{Z}$ for a spin filling W of Y . We can assume that the spin filling W is $H_1(W, \mathbb{Z}) = \{0\}$ (we say *homologically 1-connected*). In this article we mainly consider homologically 1-connected definite fillings.

Ozsváth and Szabó defined a homology cobordism invariant d in [9]. If a 3-manifold has a negative-definite filling of Y , then the d -invariant has the following restriction.

Theorem 1.1 ([9]). *Let Y be an integral homology three-sphere, then for each negative-definite four-manifold X which bounds Y , we have the inequality*

$$\xi^2 + \text{rk}(H_2(X, \mathbb{Z})) \leq 4d(Y)$$

for each characteristic vector ξ .

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Furthermore, if a homology sphere Y has an even negative-definite filling W , then $b_2(W) \leq 4d(Y)$ is satisfied. For example $\Sigma(2, 3, 5)$ is the boundary of the $-E_8$ -plumbing. Here $-E_8$ is the unique unimodular, even, negative-definite, rank 8 quadratic form. The computation $d(\Sigma(2, 3, 5)) = 2$ means that b_2 of any even negative-definite filling is at most 8. The plumbing realized a filling with $b_2 = 8$. If Y has a definite filling with intersection form nE_8 for some integer n , then the filling is called nE_8 -filling.

On the other hand, the d -invariant of $\Sigma(2, 3, 7)$ is 0. Thus, if there exists an even negative-definite filling, then the b_2 has to be 0. Hence, the b_2 of any positive-definite filling is also zero. Since $\mu(\Sigma(2, 3, 7)) = 1$, it has no homology 1-connected even definite filling. The plumbing of $\Sigma(2, 3, 7)$ with all weights -2 can give an even filling with even intersection form $(-E_8) \oplus H$, where H is the hyperbolic intersection form. This filling is a homologically 1-connected even (equivalently spin) indefinite filling.

In [14] the author defined the following invariants. If Y has an nE_8 -filling, then we define g_8 (or \underline{g}_8) to be

$$g_8(Y) = \max\{b_2(W)/8 \mid W : nE_8\text{-filling of } Y, H_1(W) = \{0\}\},$$

$$\underline{g}_8(Y) = \min\{b_2(W)/8 \mid W : nE_8\text{-filling of } Y, H_1(W) = \{0\}\}.$$

If Y has no nE_8 -fillings, then $g_8(Y) = -\infty$.

We call the invariant g_8 E_8 -genus. If Y has an nE_8 -filling, then we can immediately see the following bound

$$(1) \quad 2g_8(Y) \leq |d(Y)|.$$

For example, for any integer n , $d(\Sigma(2, 3, 12n + 5)) = 2$ holds. The author in [14] showed $g_8(\Sigma(2, 3, 12n + 5)) = 1$ when $0 \leq n \leq 13$ or $n = 15$. In [14] we gave the examples with $2g_8(Y) = |d(Y)|$. The simple question is the following:

Question 1.2. *Among homology spheres Y with non-negative E_8 -genus is $|d(Y)| - 2g_8(Y)$ bounded?*

We give families of Brieskorn homology spheres to obtain negative answers for this question.

1.2. Main results. Here we give the main result:

Theorem 1.3. *For any integer n , Brieskorn homology spheres $\Sigma(|p|, |q|, |r|)$ for a pairwise coprime positive integer triple (p, q, r) below have homologically 1-connected $-E_8$ -fillings with $g_8 = -\bar{\mu} = 1$.*

- (i) $(2, 8n - 3, 14n - 5)$, (ii) $(2, 14n + 3, 24n + 5)$
- (iii) $(2, 16n + 3, 26n + 5)$, (iv) $(2, 10n - 3, 16n - 5)$
- (v) $(5, 35n - 2, 50n - 3)$, (vi) $(5, 25n - 2, 40n - 3)$
- (vii) $(3, 15n - 2, 36n - 5)$, (viii) $(3, 9n - 2, 24n - 5)$
- (ix) $(3, 21n - 4, 36n - 7)$, (x) $(3, 27n - 4, 48n - 7)$
- (xi) $(4, 28n - 3, 64n - 7)$, (xii) $(4, 32n - 3, 76n - 7)$

The invariant $\bar{\mu}$ is the Neumann-Siebenmann invariant, which will be defined in Section 2.3. These examples can be useful for realizing desired filling restricted by gauge theory. For example, see recent Scaduto's study [12].

Theorem 1.4. *For positive integer n the correction terms of Brieskorn homology spheres (i), (ii) (iii) and (iv) in Theorem 1.3 have the following inequalities:*

$$2 \left\lceil \frac{n}{2} \right\rceil \leq d(\Sigma(2, 8n - 3, 14n - 5)), \quad 2 \left\lceil \frac{n+1}{2} \right\rceil \leq d(\Sigma(2, 14n+3, 24n+5)),$$

$$2 \left\lceil \frac{n+1}{2} \right\rceil \leq d(\Sigma(2, 16n+3, 26n+5)), \quad 2 \left\lceil \frac{n}{2} \right\rceil \leq \Sigma(2, 10n - 3, 16n - 5).$$

These theorems say that for any positive integer n , the Brieskorn homology spheres (i), (ii), (iii), and (iv) have $-E_8$ -fillings and $d(Y) - 2g_8(Y) = d(Y) + 2\bar{\mu}(Y)$ are arbitrarily large.

Remark 1.5. Let (Y, c) be a pair of Seifert rational homology sphere Y and a spin structure c . According to [18] the $\bar{\mu}(Y, c)$ is the equivalent to the Fukumoto-Furuta invariant $w(Y, c)$.

Manolescu in [6] defined homology cobordism invariants α , β , and γ in the framework of $\text{Pin}(2)$ Seiberg-Witten Floer homology. A result in [13] says that for any Brieskorn homology sphere Y (with usual orientation) $\beta(Y) = \gamma(Y) = -\bar{\mu}(Y)$ and $\alpha(Y) = d(Y)/2$ or $d(Y)/2 + 1$. Hence, our result means the existence of homology spheres that $\beta(Y) = 1$ and $\alpha(Y)$ is arbitrarily large.

Remark 1.6. As a conjecture, the inequalities in Theorem 1.4 would become the equalities actually for any positive integer n . The evidence is due to Karakurt's program [5]. Similarly, for positive integer n we predict the following equalities for other Brieskorn homology spheres in Theorem 1.3.

- For $(p, q, r) = (5, 35n - 2, 50n - 3), (5, 25n - 2, 40n - 3)$, we have

$$d(\Sigma(p, q, r)) = 6n.$$

- For $(p, q, r) = (3, 15n - 2, 36n - 5), (3, 9n - 2, 24n - 5), (3, 21n - 4, 36n - 7), (3, 27n - 4, 48n - 7)$, we have

$$d(\Sigma(p, q, r)) = 2n.$$

- For $(p, q, r) = (4, 28n - 3, 64n - 7), (4, 32n - 3, 76n - 7)$, we have

$$d(\Sigma(p, q, r)) = 4 \left(\frac{n}{2} + \left\lceil \frac{n}{2} \right\rceil \right).$$

For non-positive integer n we conjecture that for any homology sphere above $d(\Sigma(|p|, |q|, |r|))$ are all 2.

Here we compare the following result by Ue [17] with the result above.

Theorem 1.7 ([17]). *Let (S, c) be a pair of a spherical 3-manifold and a spin structure on it. Then $d(S, c) = -2\bar{\mu}(S, c)$.*

In fact, in [17] it is shown that the general correction term $d(S, t)$ of spin^c spherical 3-manifold coincides with the Fukumoto-Furuta invariant, which is defined by using the index of the Dirac operator of a line bundle over a 4-orbifold.

We remark $\bar{\mu}$ here is defined to be the μ divided by 8 in [17]. The examples in Theorem 1.3 are homology spheres that $d(\Sigma) + 2\bar{\mu}(\Sigma)$ are arbitrarily large. The relationship between correction term d and Neumann-Siebenmann invariant $\bar{\mu}$ for non-spherical 3-manifolds is known not so many things even in the case of Seifert 3-manifolds.

1.3. Other invariants related definite fillings.

1.3.1. *Homologically 1-connected fillings.* In [14] homology cobordism invariant $\mathfrak{d}\mathfrak{s}(Y)$ is defined to be

the maximal $b_2(W)/8$ among homologically 1-connected,
even definite fillings W

of Y . A homologically 1-connected even filling gives a spin filling. Any invariant related to a kind of filling is defined to be $-\infty$, if there exists no such a kind of fillings. We define $\mathfrak{o}(Y)$ to be

the maximal rank of the minimal definite lattice L that $L \oplus \langle \pm 1 \rangle^n$ is the
intersection form of a homologically 1-connected definite filling W

of Y . Here ‘minimal’ means that any square ± 1 element is not included in the lattice and n is some non-negative integer. By the definition, we have

$$(2) \quad 8g_8(Y) \leq 8\mathfrak{d}\mathfrak{s}(Y) \leq \mathfrak{o}(Y).$$

For example, in the case of $Y = \Sigma(2, 5, 9)$, according to Corollary 1.2 in [12], $g_8(Y) = \mathfrak{d}\mathfrak{s}(Y) = 1$ and $\mathfrak{o}(Y) = 12$ holds. We have the following question.

Question 1.8. *Differences of these invariants are bounded or unbounded?*

We can show that homology spheres in Theorem 1.4 satisfy the following unbounded property.

Corollary 1.9. *Among homology spheres Y , $\mathfrak{o}(Y) - 8\mathfrak{d}\mathfrak{s}(Y)$ is unbounded.*

These integer homology spheres satisfy $\mathfrak{d}\mathfrak{s}(Y) = g_8(Y)$. It is not known whether for some integer homology spheres Y the differences $\mathfrak{d}\mathfrak{s}(Y) - g_8(Y)$ are positive or unbounded.

1.3.2. *General definite fillings.* Define $E(Y)$ to be

the maximal among $b_2(W)/8$ even definite filling W

of Y . Note the filling is possibly non-spin. In the same way, we define $O(Y)$ to be

the maximal rank of minimal sub-lattice of definite fillings W
of Y and $G_8(Y)$ to be

the maximal $|n|$ among nE_8 -fillings

of Y . Since fillings for invariants $G_8(Y)$, $E(Y)$, $O(Y)$ do not assume homologically 1-connected, we have $g_8(Y) \leq G_8(Y)$, $\mathfrak{d}\mathfrak{s}(Y) \leq E(Y)$, and $\mathfrak{o}(Y) \leq O(Y)$. We have the similar inequality to (2):

$$(3) \quad 8G_8(Y) \leq 8E(Y) \leq O(Y).$$

For example, consider the case of $Y = \Sigma(2, 3, 7)$. As we mentioned above Y has no even definite fillings with homologically 1-connected, i.e., $g_8(Y) = \mathfrak{d}\mathfrak{s}(Y) = -\infty$. On the other hand, Fintushel and Stern constructed a rational ball that bounds Y in [2], i.e., $E(Y) = G_8(Y) = 0$. Let W be a negative-definite filling of Y and Ξ the set of characteristic elements in $H_2(W)$. $d(Y) = 0$ means $\max_{c \in \Xi} (c^2 + b_2(W)) \leq 0$. The Elkies theorem in [1] concludes the inequality means the equality and the negative-definite lattice must be diagonalized. For the positive-definite filling of Y one has only to consider negative-definite fillings of $-Y$. In fact the plumbing lattice of $\Sigma(2, 3, 7)$ is diagonalized, therefore, $O(Y) = \mathfrak{o}(Y) = 0$.

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2. NOTATIONS AND PRELIMINARIES

2.1. Plumbing diagram. For $i = 1, 2$ let $V_i \rightarrow S^2$ be two D^2 -bundles over S^2 or let $V \rightarrow S^2$ be a D^2 -bundle. For the D^2 -bundles V_1 and V_2 we take sub- D^2 -bundles over each disk in two base spaces S^2 . For $V \rightarrow S^2$ we take sub- D^2 -bundles over two disjoint disks in S^2 . *Plumbing process* is a surgery obtained by identifying two D^2 -bundles in such a way that one exchanges the roles of their sections and fibers. We call the plumbing of $V \rightarrow S^2$ *self-plumbing*. Actually, to define the plumbing process we need choose one of the two possibilities of the orientation of the identification as in p.201 in [4]. Since we deal with the tree-type graph only later, then we do not explain the choices.

We define a plumbing diagram (or graph) as explained in [11]. Let V be the set of vertices with a weight function $m : V \rightarrow \mathbb{Z}$. We assign for $v \in V$ the D^2 -bundle over S^2 with the Euler number $m(v)$. Let E be the set of edges. Each edge $\{v, w\} \in E$ of a plumbing diagram means the plumbing process between the D^2 -bundles over S^2 . If $v = w$, then the edge means the self-plumbing. Hence, if \mathbb{Z} -weighted graph (V, E, m) is called *plumbing graph* or *plumbing diagram*.

Let (V, E, m) be a plumbing diagram. The plumbing process along a plumbing graph $\Gamma = (V, E, m)$ gives a 4-manifold $P(\Gamma)$ and we call $P(\Gamma)$

a *plumbed 4-manifold*. The boundary $\partial P(\Gamma)$ of $P(\Gamma)$ is called a *plumbed 3-manifold*. Here $[v]$ is the class represented by the core sphere of the D^2 -bundle corresponding to the vertex v . The intersection form $(\cdot, \cdot) : H_2(P(\Gamma)) \times H_2(P(\Gamma)) \rightarrow \mathbb{Z}$ on $P(\Gamma)$ is computed from the linear extension of the following definition.

$$([v], [w]) = \begin{cases} m(v) & v = w \\ 1 & v \neq w \text{ and } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

A tree-type graph with at most one vertex of degree larger than two is called a *star-shaped graph*. A plumbed 3-manifold with a star-shaped graph is called a *Seifert manifold*. For $i = 1, 2, \dots, n$, let $(\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{r_i}^{(i)})$ be a sequence of weights of vertices of the i -th branch of a star-shaped graph. We compute the continued fraction for the sequence as follows:

$$(4) \quad a_i/b_i = [\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{r_i}^{(i)}],$$

where $\alpha_j^{(i)}$ is some integer and (a_i, b_i) are coprime integers. Here the continued fraction is defined to be

$$[c_1, c_2, \dots, c_m] = c_1 - \frac{1}{c_2 - \dots - \frac{1}{c_m}}.$$

The Seifert manifold with the rational numbers a_i/b_i ($i = 1, 2, \dots, n$) and the central weight e . We present such a Seifert manifold as

$$S(e; (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)).$$

We call these integers the *Seifert invariant*. Instead of (a_i, b_i) , we also present it as $\alpha_1^{(i)} \cdot \alpha_2^{(i)} \cdots \alpha_{r_i}^{(i)}$. Thus, we denote the plumbing process by dot \cdot . We present several consecutive integers by the power as follows:

$$\dots \overbrace{2 \cdot 2 \cdots 2}^m \dots = \dots 2^{[m]} \dots$$

Brieskorn homology sphere $\Sigma(p, q, r)$ is defined to be

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^r = 0\} \cap S^5,$$

where p, q , and r are pairwise coprime positive integers. The manifold is a plumbed 3-manifold with a branch number three star-shaped graph. The Seifert invariant is

$$S(e; (p, p'), (q, q'), (r, r')),$$

where $e - (p'/p + q'/q + r'/r) = -1/pqr$. For example, the plumbing diagram of $\Sigma(2, 3, 5)$ is described as follows:

$$S(-2; (-2)^{[4]}, (-2)^{[2]}, -2).$$

In [14], the author showed that the Brieskorn homology sphere $\Sigma(p, q, r)$ whose intersection matrix of minimal plumbed 4-manifold is isomorphic to $-E_8$ is $\Sigma(2, 3, 5)$ or $\Sigma(3, 4, 7)$.

2.2. Notations. We explain two new notations as below. The first notation is the following:

$$(5) \quad L(a_1 \cdot a_2 \cdots a_n \cdot^{(k)} b_m \cdot b_{m-1} \cdots b_1),$$

It presents the plumbing as in FIGURE 1. Here the integers a_i , b_j and k are integers. A dot with (k) means the two components with framing a_n and b_m

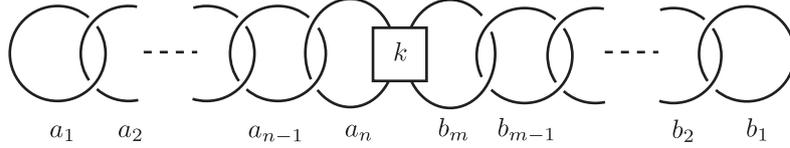


FIGURE 1. A surgery description of (5)

geometrically link k -times. Here, the box with the k means the full k -twist. We call the notation a *linear diagram*.

As an example, we consider a linear diagram of $\Sigma(2, 3, 5)$. Sliding the first branch of $S(-2; (-2)^{[4]}, (-2)^{[2]}, -2)$ to the third branch, we have the following linear diagram:

$$\begin{aligned} S(-2; (-2)^{[4]}, (-2)^{[2]}, -2) &= L((-2)^{[2]} \cdot (-2) \cdot (-2) \cdot^{(-2)} (-4) \cdot (-2)^{[3]}) \\ &= L((-2)^{[4]} \cdot^{(-2)} (-4) \cdot (-2)^{[3]}). \end{aligned}$$

The 4-manifold having the framed link as in FIGURE 1 is called *4-manifold having linear diagram (5)*.

The second notation is a linear diagram with torus knot component. Consider a linear diagram that the framing of the nearest component to (k) is zero. Then, the surgery diagram can be deformed as in FIGURE 2. We present the deformation of the linear diagram as follows:

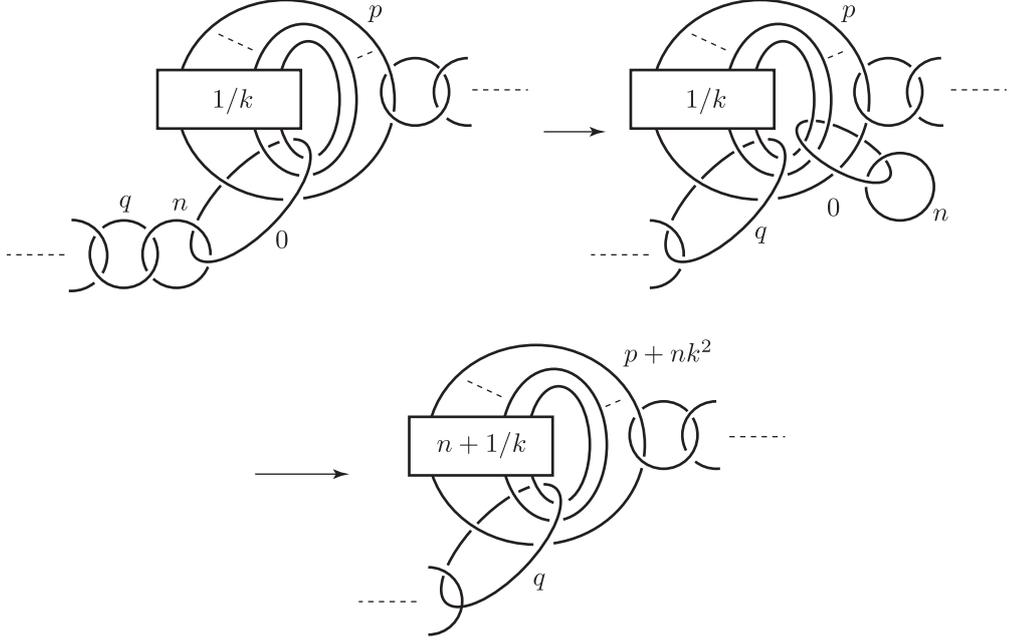
$$Y = L(\cdots q \cdot n \cdot 0 \cdot^{(k)} p \cdots) = L(\cdots q \cdot \underline{p + nk^2}_{(k, nk+1)} \cdots).$$

The component with the underline having index $(k, nk + 1)$ stands for the $(k, nk + 1)$ -torus knot with framing $p + nk^2$. The last surgery diagram gives a 4-manifold X bounded by Y . Clearly, the intersection form of X is isomorphic to the intersection form of the 4-manifold having the linear diagram as follows:

$$L(\cdots q \cdot^{(k)} (p + nk^2) \cdots).$$

2.3. An estimate of $\bar{\mu}$ -invariant. Neumann-Siebenmann's $\bar{\mu}$ -invariant in [7] is defined for any plumbed 3-manifold $M = \partial P(\Gamma)$. We set $\Gamma = (V, E, m)$. We assume that the plumbing graph is tree and $\partial P(\Gamma)$ is a homology sphere. We define the *Wu class* $w(\Gamma) \in H_2(P(\Gamma), \mathbb{Z})$ as follows:

- (1) The class $w(\Gamma)$ is written by $w(\Gamma) = \sum_{v \in V} \epsilon_v [v]$ for $\epsilon_v = 0$ or 1 .
- (2) For any $v \in V$ we have $(w(\Gamma), [v]) \equiv ([v], [v]) \pmod{2}$.

FIGURE 2. A formula of n -twisting

Let $\sigma(\Gamma)$ be the signature of the intersection form (\cdot, \cdot) associated with Γ . Then we define the $\bar{\mu}$ -invariant of M to be

$$\bar{\mu}(M) = \frac{\sigma(\Gamma) - w(\Gamma)^2}{8}.$$

The invariant $\bar{\mu}$ can be extended to any rational plumbed spin 3-manifold (M, c) naturally as in [7].

Theorem 2.1 ([17]). *Suppose that a Seifert rational homology 3-sphere M with spin structure c bounds a negative-definite spin 4-manifold Y with spin structure c_Y . Then*

$$\begin{aligned} b_2(Y) &\equiv -8\bar{\mu}(M, c) \pmod{16} \\ \bar{\mu}(M, c) &= \frac{\sigma(\Gamma) - w(\Gamma, c)^2}{8} \\ -\frac{8\bar{\mu}(M, c)}{9} &\leq b_2(Y) \leq -8\bar{\mu}(M, c). \end{aligned}$$

In particular, if a Seifert homology sphere M has a spin negative-definite filling Y , then $b_2(Y) \leq -8\bar{\mu}(M)$ holds.

3. THE FAMILIES OF BRIESKORN HOMOLOGY SPHERES IN THEOREM 1.1.

3.1. Proof of Theorem 1.3. We prove Theorem 1.3.

Proof. The Seifert invariants of

- $\Sigma(2, 14n - 5, 8n - 3), \Sigma(2, 24n + 5, 14n + 3)$
- $\Sigma(2, 26n + 5, 16n + 3), \Sigma(2, 10n - 3, 16n - 5)$

- $\Sigma(5, 35n - 2, 50n - 3), \Sigma(5, 25n - 2, 40n - 3)$
- $\Sigma(3, 15n - 2, 36n - 5), \Sigma(3, 9n - 2, 24n - 5)$
- $\Sigma(2, 14n - 5, 8n - 3), \Sigma(2, 16n - 5, 10n - 3)$
- $\Sigma(3, 21n - 4, 36n - 7), \Sigma(3, 27n - 4, 48n - 7)$
- $\Sigma(4, 28n - 3, 64n - 7), \Sigma(4, 32n - 3, 76n - 7)$

with reverse orientation are as follows: $S(1; (p_1, 1), (p_2, q_2), (p_3, q_3))$.

p_1	(p_2, q_2)	(p_3, q_3)	p_1	(p_2, q_2)	(p_3, q_3)
2	$(14n - 5, 7n - 6)$	$(8n - 3, 2)$	2	$(14n + 3, 7n - 2)$	$(24n + 5, 6)$
2	$(26n + 5, 13n - 4)$	$(16n + 3, 4)$	2	$(10n - 3, 5n - 4)$	$(16n - 5, 4)$
5	$(35n - 2, 28n - 3)$	$(50n - 3, 2)$	5	$(40n - 3, 32n - 4)$	$(25n - 2, 1)$
3	$(15n - 2, 10n - 3)$	$(36n - 5, 4)$	3	$(24n - 5, 16n - 6)$	$(9n - 2, 1)$
3	$(21n - 4, 14n - 5)$	$(36n - 7, 4)$	3	$(48n - 7, 32n - 10)$	$(27n - 4, 3)$
4	$(28n - 3, 21n - 4)$	$(64n - 7, 4)$	4	$(76n - 7, 57n - 10)$	$(32n - 3, 2)$

We deform the presentation as follows. The notation \sim stands for a deformation of presentation preserving the intersection form.

$$\begin{aligned} & S(1; 2, 2 \cdot (n + 1) \cdot 2^{[6]}, (2 - 4n) \cdot 2) = S(0, 2, (-2) \cdot n \cdot 2^{[6]}, (2 - 4n) \cdot 2) \\ & = L(2^{[6]} \cdot n \cdot 0 \cdot {}^{(2)}(4 - 4n) \cdot 2) = L(2^{[6]} \cdot {}^{(2)}\underline{4}_{(2, 2n+1)} \cdot 2) \sim L(2^{[6]} \cdot {}^{(2)}4 \cdot 2) \end{aligned}$$

$$\begin{aligned} & S(1; 2, 2 \cdot (n + 1) \cdot 4 \cdot 2, (-4n) \cdot 2^{[5]}) = S(0; 2, (-2) \cdot n \cdot 4 \cdot 2, (-4n) \cdot 2^{[5]}) \\ & = L(2^{[5]} \cdot (2 - 4n) \cdot {}^{(2)}0 \cdot n \cdot 4 \cdot 2) = L(2^{[5]} \cdot \underline{2}_{(2, 2n+1)} \cdot {}^{(2)}4 \cdot 2) \\ & \sim L(2^{[6]} \cdot {}^{(2)}4 \cdot 2) \end{aligned}$$

$$\begin{aligned} & S(1; 2, 2 \cdot (n + 1) \cdot 4 \cdot 2^{[3]}, (-4n) \cdot 2^{[3]}) = S(0; 2, (-2) \cdot n \cdot 4 \cdot 2^{[3]}, (-4n) \cdot 2^{[3]}) \\ & = L(2^{[3]} \cdot 4 \cdot n \cdot 0 \cdot {}^{(2)}(2 - 4n) \cdot 2^{[3]}) = L(2^{[3]} \cdot 4 \cdot {}^{(2)}\underline{2}_{(2, 2n+1)} \cdot 2^{[3]}) \\ & \sim L(2^{[3]} \cdot 4 \cdot {}^{(2)}2^{[4]}) \end{aligned}$$

$$\begin{aligned} & S(1; 2, 2 \cdot (n + 1) \cdot 2^{[4]}, (2 - 4n) \cdot 2^{[3]}) = S(0; 2, (-2) \cdot n \cdot 2^{[4]}, (2 - 4n) \cdot 2^{[3]}) \\ & = L(2^{[4]} \cdot n \cdot 0 \cdot {}^{(2)}(4 - 4n) \cdot 2^{[3]}) = L(2^{[4]} \cdot {}^{(2)}\underline{4}_{(2, 2n+1)} \cdot 2^{[3]}) \sim L(2^{[4]} \cdot {}^{(2)}4 \cdot 2^{[3]}) \end{aligned}$$

$$\begin{aligned} & S(0; -5, 5 \cdot n \cdot (-7), (2 - 25n) \cdot 2) = L((-7) \cdot n \cdot 0 \cdot {}^{(-5)}(-3 - 25n) \cdot 2) \\ & = L((-7) \cdot {}^{(-5)}\underline{-3}_{(-5, -5n+1)} \cdot 2) = L(2 \cdot 1 \cdot (-5) \cdot {}^{(-5)}\underline{-3}_{(-5, -5n+1)} \cdot 2) \end{aligned}$$

Here we slide the -3 -framed $(-5, -5n + 1)$ -torus knot component to the component with (-5) -framed component. Then we have a 4-manifold with intersection form of the plumbed 4-manifold for $S(1; 2, 2^{[2]}, -5)$. By doing four blow-ups and one blow-down, we have the intersection E_8 .

In the same way, the 4-manifolds that the last diagrams in the following equalities present can be deformed into 4-manifolds with intersection form

E_8 . The results of the latter four equalities are a plumbed 4-manifold that reduces to the Seifert invariant $S(2; 2^{[2]}, 2^{[4]}, 2 \cdot 4) = -\Sigma(3, 4, 7)$.

$$\begin{aligned} & S(0; -5, 5 \cdot n \cdot (-8), (2 - 25n)) = L((-3 - 25n) \cdot^{(-5)} 0 \cdot n \cdot (-5) \cdot 1 \cdot 2^{[2]}) \\ & = L(\underline{-3}_{(-5, -5n+1)} \cdot^{(-5)} (-5) \cdot 1 \cdot 2^{[2]}) \end{aligned}$$

$$\begin{aligned} & S(0; -3, 3 \cdot n \cdot (-5), (2 - 9n) \cdot 2^{[3]}) = L((-5) \cdot n \cdot 0 \cdot^{(-3)} (-1 - 9n) \cdot 2^{[3]}) \\ & = L(2 \cdot 1 \cdot (-3) \cdot^{(-3)} \underline{-1}_{(-3, -3n+1)} \cdot 2^{[3]}) \end{aligned}$$

$$\begin{aligned} & S(0; -3, 3 \cdot n \cdot (-8), (2 - 9n)) = L(2^{[4]} \cdot 1 \cdot (-3) \cdot n \cdot 0 \cdot^{(-3)} (-1 - 9n)) \\ & = L(2^{[4]} \cdot 1 \cdot (-3) \cdot^{(-3)} \underline{-1}_{(-3, -3n+1)}) \end{aligned}$$

$$\begin{aligned} & S(1; 2, 2 \cdot n \cdot (-7), (2 - 4n) \cdot 2^{[2]}) = S(0; -2, 2 \cdot n \cdot (-7), (2 - 4n) \cdot 2^{[2]}) \\ & = L(2^{[4]} \cdot 1 \cdot (-2) \cdot n \cdot 0 \cdot^{(-2)} (2 - 4n) \cdot 2^{[2]}) \\ & = L(2^{[4]} \cdot 1 \cdot (-2) \cdot^{(-2)} \underline{2}_{(-2, -2n+1)} \cdot 2^{[2]}) \end{aligned}$$

$$\begin{aligned} & S(1; 2, 2 \cdot n \cdot (-6), (2 - 4n) \cdot 2^{[3]}) = S(0; -2, 2 \cdot n \cdot (-6), (2 - 4n) \cdot 2^{[3]}) \\ & = L(2^{[3]} \cdot 1 \cdot (-2) \cdot n \cdot 0 \cdot^{(-2)} (-4n) \cdot 2^{[3]}) \\ & = L(2^{[3]} \cdot 1 \cdot (-2) \cdot^{(-2)} \underline{0}_{(-2, -2n+1)} \cdot 2^{[3]}) \end{aligned}$$

$$\begin{aligned} & S(0; -3, 3 \cdot n \cdot (-7), (2 - 9n) \cdot 4) = L(2^{[3]} \cdot 1 \cdot (-3) \cdot n \cdot 0 \cdot^{(-3)} (-1 - 9n) \cdot 4) \\ & = L(2^{[3]} \cdot 1 \cdot (-3) \cdot^{(-3)} \underline{-1}_{(-3, -3n+1)} \cdot 4) \end{aligned}$$

$$\begin{aligned} & S(0; -3, 3 \cdot n \cdot (-5) \cdot 3, (2 - 9n) \cdot 2^{[2]}) \\ & = L(2^{[2]} \cdot (-1 - 9n) \cdot^{(-3)} 0 \cdot n \cdot (-3) \cdot 1 \cdot 2 \cdot 4) \\ & = L(2^{[2]} \cdot \underline{-1}_{(-3, -3n+1)} \cdot^{(-3)} (-3) \cdot 1 \cdot 2 \cdot 4) \end{aligned}$$

$$\begin{aligned} & S(0; -4, 4 \cdot n \cdot (-7), (2 - 16n) \cdot 4) \\ & = L(4 \cdot (-2 - 16n) \cdot^{(-4)} 0 \cdot n \cdot (-4) \cdot 1 \cdot 2^{[2]}) \\ & = L(4 \cdot \underline{-2}_{(-4, -4n+1)} \cdot^{(-4)} (-4) \cdot 1 \cdot 2^{[2]}) \end{aligned}$$

$$\begin{aligned} & S(0; -4, 4 \cdot n \cdot (-6) \cdot 3, (-2 - 16n) \cdot 2) \\ & = L(2 \cdot (-2 - 16n) \cdot^{(-4)} 0 \cdot n \cdot (-4) \cdot 1 \cdot 2 \cdot 4) \\ & = L(2 \cdot \underline{-2}_{(-4, -4n+1)} \cdot^{(-4)} (-4) \cdot 1 \cdot 2 \cdot 4) \end{aligned}$$

According to the definition of $\bar{\mu}$ as above, computing the $\bar{\mu}$ -invariants for these Brieskorn homology spheres, we can see $\bar{\mu} = -1$ easily. From the description under Theorem 2.1 we obtain $g_8 \leq 1$. Namely, the homology spheres have all $g_8 = 1$.

□

4. BRIESKORN HOMOLOGY SPHERES WITH E_8 -FILLING AND ARBITRARILY LARGE CORRECTION TERMS.

4.1. Heegaard Floer homology and one preparation. In [9] for any spin^c rational homology sphere (Y, \mathfrak{s}) the Heegaard Floer homology $HF^+(Y, \mathfrak{s})$ has the following exact sequence:

$$0 \rightarrow T_{d(Y, \mathfrak{s})}^+ \rightarrow HF^+(Y, \mathfrak{s}) \rightarrow HF_{\text{red}}(Y, \mathfrak{s}) \rightarrow 0.$$

$T_{\mathfrak{s}}^+$ is isomorphic to $T^+ := \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$ with the minimal degree s . $HF_{\text{red}}(Y, \mathfrak{s})$ is a finite dimensional torsion $\mathbb{F}[U]$ -module. $d(Y, \mathfrak{s})$ is called the *correction term* of (Y, \mathfrak{s}) . We call the submodule $T_{d(Y, \mathfrak{s})}^+$ in $HF^+(Y, \mathfrak{s})$ the T^+ -part of $HF^+(Y, \mathfrak{s})$.

Here we prepare a lemma to prove Theorem 1.4. We abbreviate $d(L(p, q), i)$ by $d(p, q, i)$. Here we define the lens space $L(p, q)$ to be the p/q -surgery of the unknot. The identification of spin^c structures with $\mathbb{Z}/p\mathbb{Z}$ is due to Fig.2 in [9]. Here the p -Dehn surgery of a knot K in a homology sphere Y is the surgery $(Y \setminus S^1 \times D^2) \cup V$, where $V \cong S^1 \times D^2$. Here the attaching meridian of the new solid torus V is mapped to $p \cdot [m] + [l] \in H_1(\partial(S^1 \times D^2))$ where m is the meridian of K and l is the homologically trivial longitude of K . We denote the p -Dehn surgery of a knot in Σ by $\Sigma(p)$.

Lemma 4.1. *Let Σ be a homology sphere and $K \subset \Sigma$ a knot. For some positive integer p , if $\Sigma(p-1)$ is an L-space and $\Sigma(p)$ is a lens space $L(p, q)$, then the correction term $d(\Sigma)$ is computed as follows.*

$$(6) \quad d(\Sigma) = \max \{d(p, q, ki + c) - d(p, 1, i) \mid 0 \leq i < p\},$$

where k is the dual class of $[\tilde{K}] \in H_1(L(p, q), \mathbb{Z})$ and $c = (k+1+p)(k-1)/2$, where \tilde{K} is the surgery dual of the lens space surgery.

Let C be a core circle of the genus one Heegaard decomposition. Then the *dual class* k is defined $k[C] = [\tilde{K}] \in H_1(L(p, q), \mathbb{Z})$. The dual class is used in the situation of the integral lens space surgery of a homology sphere in [16].

Proof. We use the following surgery exact sequence (Corollary 9.13 in [9]):

$$\dots \rightarrow HF^+(\Sigma) \rightarrow HF^+(\Sigma(p-1)) \xrightarrow{G^+} HF^+(\Sigma(p)) \xrightarrow{F^+} HF^+(\Sigma) \rightarrow \dots$$

Since $\Sigma(p-1)$ and $\Sigma(p)$ are L-spaces and the corresponding map F^∞ on HF^∞ is surjective, F^+ is also surjective onto the T^+ -part in $HF^+(\Sigma)$. The map F^+ is induced from the cobordism $\Sigma(p)$ to Σ obtained by attaching a 0-framed 2-handle along the meridian of \tilde{K} . The spin^c structures on $\Sigma(p)$ are identified with $\mathbb{Z}/p\mathbb{Z}$ due to the description in p.213 in [9]. For any integer j with $0 \leq j < p$ consider the surgery exact sequence in Theorem 9.19 in [8]:

$$\dots \rightarrow HF^+(\Sigma) \rightarrow HF^+(\Sigma(0), [j]) \rightarrow HF^+(\Sigma(p), j) \xrightarrow{F_j^+} HF^+(\Sigma) \rightarrow \dots$$

F_j^+ is a component of F^+ restricted to the spin^c structure j . It is also a sum of homogeneous maps f_i^+ with respect to the spin^c cobordism from $(\Sigma(p), j)$ to the unique spin^c manifold on Σ . Namely, F_j^+ is described by the sum $F_j^+ = \sum_{j \equiv i \pmod p} f_i^+$. The degree shift of f_i^+ is $(4p - (2i - p)^2)/(4p)$ due to [9]. The maximal degree shift among $\{f_i^+ | j \equiv i \pmod p\}$ is $(4p - (2j - p)^2)/(4p) = -d(p, 1, j)$. Since F_j^+ is a surjective U -equivariant map, for $0 \leq j < p$ we have

$$d(\Sigma) \geq d(p, q, kj + c) - d(p, 1, j).$$

The 1-1 correspondence $\mathbb{Z}/p\mathbb{Z} \rightarrow \text{Spin}^c(L(p, q))$ in Corollary 7.5 in [9] is described by $ki + c$. See [15].

Suppose that $d(\Sigma) > d(p, q, kj + c) - d(p, 1, j)$ for any integer j with $0 \leq j < p$. Then any element with the minimal degree in $HF^+(\Sigma(p))$ is included in the kernel of $F^+ = \sum_{0 \leq j < p} F_j^+$. Thus the kernel of F^+ includes at least p components. On the other hand, for a sufficient large number N , $\ker(F^+)/(U^N = 0)$ is $(p - 1)$ -fold direct sum of T^+ from the exact sequence of the version of HF^∞ . Hence, this implies that in the image of G^+ there is a torsion $\mathbb{F}[U]$ -module by at least one component. However, since $\Sigma(p - 1)$ is an L-space, the image of G^+ has no torsion $\mathbb{F}[U]$ -module. This is a contradiction. Therefore for some j , $d(\Sigma) = d(p, q, kj + c) - d(p, 1, j)$ holds. \square

4.2. The d -invariants for the four families of Brieskorn homology spheres. We prove Theorem 1.4.

Proof. The Seifert presentations of Brieskorn homology spheres from (i) to (iv) in Theorem 1.3 are the below:

(i)	$S(1; 2, 2 \cdot (-n + 1) \cdot 7, (4n - 1) \cdot 2)$
(ii)	$S(1; 2, 2 \cdot (-n) \cdot (-3) \cdot 2, (4n + 1) \cdot 6)$
(iii)	$S(1; 2, 2 \cdot (-n) \cdot (-3) \cdot 4, (4n + 1) \cdot 4)$
(iv)	$S(1; 2, 2 \cdot (-n + 1) \cdot 5, (4n - 1) \cdot 4)$

Let Σ_n be one of Brieskorn homology spheres parametrized by n in the list above. We do 0-surgery and +1-surgery of the homology sphere along the meridian of the singular fiber of multiplicity 2. We call the meridian K_n . Note the coefficients 0 and 1 are the framing of the unknot K_n in the diagram. The 0-surgery and 1-surgery give lens spaces $L(r_n, s_n)$ and $L(p_n, q_n)$. The results are the lens spaces in the list below.

	0-surgery (r_n, s_n)	1-surgery (p_n, q_n)
(i)	$(56n^2 - 41n + 7, 8n^2 - 7n + 2)$	$(56n^2 - 41n + 8, 8n^2 - 7n + 1)$
(ii)	$(168n^2 + 71n + 7, 72n^2 + 27n + 4)$	$(168n^2 + 71n + 8, 72n^2 + 27n + 1)$
(iii)	$(208n^2 + 79n + 7, 48n^2 + 17n + 2)$	$(208n^2 + 79n + 8, 48n^2 + 17n + 1)$
(iv)	$(80n^2 - 49n + 7, 16n^2 - 13n + 4)$	$(80n^2 - 49n + 8, 16n^2 - 13n + 1)$

These examples satisfy $p_n = r_n + 1$. As a result, the 0-surgery means a positive r_n -Dehn surgery along K_n .

We set $d_n := d(\Sigma_n)$. Here using Lemma 4.1, we compute the lower bound of d_n . We argue the case of (i) only. Other cases are able to be proven by

similar arguments. Let Σ_n be the Brieskorn homology sphere in the type (i). We set $p_n = 56n^2 - 41n + 8$, $q_n = 8n^2 - 7n + 1$, $k_n = 14n - 5$ and $c_n \equiv 42n^2 - 29n + 4 \pmod{p_n}$. The k_n is the dual class in the lens space $L(p_n, q_n)$ which presents K_n .

Here we set $i = \lfloor \frac{q_n+1}{2} \rfloor - n$. Then by modulo p_n we have

$$k_n i + c_n \equiv \begin{cases} \frac{7n-5}{2} & n: \text{ odd} \\ -\frac{7n}{2} & n: \text{ even.} \end{cases}$$

If n is an odd number, then by using the reciprocity formula in [9], we have

$$d\left(L(p_n, q_n), \frac{7n-5}{2}\right) = \frac{224n^3 + 8n^2 - 95n + 25}{4p_n}$$

and

$$d(L(p_n, 1), i) = -\frac{52n^2 - 37n + 7}{4p_n}$$

Thus we have

$$d(L(p_n, q_n), k_n i + c_n) - d(L(p_n, 1), i) = n + 1.$$

If n is an even number, then we have

$$d\left(L(p_n, q_n), -\frac{7n}{2}\right) = \frac{224n^3 - 216n^2 + 73n - 8}{4p_n}$$

$$d(L(p_n, 1), i) = -\frac{52n^2 - 41n + 8}{4p_n}.$$

Thus we have

$$d(L(p_n, q_n), k_n i + c_n) - d(L(p_n, 1), i) = n.$$

Therefore we have $d_n \geq 2\lceil \frac{n}{2} \rceil$. \square

4.3. Proof of Corollary 1.9. Here we prove Corollary 1.9.

Proof. Let $\{\Sigma_n\}$ be one family of homology spheres in Theorem 1.4. Since the correction term is positive, Σ_n is no even positive-definite filling of Σ_n . A Seifert homology sphere Σ_n has a negative-definite plumbing $\Sigma_n = \partial P(\Gamma_n)$. Due to Corollary 1.5 in [10], $4d(\Sigma_n) = \max_{c \in \Xi} (c^2 + \text{rank}(\Gamma_n))$ holds, where Ξ_n is the set of characteristic classes in Γ_n . If for a non-negative integer N , $\Gamma_n = \Gamma'_n \oplus \langle -1 \rangle^N$ and Γ'_n is a minimal sub-lattice, then $\max_{c \in \Xi} (c^2 + \text{rank}(\Gamma_n))$ is decomposed into the sum of the two maximal values according to the direct sum. Thus, we have

$$\max_{c \in \Xi} (c^2 + \text{rank}(\Gamma_n)) = \max_{c \in \Xi'} (c^2 + \text{rank}(\Gamma'_n)) \leq \text{rank}(\Gamma'_n),$$

where Ξ' is the set of characteristic elements of L' . Since $d(\Sigma_n)$ has no upper bound, the maximal of the rank of minimal lattice Γ'_n is also unbounded. On the other hands, $\bar{\mu}(Y_n) = 1$ and our construction of filling of Σ_n implies that $\partial \mathfrak{s}(Y_n) = 1$. Therefore $\mathfrak{o}(\Sigma_n) - \partial \mathfrak{s}(Y_n)$ is unbounded. \square

REFERENCES

- [1] N. E. Elkies, *A characterization of the \mathbb{Z}^n lattice*, Math. Res. Lett. 2 (1995), no. 3, 321–326.
- [2] R. Fintushel and R. Stern, *A μ -invariant one homology 3-sphere that bounds an orientable rational ball*, Four-manifold theory (Durham, N.H., 1982), 265–268, Contemp. Math., 35, Amer. Math. Soc., Providence, RI, 1984.
- [3] M. Golla and C. Scaduto, *On definite lattices bounded by integer surgeries along knots with slice genus at most 2*, arXiv:1807.11931.
- [4] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999. xvi+558 pp.
- [5] C. Karakurt, HFNem2, (Computer program).
- [6] C. Manolescu, *Pin(2)-equivariant Seiberg-Witten Floer homology and the Triangulation Conjecture*, Journal of the American Mathematical Society, Vol. 29 (2016), 147–176.
- [7] W. Neumann, *An invariant of plumbed homology spheres*, Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979), pp. 125–144, Lecture Notes in Math., 788, Springer, Berlin, 1980.
- [8] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) 159 (2004), no. 3, 1159–1245.
- [9] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003), no. 2, 179–261.
- [10] P. Ozsváth and Z. Szabó, *On the Floer homology of plumbed three-manifolds*, Geom. & Top. Vol. 7 (2003) 185–224.
- [11] N. Saveliev, *Invariants for homology 3-spheres*, Encyclopaedia of Mathematical Sciences, 140. Low-Dimensional Topology, I. Springer-Verlag, Berlin, 2002. xii+223 pp.
- [12] C. Scaduto, *On definite lattices bounded by a homology 3-sphere and Yang-Mills instanton Floer theory*, arXiv:1805.07875.
- [13] M. Stoffregen *Pin(2)-equivariant Seiberg-Witten Floer homology of Seifert fibrations*, arXiv:1505.03234.
- [14] M. Tange, *The E_8 -boundings of homology spheres and negative sphere classes in $E(1)$* , Topol. Appl. 202 (2016), 160–182.
- [15] M. Tange, *Ozsváth-Szabó’s correction term and lens surgery*, Math. Proc. Cambridge Philos. Soc. 146 (2009), no. 1, 119–134.
- [16] M. Tange, *Homology spheres yielding lens lens spaces*, Proceedings of 24th Gökova Geometry-Topology Conference pp.73–121.
- [17] M. Ue, *The Fukumoto-Furuta and the Ozsváth-Szabó invariants for spherical 3-manifolds*, Algebraic topology old and new, 121–139, Banach Center Publ., 85, Polish Acad. Sci. Inst. Math., Warsaw, 2009.
- [18] M. Ue, *The Neumann-Siebenmann invariant and Seifert surgery*, Math. Z. 250 (2005), 475–493.

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