THE E_8-BOUNDINGS OF HOMOLOGY SPHERES AND NEGATIVE SPHERE CLASSES IN $E(1)$.

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ABSTRACT. We define topological invariants of homology 3-sphere, $\mathfrak{ds}$ and $\underline{\mathfrak{ds}}$, which are the maximal and minimal second Betti number divided by 8 among definite spin boundings of the homology sphere. We also define similar invariants g_8 and $\underline{g_8}$ by the maximal (or minimal) product sum of the quadratic form E_8 of bounding 4-manifolds. The aim of these invariants is to measure the size of bounding definite spin 4-manifold. We give several ways to construct definite spin boundings. In particular, we construct uncommon E_8-boundings for $\Sigma(2,3,12n+5)$ by using handle decomposition. As a by-product of this construction, we show that some negative 2nd homology classes $k[f]-[s]$ in $E(1)$ are represented by a sphere, where f and s are a fiber and sectional class of $E(1)$.

1. INTRODUCTION

1.1. **The spin definite bounding and related invariants.** It is well-known that any closed 3-manifold Y is the boundary of a spin 4-manifold X. Furthermore, if we set some conditions of the intersection form of X, it becomes unclear whether there exists the bounding with those conditions. The Rokhlin theorem says that homology sphere Y with the Rokhlin invariant $\mu(Y)=1$ cannot bound any smooth spin 4-manifold with $\sigma(X)\equiv 0 \bmod 16$.

Let X be a spin bounding of a homology sphere Y. We can construct a new spin bounding increasing the one positive and negative eigenvalues of the intersection form by taking connected-sum $X\#S^2\times S^2$. In this paper we focus on the construction of spin boundings without positive or negative eigenvalues of the intersection matrix, i.e., $b_2(X)=|\sigma(X)|$. Such boundings of homology spheres are called *spin negative- (or positive-) definite boundings.*

Ozsváth and Szabó in [17] defined the integer-valued homology cobordism invariant $d(Y)$ for any homology sphere Y. It is called *the correction term* or *d-invariant.* By using this homology cobordism invariant they obtained the following:

Date: January 14, 2016.

1991 *Mathematics Subject Classification.* 57R55, 57R65.

Key words and phrases. Definite spin 4-manifold, Brieskorn homology 3-sphere, minimal genus surface, E_8-plumbing.

The author was partially supported by JSPS KAKENHI Grant Number 24840006.

Theorem 1.1 ([17])**.** *Let Y be an integral homology 3-sphere. Then any negative-definite bounding X of Y satisfies the inequality*

$$\xi^2 + rk(H^2(X;\mathbb{Z})) \leq 4d(Y) \tag{1}$$

for each characteristic vector $\xi \in H^2(X,\mathbb{Z})$.

Furthermore, if Y has a spin negative-definite bounding X, then the inequality (1) implies

$$b_2(X) \leq 4d(Y). \tag{2}$$

Here, Elkies' result in [4] guarantees that for an n-dimensional negative-definite unimodular quadratic form (V,Q), the inequality

$$\max_{\xi\in\Xi(V)} Q(\xi,\xi) + n \geq 0$$

holds, where $\Xi(V)$ is the set of the characteristic vectors in the from (V,Q).

Hence, the condition $d(Y) \geq 0$ is a necessary condition for the integral homology 3-sphere Y to have a negative-definite bounding.

We introduce another condition for spin negative-definite bounding. Let $\overline{\mu}$ be the Neumann-Siebenmann invariant defined in [15]. In [18], Ue shows the following:

Theorem 1.2 ([18])**.** *Suppose that a Seifert rational homology 3-sphere Y with spin structure c bounds a negative-definite 4-manifold X with spin structure c_X. Then*

$$b_2(X) \equiv -8\bar{\mu}(Y,c) \mod 16,$$

$$-\frac{8\bar{\mu}(Y,c)}{9} \leq b_2(X) \leq -8\bar{\mu}(Y,c).$$

Hence, $\overline{\mu}(Y,c) \leq 0$ is a necessary condition for a Seifert spin rational homology 3-sphere to have a spin negative-definite bounding. On the other hand, the inequality does not guarantee the existence of such bounding X.

A topological space X is said to be *homologically 1-connected*, if it is connected and $H_1(X,\mathbb{Z}) = \{0\}$. In this article, we assume that the bounding 4-manifolds are homologically 1-connected. If a 4-manifold X is homologically 1-connected, then what X is a spin manifold is equivalent to what X has an even intersection form. There exists a non homologically 1-connected 4-manifold which is non-spin and has an even intersection form.

We define the following invariants.

Definition 1.1. *Let Y be a homology 3-sphere. If Y has a definite spin bounding, then we define $\epsilon(Y)$ as follows:*

$$\epsilon(Y) = \begin{cases} 1 & Y \textit{ has a positive-definite spin bounding } X \textit{ with } b_2(X) > 0 \\ -1 & Y \textit{has a negative-definite spin bounding } X \textit{ with } b_2(X) > 0 \\ 0 & Y \textit{ has a bounding } X \textit{ with } b_2(X) = 0. \end{cases}$$

If Y does not have any definite spin bounding, then we define $\epsilon(Y) = \infty$. Here, the boundings are all assumed homologically 1-connected.

The invariant ϵ is well-defined. In fact, if a homology 3-sphere Y has two boundings X_1, X_2 for two among $\{1,-1,0\}$, then $X = X_1 \cup (-X_2)$ is a definite spin closed 4-manifold with $b_2(X) > 0$. Donaldson's diagonalization theorem in [3] denies the existence of X.

Definition 1.2. *Let Y be a homology 3-sphere. We define invariants $\mathfrak{ds}, \underline{\mathfrak{ds}}$ on homology 3-spheres as follows:*

$$\mathfrak{ds}(Y) = \underline{\mathfrak{ds}}(Y) = \infty \Leftrightarrow \epsilon(Y) = \infty.$$

and otherwise,

$$\mathfrak{ds}(Y) = \max\left\{\frac{b_2(X)}{8}|\partial X = Y,\ b_2(X) = |\sigma(X)|,\ w_2(X) = 0\right\}$$

$$\underline{\mathfrak{ds}}(Y) = \min\left\{\frac{b_2(X)}{8}|\partial X = Y,\ b_2(X) = |\sigma(X)|, w_2(X) = 0\right\}.$$

We assume the spin boundings are all homologically 1-connected.

These invariants $\mathfrak{ds}$ and $\underline{\mathfrak{ds}}$ measure the size of definite spin bounding 4-manifolds. The rank of unimodular definite quadratic forms with even type is divisible by 8. Even type means that the square for any element is even. Thus, the values of these invariants are in $\mathbb{N} \cup \{0,\infty\}$. By the defintion $0 \le \underline{\mathfrak{ds}}(Y) \le \mathfrak{ds}(Y)$ holds. We do not know whether there exists a homology 3-sphere with $\underline{\mathfrak{ds}}(Y) \neq \mathfrak{ds}(Y)$. The property (14) in Theorem 2.1 in Section 2 proves that the difference is bounded by $\underline{\mathfrak{ds}}$.

The invariants $\mathfrak{ds}$ can be taken arbitrarily large. The examples below will be computed in Section 3.2.

For positive integer n, Brieskorn homology 3-spheres

$$\Sigma(4n-2, 4n-1, 8n-3),\ \Sigma(4n-1, 4n, 8n-1)$$

$$\Sigma(4n-2, 4n-1, 8n^2-4n+1),\ \Sigma(4n-1, 4n, 8n^2-1)$$

have $\mathfrak{ds} = n$. This will be proven in Theorem 1.3.

Suppose that a homology 3-sphere Y has a homologically 1-connected bounding X satisfying

$$\partial X = Y,\ Q_X = nE_8,$$

where n is a negative integer, then nE_8 is a direct product of $(-n)$-copies of the negative-definite quadratic form with E_8-type. Then, we call the spin bounding X *(homologically 1-connected) E_8-bounding.* If the bounding is positive-definite, we call the bounding *positive E_8-bounding*, and if the bounding is negative-definite, the bounding *negative E_8-bounding.*

Definition 1.3 (E_8-genera)**.** *Let Y be a homology 3-sphere with finite $\epsilon(Y)$. If Y has an E_8-bounding, then we define the E_8-genera as follows:*

$$g_8(Y) = \max\{|n||Y = \partial X, w_2(X) = 0, H_1(X) = \{0\},\ \textit{and}\ Q_X = nE_8\}$$

$$\underline{g_8}(Y) = \min\{|n||Y = \partial X, w_2(X) = 0, H_1(X) = \{0\},\ \textit{and}\ Q_X = nE_8\},$$

If Y does not have any E_8-bounding, then we define $g_8(Y)$ to be

$$g_8(Y) = +\infty.$$

Even if a homology 3-sphere Y has finite $\epsilon(Y)$, it is not known whether Y has an E_8-bounding or not.

These invariants, $g_8, \underline{g_8}$ are different from $\mathfrak{ds}, \underline{\mathfrak{ds}}$ in terms of the classification of the quadratic unimodular form, because there are tremendous variations of even definite unimodular forms, unlikely even indefinite unimodular forms, which are classified as $aE_8 \oplus bH$ for some integers a, b. In the 8, 16, and 24-dimensions, there are 1, 2 and 24 isomorphism types respectively. However, for example in 32-dimension, the number of isomorphism types of even, definite, unimodular forms are more than 8×10^7. It is non-trivial whether Y has a bounding with $Q_X \cong nE_8$. That is the reason why we define two types of invariants that measures the dimension of even definite 4-manifolds bounding a homology sphere.

We introduce other related invariants due to the definition by Y. Matsumoto.

Definition 1.4 (Matsumoto,[13]). *Let Y be a homology 3-sphere. Then the bounding genus $|Y|$ of Y is defined to be*

$$|Y| := \begin{cases} \min\{n | \partial X = Y, Q_X = nH\} & \mu(Y) = 0, \\ \infty & \mu(Y) = 1, \end{cases}$$

where the bounding 4-manifold X is restricted to homologically 1-connected 4-manifold and H is the quadratic form represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

This invariant is considered as an homology cobordism invariant

$$|\cdot| : \Theta^3_{\mathbb{Z}} \to \mathbb{N} \cup \{0, \infty\}.$$

Even when we impose to being simply-connected, we can give a homology cobordism invariant. That is, we define $|Y|_\pi$ to be the minimal n which $Q_X = nH$ and X is simply-connected. If we have a homologically 1-connected bounding with $Q_X = nH$, by a surgery of X, we can give a simply-connected bounding X' with the intersection form several direct sum of H. However, in general $b_2(X)$ may not be different from $b_2(X')$. Since by the surgery we may have to increase the H-component of the intersection form of the bounding 4-manifold, in general the following is satisfied:

$$|Y| \leq |Y|_\pi.$$

In [12] the ξ-invariant for any homology sphere Y is defined as follows:

$$\xi(Y) = \max\{p - q | p, q \in \mathbb{Z}, q > 0, p(-E_8) \oplus qH = Q_X \text{ and } w_2(X) = 0, \partial X = Y\}.$$

Bohr and Lee's m in [2] and $\overline{m}$ are defined as follows:

$$m(Y) = \max\left\{\frac{5}{4}\sigma(X) - b_2(X) | \partial X = Y, \text{ and } w_2(X) = 0\right\}$$

$$\overline{m}(Y) = \min\left\{\frac{5}{4}\sigma(X) + b_2(X) | \partial X = Y, \text{ and } w_2(X) = 0\right\}$$

Here, the relationship between m and ξ are as follows:

$$m(-Y)/2 = \max\left\{\frac{b_2(N)}{8} - q | q \in \mathbb{Z}, \partial X = Y, Q_X \cong N \oplus qH, w_2(X) = 0 \right.$$
$$\left. \text{and } N : \text{even negative-definite form}\right\}.$$

Thus we have

$$m(-Y)/2 \leq \xi(Y) + 1,$$

as seen in [12].

1.2. **Some connection to $(11/8)$-conjecture.** We state the $(11/8)$-conjecture by Y. Matsumoto and a similar conjecture in terms of $\mathfrak{ds}$ and bounding genus.

Conjecture 1.1 (Y. Matsumoto ($(11/8)$-conjecture))**.** *If X is a closed, oriented, smooth, spin 4-manifold and Q_X is equivalent to $2k(-E_8) \oplus lH$, then $l \geq 3|k|$ holds.*

Conjecture 1.2. *Suppose that Y is a homology 3-sphere with $\mu(Y) = 0$ and $\mathfrak{ds}(Y) < \infty$. Then the following is satisfied:*

$$|Y| \geq \frac{3}{2}\mathfrak{ds}(Y).$$

Proposition 1.1. *Suppose that Y is a homology 3-sphere with $\mu(Y) = 0$ and $\mathfrak{ds}(Y) < \infty$. If $(11/8)$-conjecture is true, then the following is satisfied:*

$$|Y| \geq \frac{3}{2}\mathfrak{ds}(Y).$$

Under the same condition of Y as above, if Conjecture 1.2 is true, then for any simply-connected 4-manifolds $(11/8)$-conjecture holds.

These invariants $\mathfrak{ds}$, $\underline{\mathfrak{ds}}$, g_8 and $\underline{g_8}$ might be useful to construct counterexamples for $(11/8)$-conjecture, by finding an example not satisfying the above inequality. Conversely, for the $(11/8)$-conjecture to be true, homology 3-spheres with bounded bounding genus must have at least bounded $\mathfrak{ds}$, or $\mathfrak{ds} = \infty$ (Proposition 1.1).

1.3. **Examples of negative-definite spin boundings.** The aim of this paper is to find negative-definite spin boundings or E_8-boundings for some types of Brieskorn homology 3-spheres $\Sigma(a_1, a_2, \cdots, a_n)$. In this section we list the several results below which are proven later. The $\mathfrak{ds}$-invariants of all the examples are $0 \leq \mathfrak{ds} < \infty$.

In the subsection 3.1 we obtain examples of spin definite boundings by the Milnor-fiber construction, we get the following:

Theorem 1.3. *For any integer n, we set $M_n = \Sigma(2,3,6n-1)\#(-\Sigma(2,3,6n-5))$ then $\epsilon(M_n) = -1$ and $g_8(M_n) = 1$.*

The minimal resolutions of Brieskorn singularities give definite boundings for the homology 3-spheres. We will classify all the minimal resolutions of Brieskorn singularities with boundings with $g_8 = 1$ and $\epsilon = -1$.

Theorem 1.4. *If the minimal resolution of Brieskorn singularity gives a bounding with $g_8 = 1$ and $\epsilon = -1$, then the homology 3-sphere is one of $\Sigma(2,3,5)$, $\Sigma(3,4,7)$, $\Sigma(2,3,7,11)$, $\Sigma(2,3,7,23)$ or $\Sigma(3,4,7,43)$.*

We give some examples of minimal resolutions of the Brieskorn singularity with large $\mathfrak{ds}$:

Theorem 1.5. *For any integer n, we have*

$$\mathfrak{ds}(\Sigma(4n-2, 4n-1, 8n-3)) = \mathfrak{ds}(\Sigma(4n-1, 4n, 8n-1)) = n$$

$$\mathfrak{ds}(\Sigma(4n-2, 4n-1, 8n^2-4n+1)) = \mathfrak{ds}(\Sigma(4n-1, 4n, 8n^2-1)) = n.$$

In the last section we will post a question (Question 5.5) related to the (11/8)-conjecture and the bounding genus.

Even if the minimal resolution itself of a Brieskorn singularity does not give a spin 4-manifold, in some cases the additional blow-downs of the 4-manifold can give a spin manifold.

Let (G, a, b, c) be a 1-cycled weighted graph G as in the left of Figure 9. The labels on two edges on G are given by 3 integers labeled by a, b with $\gcd(a, b) = 1$ as drawn in the figure and the other (unlabeled) edges are labeled by 1. The weight on the vertex intersected by the two edges with a and b is $-2c$ and the other (unweighted) vertices are weighted by -2. Such a graph can give a smooth 4-manifold with a boundary. This description by a weighted graph is just a short hand to describe a bounding 4-manifold. It is similar to the plumbing graph with cycles, however, they are different objects each other.

The handle diagram of the manifold is drawn in Figure 9. The component weighted by $-2c$ is the (a, b)-torus knot.

Theorem 1.6. *A quadruple $(G; a, b, c)$ in* Table *1 with $\gcd(a, b) = 1$ gives a Brieskorn homology 3-sphere Σ with $g_8(\Sigma) = 1$ and $\epsilon(\Sigma) = -1$. Here G (from (1) to (8)) is one of graphs in* Figure *1.*

In the case of $((1); 1, b, c)$, for some non-negative integer m the homology 3-spheres $\Sigma(p, q, r)$ with the pairs p, q, r in Table *2 have boundings with $g_8 = 1$ and $\epsilon = -1$.*

Hence, any Brieskorn 3-sphere above satisfies $g_8(\Sigma) = 1$.

These E_8-boundings are constructed by blow-downs of minimal, negative-definite resolutions of Brieskorn singularities.

1.4. **Other examples.** Let Y_n^- denote $\Sigma(2, 3, 6n-1)$. Then the Neumann-Siebenmann invariant $\bar{\mu}$ is computed as follows:

$$\bar{\mu}(Y_n^-) = \begin{cases} -1 & n \equiv 1 \bmod 2 \\ 0 & n \equiv 0 \bmod 2. \end{cases} \tag{3}$$

As a corollary of Theorem 1.2 [18], if $\mathfrak{ds}(Y_n^-) < \infty$, then $g_8(Y_{2k+1}^-) = 1$ and $g_8(Y_{2k}^-) = 0$ hold. In this paper we show the existence of negative-definite spin boundings of Y_{2k+1}^- for some of k.

G	a	b	c
(1)	$3k-2\ell\pm 2$	$-2k+3\ell\mp 2$	$3k^2-4k\ell+3\ell^2\pm 2(2k-2\ell)+2$
(2)	$4k-\ell\pm 2$	$-3k+2\ell\mp 2$	$6k^2-3k\ell+\ell^2\pm 2(3k-\ell)+2$
(3)	$4k-3\ell\pm 2$	$-3k+4\ell\mp 2$	$6k^2-9k\ell+6\ell^2\pm 2(3k-3\ell)+2$
(4)	$5k-2\ell\pm 2$	$-4k+3\ell\mp 2$	$10k^2-8k\ell+3\ell^2\pm 2(4k-2\ell)+2$
(5)	$6k-\ell\pm 2$	$-5k+2\ell\mp 2$	$15k^2-5k\ell+\ell^2\pm 2(5k-\ell)+2$
(6)	$12k-4\ell\pm 3$	$-10k+6\ell\mp 3$	$60k^2-40k\ell+12\ell^2\pm 6(5k-2\ell)+4$
(6)	$12k-4\ell\pm 5$	$-10k+6\ell\mp 5$	$60k^2-40k\ell+12\ell^2\pm 10(5k-2\ell)+11$
(6)	$12k-4\ell\pm 1$	$-10k+6\ell$	$60k^2-40k\ell+12\ell^2\pm 10k+1$
(6)	$12k-4\ell\pm 3$	$-10k+6\ell\mp 2$	$60k^2-40k\ell+12\ell^2\pm 2(15k-4\ell)+4$
(7)	$14k-2\ell\pm 3$	$-12k+4\ell\mp 3$	$84k^2-24k\ell+4\ell^2\pm 6(6k-\ell)+4$
(7)	$14k-2\ell\pm 5$	$-12k+4\ell\mp 5$	$84k^2-24k\ell+4\ell^2\pm 10(6k-\ell)+11$
(7)	$14k-2\ell\pm 2$	$-12k+4\ell\mp 1$	$84k^2-24k\ell+4\ell^2\pm 2(12k-\ell)+2$
(7)	$14k-2\ell\pm 4$	$-12k+4\ell\mp 3$	$84k^2-24k\ell+4\ell^2\pm 6(8k-\ell)+7$
(8)	$4k-3\ell$	$4\ell\pm 1$	$16k^2+6\ell^2\pm 3(2k+\ell)+1$
(8)	$4k-3\ell\pm 1$	$4\ell\pm 3$	$16k^2+6\ell^2\pm (26k+9\ell)+14$

TABLE 1. The negative-definite E_8-boundings for $(G; a, b, c)$ in FIGURE 1

p	q	r
$10i+7$	$15i+8$	$120i^2+148i+45$
$10i+3$	$15i+2$	$120i^2+52i+5$
$20i-8$	$30i-17$	$480i^2-464i+109$
$20i+8$	$30i+7$	$480i^2+304i+45$
$30i-13$	$45i-27$	$1080i^2-1116i+281$
$30i-7$	$45i-18$	$1080i^2-684i+101$
$30i+7$	$45i+3$	$1080i^2+324i+17$
$30i+13$	$45i+12$	$1080i^2+756i+125$
$20i+2$	$30i-7$	$480i^2-64i-11$
$20i-2$	$30i-23$	$480i^2-256i+21$
$10i+7$	$15i-2$	$120i^2+68i-365$
$10i+13$	$15i+7$	$120i^2+212i+73$
$60i-28$	$90i-57$	$4320i^2-4752i+1277$
$60i-8$	$90i-27$	$4320i^2-1872i+173$
$60i+8$	$90i-3$	$4320i^2+432i-19$
$60i+28$	$90i+27$	$4320i^2+3312i+605$

TABLE 2. Brieskorn homology 3-spheres $\Sigma(p, q, r)$ from the blow-downs of the minimal resolution of negative-definite plumbings.

Theorem 1.7. *For $0 \leq k \leq 12$ or $k = 14$, we have $\mathfrak{ds}(Y^-_{2k+1}) < \infty$. In particular, for these integers k we have $g_8(Y^-_{2k+1}) = 1$.*

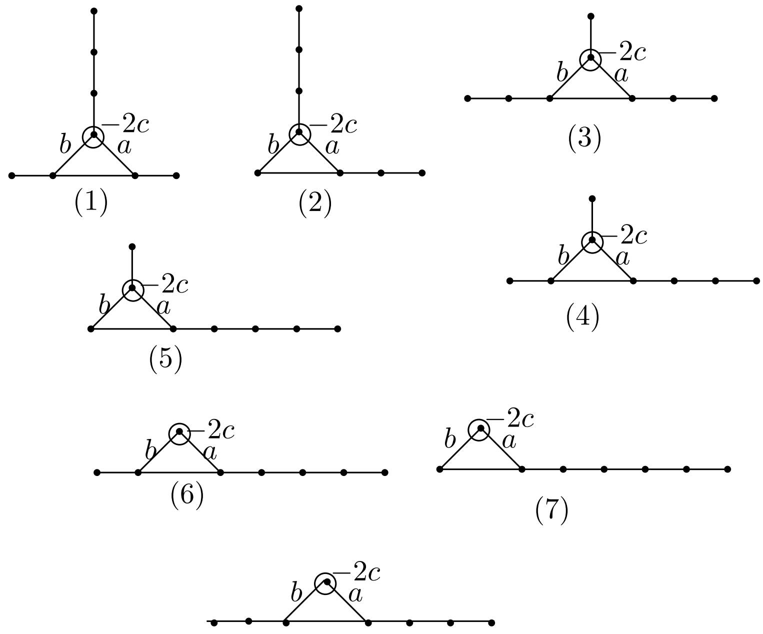


FIGURE 1. The 8 possible configurations with $-E_8$-intersection form. All the unweighted components are -2 and all the labels with unlabeled is $+1$.

The boundings cannot be obtained by the minimal resolution or blow-downs of minimal resolutions. Actually, these boundings can be embedded in $E(1)$ and the complements are Gompf's nuclei N_{2k+1}.

1.5. **Embedded spheres in** $E(1)$**.** Let $E(1)$ be an elliptic fibration diffeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. According to Li and Li's result in [7], the spherical realization of the following negative classes in $E(1)$ has been studied:

Theorem 1.8 (Li-Li [7])**.** *In $H_*(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2})$ with $1\le n\le 9$ all classes ξ with $0>\xi^2>-(n+7)$, have minimal genus 0.*

As a by-product of Theorem 1.7 we can obtain the following theorem:

Theorem 1.9. *Let f and s be the general fiber and a section of elliptic fibration in $E(1)$. We put $a_k:=k[f]-[s]\in H_2(E(1))$. For any $0\le k\le 12$ or $k=14$, the class a_k represents an embedded sphere in $E(1)$.*

This intersection number of a_k is $-2k-1$. Theorem 1.9 can be also compared with following Finashin and Mikhalkin's theorem:

Theorem 1.10 (Finashin-Mikhalkin[9])**.** *There exists a smooth embedding of S^2 into an $E(2)$ with the normal Euler number equal to n for any negative even $n\ge -86$.*

In particular, for the general fiber f and a section s in the K3-surface, the class $k[f]-[s]\in H_2(E(2))$ can be represented by an embedded S^2 for

$k \leq 42$. We will post a question (Question 5.7) on the sphere class of a_k in $E(n)$ in the last section.

Acknowledgements. The results in this article are partially done when I visited in Michigan State University in 2013 spring. I am grateful for useful comments by S. Akbulut and the hospitality of the institute.

2. Basic properties of invariants $\mathfrak{ds}$ and g_8.

We will prove the basic properties on $\mathfrak{ds}$ and g_8. Let $\Theta^3_{\mathbb{Z}}$ denote the group of the homology 3-spheres up to homology cobordism.

Theorem 2.1. *Let $\mathfrak{ds}'$ be one of $\mathfrak{ds}$, $\underline{\mathfrak{ds}}$ and g_8' denote g_8, or $\underline{g_8}$. Then the following properties are satisfied:*

(1) *The $\mathfrak{ds}'$ and g_8' are homology cobordism invariants i.e., $\mathfrak{ds}' : \Theta^3_{\mathbb{Z}} \to \mathbb{N} \cup \{0, \infty\}$.*
(2) *$\underline{\mathfrak{ds}}(Y) = 0$ or $\underline{g_8}(Y) = 0$, if and only if $[Y] = 0$ in $\Theta^3_{\mathbb{Z}}$.*
(3) *If $\mathfrak{ds}(Y), g_8(Y) < \infty$, then $\mu(Y) \equiv \mathfrak{ds}'(Y) \equiv g_8'(Y) \equiv 0 \bmod 2$*
(4) *If $\epsilon(Y_1)\epsilon(Y_2) = 1$, then $\mathfrak{ds}(Y_1) + \mathfrak{ds}(Y_2) \leq \mathfrak{ds}(Y_1 + Y_2)$.*
(5) *If $\epsilon(Y_1)\epsilon(Y_2) = 1$, then $\underline{\mathfrak{ds}}(Y_1 + Y_2) \leq \underline{\mathfrak{ds}}(Y_1) + \underline{\mathfrak{ds}}(Y_2)$.*
(6) *If $\mathfrak{ds}(Y) = 1$, then $g_8(Y) = 1$.*
(7) *$\mathfrak{ds}(-Y) = \mathfrak{ds}(Y)$ and $\underline{\mathfrak{ds}}(-Y) = \underline{\mathfrak{ds}}(Y)$.*
(8) *$g_8(-Y) = g_8(Y)$ and $\underline{g_8}(-Y) = \underline{g_8}(Y)$.*
(9) *If $0 < \mathfrak{ds}(Y) < \infty$, then $\epsilon(Y)d(Y) < 0$ and $\mathfrak{ds}(Y) \leq |d(Y)|/2$.*
(10) *If $\mathfrak{ds}'(Y)$ or $g_8'(Y)$ is odd, then $|Y| = \infty$.*
(11) *If $\mathfrak{ds}(Y)$ is even, then we have $\mathfrak{ds}(Y) + 1 \leq |Y|$.*
(12) *If $|Y| = 1, 2$, then $\mathfrak{ds}(Y) = \infty$.*
(13) *If $\epsilon(Y) \neq \infty$, then $2\mathfrak{ds}(Y) \leq m(\epsilon(Y)Y) \leq \overline{m}(\epsilon(Y)Y) \leq 18\underline{\mathfrak{ds}}(Y)$.*
(14) *If $\epsilon(Y)$ is finite and $\underline{\mathfrak{ds}}(Y) < \mathfrak{ds}(Y)$, then $\mathfrak{ds}(Y) - \underline{\mathfrak{ds}}(Y) \leq 8(\underline{\mathfrak{ds}}(Y) + 1)$ holds.*
(15) *Suppose that Y is a Seifert homology 3-sphere. If $\mathfrak{ds}(Y) < \infty$, then $\bar{\mu}(Y)\epsilon(Y) > 0$ and $\mathfrak{ds}(Y) \leq |\bar{\mu}(Y)|$.*

Proof. (1) Suppose that Y, Y' are homology cobordant homology 3-spheres. If $\mathfrak{ds}'(Y) < \infty$, then there exists a definite spin bounding W of Y with maximal (or minimal) b_2. Connecting between Y and Y' by the cobordism, we get bounding W' of Y' with a maximal (or minimal) b_2. If $\mathfrak{ds}'(Y) = \infty$ and $\mathfrak{ds}'(Y')$ is finite, then we get a definite spin bounding of Y. This is contradiction. Thus, if $\mathfrak{ds}'(Y) = \infty$, then $\mathfrak{ds}'(Y') = \infty$

(2) Suppose Y is a homology 3-sphere with $\underline{\mathfrak{ds}}(Y) = 0$. Then Y bounds a homology 4-ball W. Puncturing W, we get a homology cobordism between Y and S^3.

(3) Suppose that W is any definite spin bounding of Y. Then by the definition of μ we have $b_2(W)/8 \equiv \mu(Y) \bmod 2$.

(4,5) From the properties of maximal and minimal, we have the inequalities by taking the boundary sum of the two definite bounding.

(6) From the property that the unimodular even definite quadratic form with rank 8 is isomorphic to $\pm E_8$.

(7,8) The definition of $\mathfrak{ds}$ and g_8 does not depend on the orientation.

(9) From the inequality (2) the inequalities hold.

(10) If $\mathfrak{ds}'(Y)$ or $g_8'(Y)$ is odd, then $\mu(Y) = \mathfrak{ds}(Y) \neq 0 \bmod 2$, thus we have $|Y| = \infty$.

(11) If $\mathfrak{ds}(Y)$ is even, then $|Y| < \infty$ holds. Then we get a closed spin 4-manifold by gluing the two boundings. The intersection form is isomorphic to $\mathfrak{ds}(Y) \cdot (-E_8) \oplus |Y| \cdot H$. Furuta's inequality implies $|Y| \geq \mathfrak{ds}(Y) + 1$.

(12) If $|Y| = 1, 2$ and $\mathfrak{ds}(Y)$ is finite, then due to Furuta's inequality $2 \geq \mathfrak{ds}(\pm Y) + 1$ holds. Since $\mathfrak{ds}(Y)$ is even, then $\mathfrak{ds}(Y) = 0$ namely $[Y] = 0$ in $\Theta^3_{\mathbb{Z}}$. This contradicts to $|Y| > 0$.

(13) Let Y be a homology sphere with $\epsilon(Y) \neq \infty$. We denote $||Y|| := \epsilon(Y)(Y)$. Then $||Y||$ has a positive definite spin bounding X and

$$\begin{aligned} m(||Y||)/2 &\geq \max\{5\sigma(X)/8 - b_2(X)/2 | \partial X = ||Y|| \text{ and } \sigma(X) = b_2(X)\} \\ &= \mathfrak{ds}(||Y||) = \mathfrak{ds}(Y) \end{aligned}$$

$$\begin{aligned} \overline{m}(||Y||)/2 &\leq \min\{5\sigma(X)/8 + b_2(X)/2 | \partial X = ||Y|| \text{ and } \sigma(X) = b_2(X)\} \\ &= 9\underline{\mathfrak{ds}}(||Y||) = 9\underline{\mathfrak{ds}}(Y) \end{aligned}$$

(14) Let X_1, X_2 be two negative-definite spin boundings with $b_2(X_i) = \beta_i$ and $0 < \beta_1 < \beta_2$. Then the invariants of the capped closed spin manifold $X = X_2 \cup (-X_1)$ are $b_2(X) = \beta_1 + \beta_2$ and $\sigma(X) = \beta_2 - \beta_1$. From Furuta's inequality in [5], we have $\beta_2 \leq 9\beta_1 + 8$.

$$\underline{\mathfrak{ds}}(Y) < \mathfrak{ds}(Y) \leq 9\underline{\mathfrak{ds}}(Y) + 8$$

Consequently, $\mathfrak{ds}(Y) - \underline{\mathfrak{ds}}(Y) \leq 8(\underline{\mathfrak{ds}}(Y) + 1)$ holds.

(15) By the result in Theorem 1.2, we get the bound of the $\mathfrak{ds}$-invariant. □

Here one of motivations for studying invariants $\mathfrak{ds}$, and g_8 is to give counterexamples to the (11/8)-conjecture. We give a proof of Proposition 1.1.

Proof of Proposition 1.1. Suppose Y is a homology 3-sphere with $\mu(Y) = 0$, $\mathfrak{ds}(Y) < \infty$ and $2|Y| < 3\mathfrak{ds}(Y)$. Let X_1, X_2 be two spin bounding 4-manifolds satisfying $\partial X_1 = Y$ and $\partial X_2 = -Y$, where X_1 is a definite spin 4-manifold and $Q_{X_2} \cong nH$. Gluing X_1 and X_2 along Y we get a closed spin 4-manifold with $Q_{X_1 \cup X_2} \cong mE_8 \oplus nH$. In particular, we may assume $n = |Y|$ and $m = \mathfrak{ds}(Y)$. Thus, $n < \frac{3|m|}{2}$ holds. The manifold X violates the (11/8)-conjecture.

Let X be a closed, smooth, oriented, spin, simply-connected 4-manifold with $Q_X = 2k(-E_8) \oplus lH$. Then according to [10] there exists a homology 3-sphere Y cutting the intersection form, i.e., $X = X_1 \cup_Y X_2$ and $Q_{X_1} \cong 2k(-E_8)$ and $Q_{X_2} \cong lH$ and $\partial X_2 = Y$. Thus Y satisfies $\mu(Y) = 0$ and $\mathfrak{ds}(Y) < \infty$ and X_i is homologically 1-connected. Hence, $l \geq |Y| \geq \frac{3}{2}\mathfrak{ds}(Y) \geq \frac{3}{2}|2k| = 3|k|$. This implies (11/8)-conjecture for simply-connected 4-manifolds. □

3. The negative E_8-boundings

3.1. **Milnor-fiber construction.** The Milnor-fiber $M(p,q,r)$ is the 4-manifold defined as the compactification of

$$\{(x,y,z)\in\mathbb{C}^3 | x^p+y^q+z^r=\epsilon\},$$

where ϵ is some constant. The boundary is the Brieskorn rational homology 3-spheres $\Sigma(p,q,r)$. If each two elements in $\{p,q,r\}$ are relatively prime, then the Brieskorn 3-sphere is a homology 3-sphere. The Milnor-fibers are nice examples of spin bounding. As mentioned at the Exercise 7.3.18. in Section 7.3, in [11] p.265, for integers p,q,r,p',q',r' with $p\le p'$, $q\le q'$ and $r\le r'$, there exists the inclusion $M(p,q,r)\hookrightarrow M(p',q',r')$. This gives a cobordism between $\Sigma(p,q,r)$ and $\Sigma(p',q',r')$.

Proof of Theorem 1.3. Here, consider the following natural inclusion:

$$M(2,3,6n-5)\hookrightarrow M(2,3,6n-1).$$

The induced cobordism X_n between $\Sigma(2,3,6n-5)$ and $\Sigma(2,3,6n-1)$ has intersection form $-E_8$. In fact, from Novikov's additivity, $\sigma(M(2,3,6n-1)) = \sigma(M(2,3,6n-5)+\sigma(X_n)$ holds, hence, $\sigma(X_n)=-8$. Since the boundary of X_n is diffeomorphic to $-\Sigma(2,3,6n-5)\cup\Sigma(2,3,6n-1)$, and X_n is spin, Q_{X_n} must be isomorphic to $-E_8$ from the classification of unimodular definite even quadratic forms.

By removing one 3-handle from X_n, we get a cobordism W_n from a punctured $\Sigma(2,3,6n-5)$ to punctured $\Sigma(2,3,6n-1)$. The manifold W_n satisfies $\partial W_n = M_n$ and $Q_{W_n}\cong -E_8$. From the construction of W_n we have $H_1(W_n,\mathbb{Z})\cong H_1(X_n,\mathbb{Z})$ clearly. Since X_n is obtained by attaching 8 2-handles on the boundary of $\Sigma(2,3,6n-5)\times I$, there exists a surjection $\{0\}=H_1(\Sigma(2,3,6n-5),\mathbb{Z})\to H_1(X_n,\mathbb{Z})$. Therefore W_n is homologically 1-connected.

On the other hand, since $d(M_n)=d(\Sigma(2,3,6n-1)-d(\Sigma(2,3,6n-5)=2-0=2$, we get $\mathfrak{ds}(M_n)=g_8(M_n)=\epsilon(M_n)=1$. $\square$

3.2. **The minimal resolution.** Let $W(G)$ be a plumbed 4-manifold associated with a graph G, which is a tree weighted by integers.

Definition 3.1. *Let G be a connected star-shaped graph as in Figure 2. The 'star-shaped' means the graph has at most one n-valent vertex with $n\ge 3$. Let $\{v_0,v_i^j\}$ be the vertices and m_0 and m_i^j be the weights of the vertices v_0 and v_i^j. That is, the unique vertex v_0 is at least 3-valent and the valencies of the other vertices are all 1 or 2. If G satisfies the following properties, we call the graph G is minimal:*

(1) *The incidence matrix is negative-definite.*
(2) $m_0\le -1$.
(3) $m_i^j\le -2$.

The minimal graph gives a negative-definite plumbing 4-manifold with a Seifert rational homology 3-sphere boundary. Furthermore, if all the weights are even, then the plumbing 4-manifold is a spin negative-definite bounding.

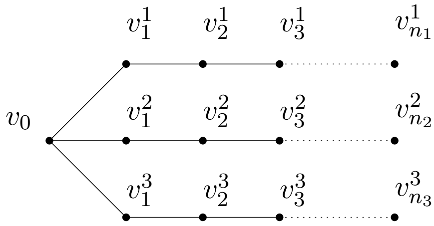


FIGURE 2. A Seifert diagram with three branches.

Let G be a weighted graph. We give a lemma for finding types of G which give homology spheres. Let $\mathrm{Vert}(G)$ be the set of vertices of G. Let $\det(G)$ be the determinant of the incident matrix of G.

Lemma 3.1. *Let $v \in Vert(G)$ be a 1-valent vertex with weight $m(v)$ and G_p the graph from G obtained by adding the weight $m(v)$ by $p > 0$. Then* $\det(G) \equiv \det(G_p) \bmod p$.

Let w be the vertex connecting to v and G_0 a graph obtained by deleting v and w. If $m(v) \equiv 0 \bmod p$, $\det(G) \equiv -\det(G_0) \bmod p$.

Proof. Let $m(v)$ be the weight of v and $\tilde{G}$ the graph obtained by deleting v from G. Expanding the determinant $\det(G)$, we have

$$\det(G) = m(v)\det(\tilde{G}) - \det(G_0) \overset{\bmod p}{\equiv} (m(v)+p)\det(\tilde{G}) - \det(G_0) = \det(G_p).$$

The latter assertion follows from

$$\det(G) = m(v)\det(\tilde{G}) - \det(G_0) \overset{\bmod p}{\equiv} -\det(G_0).$$

□

If the lengths of branches of the Seifert diagram are $n_1, n_2, \cdots, n_k$, we say that the Seifert diagram has *length type* $(n_1, n_2, \cdots, n_k)$. For example, FIGURE 2 has length diagram (n_1, n_2, n_3). Let $G_{(n_1,\cdots,n_k)}$ be a weighted graph of the length type $(n_1, \cdots, n_k)$ with all weights even. $G_{(1)}$ is a weighted graph having one vertex with weight even, namely $\det(G_{(1)}) \equiv 0 \bmod 2$. Note that the following lemma holds.

Lemma 3.2.

$$\det(G_1 \cup G_2) = \det(G_1)\det(G_2).$$

$$\det(G_{(n,1,1,\cdots,1)}) \equiv 0 \bmod 2$$

Proof. Since any $v \in G_1$ and $w \in G_2$ are not connected, the determinant is computed as $\det(G_1 \cup G_2) = \det(G_1)\det(G_2)$. By using Lemma 3.1, we have

$$\det(G_{(n,1,1,\cdots,1)}) \equiv \det(G_{(1,1,\cdots,1)}) \equiv \det(G_{(1)} \cup \cdots \cup G_{(1)}) \equiv 0 \bmod 2.$$

□

We prove the following:

Proposition 3.1. *Let $\Sigma(p,q,r)$ be a Brieskorn homology 3-sphere whose minimal resolution with negative-definite gives an E_8-bounding with $b_2 = 8$. Then $\Sigma(p,q,r) = \Sigma(2,3,5)$ or $\Sigma(3,4,7)$.*

Proof. The minimal resolution graph of the Seifert structure we require is rank$= 8$, unimodular, negative-definite and even. Since the graph is even, the weight of the central vertex is -2.

In general, consider negative definite resolutions graph for $\Sigma(a_1, \cdots, a_n)$ with all vertices ≤ -2 except at the central vertex. Let e be the weight of the central vertex of the graph. It is well-known that e has the restriction $-1-n < e \leq -1$. This inequality is obtained by computing the 1st homology of the Brieskorn homology sphere. In this case since $n = 3$ and e is even, e must be -2.

The three possible lengths $n_1 \geq n_2 \geq n_3$ of branches are $(n_1, n_2, n_3) = (4,2,1), (3,2,2)$, in fact other ones $(5,1,1), (3,3,1)$ cannot be unimodular. Because $\det(G_{(5,1,1)}) \equiv 0 \bmod 2$ and $G_{(3,3,1)} \equiv G_{(1,1,1)} \equiv 0 \bmod 2$ (Lemma 3.1).

Let us consider Seifert manifolds with length type $(3,2,2)$ as in Figure 3, where b, c, d, e, f, g, h are positive integers. Then we put an integer D as follows:

$$X := 2 - \frac{4gh-1}{2(4fgh-f-h)} - \frac{2c}{4bc-1} - \frac{2e}{4de-1} = \frac{D}{pqr}.$$

D is the determinant of the resolution graph and p, q, r are the multiplicities of the Seifert manifolds, namely, $p = 2(4fg - f - h)$, $q = 4bc - 1$ and $r = 4de - 1$. We find cases where the Seifert manifold is a homology sphere. Since the Seifert manifold is a homology sphere, any two of p, q, r is coprime and D is one. Here X is a function of variables b, c, d, e, f, g, h. The maximal value of D/pqr for the Brieskorn homology spheres $\Sigma(p,q,r)$ is $1/30$, which is the case of the Poincaré homology sphere.

Here $-\frac{4gh-1}{2(4fgh-f-h)}$, $-\frac{2c}{4bc-1}$, and $-\frac{2e}{4de-1}$ are increasing functions, because all partial derivatives in b, c, d, e, f, g are positive functions on the each point. Since f, g, h are natural numbers, $-\frac{4gh-1}{2(4fgh-f-h)} \geq -\frac{3}{4}$ holds. Now, we assume $bc \geq 2$ and $de \geq 2$. then we have

$$X \quad \geq \quad \frac{5}{4} - \frac{4}{7b} - \frac{4}{7d} \geq \frac{5}{4} - \frac{8}{7} = \frac{3}{28} > \frac{1}{30}$$

Thus, this case is not a homology sphere. From the symmetry of the graph we may assume $d = e = 1$. This means that $r = 3$ holds.

Then, further, if $b \geq 2$, then

$$X \geq \frac{5}{4} - \frac{2c}{8c-1} - \frac{2}{3} \geq \frac{7}{12} - \frac{2}{7} = \frac{25}{84} > \frac{1}{30}.$$

Thus, this case does not occur.

Suppose that $b = d = e = 1$. If $c > 2$, then we have

$$X \geq \frac{7}{12} - \frac{2c}{4c-1} \geq \frac{7}{12} - \frac{6}{11} \geq \frac{5}{132} > \frac{1}{30}.$$

Thus we have $c = 1, 2$. If $c = 2$ holds, then $q = 7$ and

$$X \geq \frac{7}{12} - \frac{4}{7} = \frac{1}{84},$$

hence $p \leq 4$ holds. Further, since $p = 2(4fgh - f - h) \geq 4$, we have $p = 4$. Hence $f = g = h = 1$. This case corresponds to the Brieskorn 3-sphere $\Sigma(3, 4, 7)$.

If we suppose $c = 1$, then $q = 3$ holds. This is contradiction to what p, q are relatively prime.

In the case of $(4, 2, 1)$, X takes the minimal value among negative definite quadratic even forms, when all weights are -2. This is the $\Sigma(2, 3, 5)$ case. On the other hand, the $\Sigma(2, 3, 5)$ takes the maximal value $1/30$ in all the Brieskorn homology spheres $\Sigma(p, q, r) \neq S^3$. Thus, this means that all Seifert homology spheres with even form and lengths $(4, 2, 1)$ is $(p, q, r) = (2, 3, 5)$ only. □

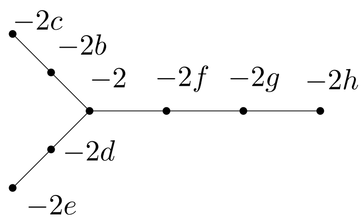


FIGURE 3. The resolution graph with type $(3, 2, 2)$.

Proposition 3.2. *Let $\Sigma(a_1, a_2, \cdots, a_n)$ be a Brieskorn homology 3-sphere with $n \geq 4$. Suppose that the minimal resolution graph is even and rank 8. Then $n = 4$ and the Brieskorn 3-spheres are $\Sigma(2, 3, 7, 11)$, $\Sigma(2, 3, 7, 23)$, or $\Sigma(3, 4, 7, 43)$.*

Proof. The partitions of 7, the number of whose parts is more than 3, are the following 7 types. $(4, 1, 1, 1)$, $(3, 2, 1, 1)$, $(2, 2, 2, 1)$, $(3, 1, 1, 1, 1)$, $(2, 2, 1, 1, 1)$, $(2, 1, 1, 1, 1, 1)$, and $(1, 1, 1, 1, 1, 1, 1)$. By using Lemma 3.2, the determinants of $G_{(4,1,1,1)}, G_{(3,1,1,1,1)}, G_{(2,1,1,1,1,1)}, G_{(1,1,1,1,1,1,1)}$ are even. In the cases $G_{(3,2,1,1)}$ and $G_{(2,2,1,1,1)}$, since

$$\det(G_{(3,2,1,1)}) \equiv \det(G_{(3,1,1)}) \equiv 0 \bmod 2$$
$$\det(G_{(2,2,1,1,1)}) \equiv \det(G_{(2,1,1,1)}) \equiv 0 \bmod 2,$$

these do not occur. The type $(2, 2, 2, 1)$ has det $= 1 \bmod 2$. Let D be the determinant for the graph of type $(2, 2, 2, 1)$ in FIGURE 4. We may assume the type $(2, 2, 2, 1)$. The parameters a, b, c, d, e, f, g, h are positive numbers.

$$X := 2a - \frac{1}{2h} - \frac{2c}{4bc - 1} - \frac{2e}{4de - 1} - \frac{2g}{4fg - 1} = \frac{D}{pqrs} \tag{4}$$

where $p = 2h$, $q = 4bc - 1$, $r = 4de - 1$, and $s = 4fg - 1$. Since any two of p, q, r, s is coprime, we have $pqrs \geq 2 \times 3 \times 7 \times 11 = 462$.

D is the determinant of the Seifert manifolds as in FIGURE 4. We find the parameters a, b, c, d, e, f, g, h with $D = 1$.

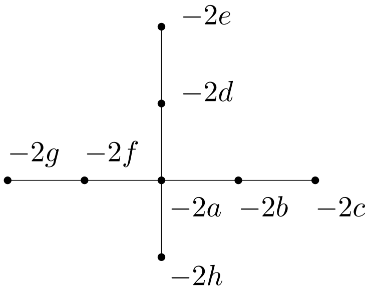


FIGURE 4. Resolution graph with type $(2, 2, 2, 1)$.

First, the central weight $-2a$ is -2 or -4, because the above inequality $-1-4 < -2a < 0$ implies $a = 1$ or 2. We may assume that $q \leq r \leq s$ from symmetry of the graph.

[The case of $a = 2$**.]** Suppose that $a = 2$. Note that $-\frac{2g}{4fg-1}$ is an increasing function of f, g. By using $b, c, d, e, f, g, h \geq 1$, we have

$$X \geq 4 - \frac{1}{2} - 3 \times \frac{2}{3} \geq \frac{3}{2}.$$

This case does not satisfy $D = 1$.

[The case of $a = 1$**.]**
Suppose that $h \geq 3$. Then we have

$$X \geq 2 - \frac{1}{6} - \frac{2}{3} - \frac{4}{7} - \frac{6}{11} \geq \frac{23}{462} > \frac{1}{924}.$$

Thus we have $h = 1, 2$.

We assume that $h = 2$, i.e., $p = 4$. Suppose that $bc \geq 2$. Then we have $de \geq 3$, $fg \geq 4$ and

$$X \geq \frac{7}{4} - \frac{4}{7} - \frac{6}{11} - \frac{8}{15} \geq \frac{461}{4620} > \frac{1}{924}.$$

Hence, we have $bc = 1$ holds, then $b = c = 1$ and $q = 3$. If $de \geq 3$, then we have $fg \geq 4$ and

$$X \geq \frac{7}{4} - \frac{2}{3} - \frac{6}{11} - \frac{8}{15} \geq \frac{1}{220} > \frac{1}{924}.$$

Hence, we have $de = 2$. Thus $r = 7$ and $(d, e) = (1, 2)$ or $(2, 1)$. If $(d, e) = (2, 1)$, then $-\frac{2e}{4de-1} = -\frac{2}{7}$. Since $fg \geq 3$,

$$X \geq \frac{7}{4} - \frac{2}{3} - \frac{2}{7} - \frac{6}{11} \geq \frac{233}{924} > \frac{1}{924}.$$

Thus we must have $(d, e) = (1, 2)$. Suppose that $fg \geq 3$ and $f \geq 2$, then

$$X \geq \frac{7}{4} - \frac{2}{3} - \frac{4}{7} - \frac{3}{11} \geq \frac{221}{924} > \frac{1}{924}.$$

Thus $f = 1$ holds. If $g \geq 12$ holds, then

$$X \geq \frac{7}{4} - \frac{2}{3} - \frac{4}{7} - \frac{24}{47} \geq \frac{5}{3948} > \frac{1}{924}.$$

When $3 \leq g \leq 11$, in the case of $D = 1$ we have $g = 11$ only. The case of $(b, c, d, e, f, g, h) = (1, 1, 1, 2, 1, 11, 2)$ corresponds to $\Sigma(3, 4, 7, 43)$.

We assume that $h = 1$. Then the maximal value of $D/pqrs$ of homology spheres with type $(2, 2, 2, 1)$ is $1/(2 \cdot 3 \cdot 7 \cdot 11) = 1/462$.

Suppose that $bc \geq 2$. Then we have $de \geq 3$, $fg \geq 4$.

Furthermore, we suppose that $b \geq 2$, $d \geq 2$, or $f \geq 2$.

$$X \geq \frac{3}{2} - \frac{4}{7b} - \frac{6}{11d} - \frac{8}{15f} > \frac{1}{462}.$$

Thus $b = d = f = 1$ holds, and $c \leq e \leq g$ holds. Then, we have

$$X = \frac{3}{2} - \frac{2c}{4c-1} - \frac{2e}{4e-1} - \frac{2g}{4g-1} \leq \frac{3}{2} - \frac{6g}{4g-1} < 0.$$

Hence, $bc = 1$ holds, i.e., $q = 3$ holds.

Suppose that $bc = 1$, i.e., $b = c = 1$ and $q = 3$. If $de \geq 3$, then we have $fg \geq 4$ and

$$X \geq \frac{3}{2} - \frac{2}{3} - \frac{6}{11} - \frac{8}{15} \geq \frac{1}{220} > \frac{1}{462}.$$

Hence, we have $de = 2$. Thus $q = 7$ and $(d, e) = (1, 2)$ or $(2, 1)$.

If $(d, e) = (2, 1)$, then $-\frac{2e}{4de-1} = -\frac{2}{7}$. Since $fg \geq 3$,

$$X \geq \frac{3}{2} - \frac{2}{3} - \frac{2}{7} - \frac{6}{11} = \frac{1}{462}.$$

Hence, the required only case is the one satisfying the equality. Thus $(b, c, d, e, f, g, h) = (1, 1, 2, 1, 1, 3, 1)$ and $(p, q, r, s) = (2, 3, 7, 11)$.

If $(d, e) = (1, 2)$ and $f = 1$, then

$$X = \frac{3}{2} - \frac{2}{3} - \frac{4}{7} - \frac{2g}{4g-1} < \frac{11}{42} - \frac{1}{2} = \frac{-5}{21} < 0.$$

Thus we assume that $f \geq 2$. If $fg \geq 7$, then we have

$$X \geq \frac{3}{2} - \frac{2}{3} - \frac{4}{7} - \frac{14}{27f} \geq \frac{1}{378} > \frac{1}{462}.$$

Hence, this case does not occur. Suppose that $f \geq 2$ and $fg \leq 6$. Computing X, we obtain $(f, g) = (2, 3)$ only. This case corresponds to $(p, q, r, s) = (2, 3, 7, 23)$.

□

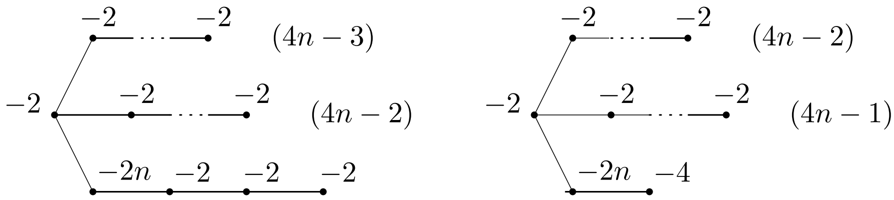
Proof of Theorem 1.4. The rest part in the assertion is the computation of $\mathfrak{ds}$. From the inequality (2) by Ozsváth and Szabó, the required assertion follows. In fact, the d-invariants of those Brieskorn homology 3-spheres below are all 2 by Némethi's algorithm in [14],

$$\Sigma(2, 3, 5),\ \Sigma(3, 4, 7),\ \Sigma(2, 3, 7, 11),\ \Sigma(2, 3, 7, 23),\ \Sigma(3, 4, 7, 43).$$

Hence, these manifolds are all $g_8 = 1$. □

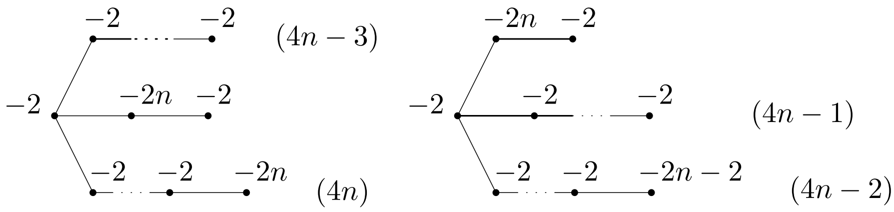
In the following, we prove Theorem 1.5.

Proof of Theorem 1.5. The minimal resolution graphs of $\Sigma(4n-2, 4n-1, 8n-3)$, $\Sigma(4n-1, 4n, 8n-1)$, $\Sigma(4n-2, 4n-1, 8n^2-4n+1)$, and $\Sigma(4n-1, 4n, 8n^2-1)$ are FIGURE 5 and 6. The numbers of the parentheses are the lengths of the branches. □



$\Sigma(4n-2, 4n-1, 8n-3)$ $\Sigma(4n-1, 4n, 8n-1)$

FIGURE 5. The minimal resolution graphs.



$\Sigma(4n-2, 4n-1, 8n^2-4n+1)$ $\Sigma(4n-1, 4n, 8n^2-1)$

FIGURE 6. The minimal resolution graphs.

The intersection forms of these minimal resolution graphs are not isomorphic to $n(-E_8)$ for $n > 1$. It follows from the argument below. Any vector with square -2 in $n(-E_8)$ is contained in a $-E_8$-component, because $-E_8$ is a definite matrix. Furthermore, there exists no 9 vectors $x_1, \cdots, x_9$ in $-nE_8$ satisfying $x_i^2 = -2$ and $x_i \cdot x_{i+1} = 1$ $(i = 1, \cdots, 8)$. In fact such vectors must be in a common $-E_8$-component in $-nE_8$, because $x_i \cdot x_{i+1} = 1$ implies that x_i and x_{i+1} are in a same $-E_8$-component. However since these vectors are linearly independent, they cannot be embedded in a $-E_8$. Therefore, any of 4 minimal resolution graphs in FIGURE 5 and 6 is not isomorphic to $n(-E_8)$ when $n > 1$. We do not know whether the homology 3-spheres have other boundings with $g_8 = n$ and $\epsilon = -1$.

3.3. **Blow-downs of the minimal resolution.** In general, any minimal resolution is a negative-definite bounding with possibly not even. But there are some -1-spheres in the bounding 4-manifold. By performing blow-downs of the spheres we can get a smaller bounding. The new bounding is not a resolution any more. In this section, we give several E_8-boundings with $g_8 = 1$ and $\epsilon = -1$ by using the blow-down of the minimal resolutions of Brieskorn homology 3-spheres. These strategies can be also seen in [16].

The blow-down process of a Brieskorn homology 3-sphere is described as follows. As an example, let us consider a plumbing graph with 3 singular fibers as in the first diagram in FIGURE 7. By doing the blow-down at the central component, we get the next diagram. The (unlabeled) edge presents the $+1$-linking between corresponding components. In the next diagram, doing the further blow-down at the -1-framed component, we get the third diagram. The integers nearby the edge are the linking number between the two components. In the same way we get the fourth diagram. Here the $(x+6)$-framed component is the $(2,3)$-torus knot. Here we deal with the diagram as in the left of FIGURE 9. This diagram stands for the handle diagram in the right of the FIGURE 9. In this paper such a graph is called *a configuration* and any graph obtained by several blow-downs of a Seifert plumbing graph is called *a blow-downed configuration*. The integer associated with any vertex is called a *weight* and that with any edge is called a *label*. The incidence matrix for the configuration naturally gives the quadratic form for the bounding 4-manifold.

Each step of the blow-down performances is based on the formula in FIGURE 8. Here, if the x-framed component in the figure is the (a,b)-torus knot, then the next $(x+b^2)$-framed component is the $(a+b,b)$-torus knot.

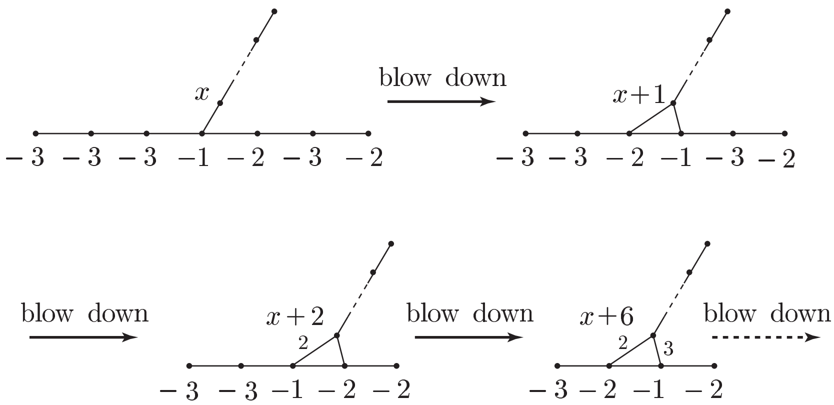


FIGURE 7. Blow-down process.

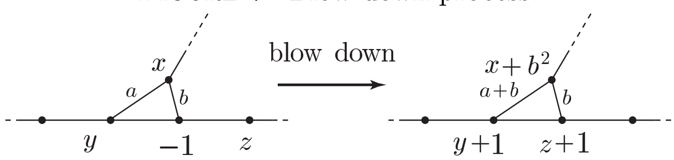


FIGURE 8. A blow-down formula on configurations.

Let G_0 be a 1-cycled graph with three edges with labels $\{a, b, 1\}$. The weight of the vertex intersecting two edges with a, b is $-2c$. The graph G is the union of G_0 and linear edges connecting the three vertices. See the right of FIGURE 9. We call the graph G a *branched triangular configuration*. Here, we must note that this graph represents *not* a usual plumbing graph *but* a simply-connected 4-manifold constructed by attaching 2-handles according to the graph.

Proposition 3.3. *Let G be a branched triangular configuration. The pair $(G; a, b, c)$ with* $\gcd(a,b) = 1$ *in* TABLE *1 is the blow-downed configurations with type (1) to (8) in* FIGURE *1, whose intersection form is presented by $-E_8$.*

Note that configurations (1) to (8) are not all the blow-downed, branched triangular configurations with $-E_8$ intersection form.

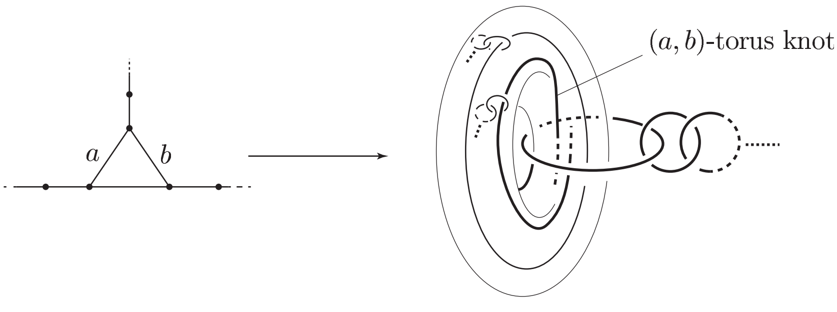


FIGURE 9. The actual handle diagram with branched triangular configuration.

Proof. We consider configurations in FIGURE 1. Other branched triangular configurations with rank 8 cannot be a unimodular form. Let $G(-2c, a, b; n_1, n_2, n_3)$ be a branched triangular configuration as in FIGURE 10 with all weights -2. Then in the case of $G(-2c, a, b; 5, 0, 0)$, we have

$$\det(G(-2c, a, b; 5, 0, 0)) = -15 - 12a^2 - 12ab - 12b^2 + 36c \equiv 0 \bmod 3.$$

This case does not occur. In the case of $G(-2c, a, b; 4, 1, 0)$, we have

$$\det(G(-2c, a, b; 4, 1, 0)) = -16 - 15b^2 - 20ab - 20a^2 + 40c \equiv -1 \bmod 5.$$

This case also does not occur. In the cases of $G(-2c, a, b; 2, 3, 0)$ or $G(-2c, a, b; 2, 2, 1)$, the determinants are all divisible by 3. Thus these cases do not occur.

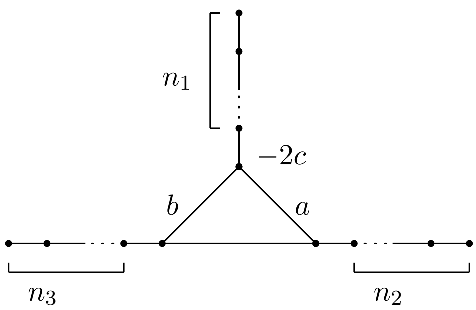


FIGURE 10. The definition of $G(-2c, a, b; n_1, n_2, n_3)$.

Computing the determinants of the 8 examples, we obtain the equations:

$$(1): 3a^2 + 4ab + 3b^2 = 5c - 2; \ (2): 3a^2 + 3ab + 2b^2 = 5c - 2$$

$$(3): 6a^2 + 9ab + 6b^2 = 7c - 2; \ (4): 6a^2 + 8ab + 5b^2 = 7c - 2$$

$$(5): 5a^2 + 5ab + 3b^2 = 7c - 2; \ (6): 15a^2 + 20ab + 12b^2 = 16c - 1$$

$$(7): 12a^2 + 12ab + 7b^2 = 16c - 1, \ (8): 16a^2 + 24ab + 15b^2 = 16c - 1.$$

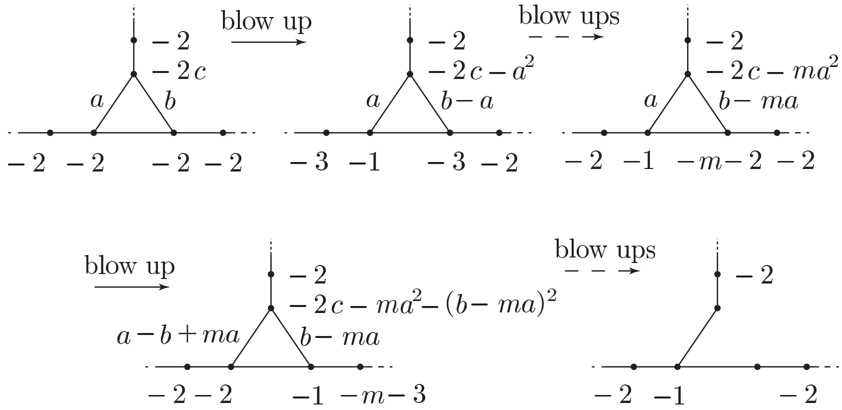


FIGURE 11. Blow-up process by the Euclidean algorithm.

The positive integral solutions $\{a, b, c\}$ in these equations give the negative-definite E_8-boundings with configurations from (1) to (8). If a and b are relatively prime, then these pairs $(G; a, b, c)$ are blow-downed configurations by iterating several blow-ups in accordance with the Euclidean algorithm for relatively prime (a, b) as in FIGURE 11.

Suppose that $\{a, b\}$ is a relatively prime solution with $a < b$. Let m denote the minimum positive number satisfying $b - ma < a$. We iterate the blow-up process (the inverse of Figure 8) m-times at the left bottom angle in the triangle as in the first configuration in FIGURE 11. Next, exchanging the role of a and $b - ma$, we continue to perform the blow-up at the right bottom angle. Applying the Euclidean algorithm to this blow-up process in this way, we obtain the star-shaped graph which all labels are $+1$ and all weights are smaller than or equal to -2 except for the central vertex of weight -1.

In consequence, the pair (a, b, c) in the TABLE 1 with relatively prime a, b can give a Brieskorn homology 3-sphere with E_8-bounding with $\epsilon = -1$. □

Proposition 3.4. *Let G be the configuration (1) in Figure 1. The integral solutions a, b, c in* TABLE *1 with $a \leq 6$ are* TABLE *3:*

Proof. Let us take $a = 1$ in the case of (1). Then for some integer m we have $k = 2m + 1$, $\ell = 3m \pm 1 + 1$ and $b = 5m + 1 \pm 1$. Thus we get

$$c = 15m^2 + 16m + 5, \text{ or } 15m^2 + 4m + 1$$

from TABLE 1. In this way we get the expressions of b, c as in TABLE 3.

Let Y be one of Brieskorn homology 3-spheres for resolution graphs obtained by doing several blow-ups of branched triangular configurations (a, b, c) in TABLE 3. According to the formula of $\overline{\mu}$ in [6], we have $\overline{\mu}(Y) = -1$. From Ue's inequality (Theorem 1.2), we get $\mathfrak{ds} = 1$. In particular they have $g_8 = 1$.

a	b	c	a	b	c
1	$5i-3$	$15i^2-14i+4$	1	$5i$	$15i^2+4i+1$
2	$10i-7$	$60i^2-68i+21$	2	$10i+1$	$60i^2+28i+5$
3	$15i-11$	$135i^2-162i+52$	3	$15i-8$	$135i^2-108i+25$
3	$15i-1$	$135i^2+18i+4$	3	$15i+2$	$135i^2+72i+13$
4	$10i-5$	$60i^2-28i+9$	4	$10i-7$	$60i^2-52i+17$
5	$5i-4$	$15i^2-4i+9$	5	$5i-1$	$15i^2+14i+12$
6	$30i-23$	$540i^2-684i+229$	6	$30i-13$	$540i^2-324i+61$
6	$30i-5$	$540i^2-36i+13$	6	$30i+5$	$540i^2+324i+61$

TABLE 3. The pairs (a,b,c) $(i \geq 0)$ are blow-downed configurations with (1) with $-E_8$ intersection form and with $a \leq 6$.

Thus, by using Theorem 1.2 we have $g_8 = \underline{g_8} = \mathfrak{ds} = \underline{\mathfrak{ds}} = 1$. These data give Brieskorn homology 3-spheres as in TABLE 2. $\square$

3.4. **The negative E_8 boundings of $\Sigma(2,3,6n-1)$.** We restrict ourselves to $\Sigma(2,3,6n\pm 1)$. Let denote $Y_n^- = \Sigma(2,3,6n-1)$ and $Y_n^+ = \Sigma(2,3,6n-5)$. The invariants μ, $\overline{\mu}$ and d for $Y_n^\pm$ are represented as in TABLE 4. We focus on bounding 4-manifolds of Y_{2k+1}^-. The minimal resolution R_n for Y_n^- is

	μ	$\overline{\mu}$	d	definite bounding
Y_{2k}^+	1	1	0	$\mathfrak{ds} = \infty$
Y_{2k}^-	0	0	2	$\mathfrak{ds} = \infty$
Y_{2k+1}^+	0	0	0	must be $b_2(X) = 0$
Y_{2k+1}^-	1	-1	2	must be $b_2(X) = 8$

TABLE 4. Invariants of $Y_n^\pm$.

Figure 12. The intersection form of R_n is isomorphic to $-E_8 \oplus^{n-1} \langle -1 \rangle$.

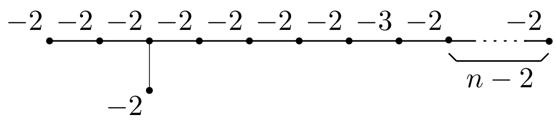


FIGURE 12. The minimal resolution graph of Y_n^-.

Any square -1 class in R_n cannot be realized as a sphere, in other words the following holds:

Proposition 3.5. *The 4-manifold R_n can be never blow-downed any more. Namely, the minimal genus of any square -1 class in R_n is positive.*

Proof. Since by replacing any component in FIGURE 12 with a Legendrian knot as in FIGURE 13, we can get a Stein surface on R_n.

As explained in [11], the union of one 0-handle and 2-handles along a Legendrian link with framings $tb-1$ admits a Stein structure, where tb is the Thurston-Bennequin invariant. Here we use the formula

$$tb(K) = w(K) - \lambda(K),$$

for the computation of $tb(K)$. Here w is the writhe of the Legendrian knot and λ is the number of left corners of the Legendrian knot. For example, in the cases of FIGURE 13, the first and the second examples are as follows:

$$tb(K_1) = 0 - 1 = -1,\ tb(K_2) = 0 - 2 = -2.$$

On the other hand, any Stein structure does not contain any (-1)-sphere due to the result in [8] by Lisca and Matić. This means that R_n can be never blow-downed any more. $\square$

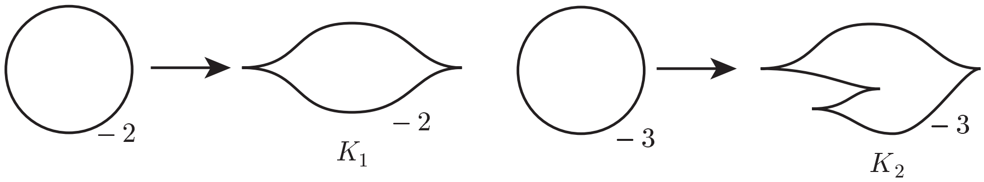


FIGURE 13. Deformations into Stein structure.

Y^-_{2k+1} has another spin bounding S_k as in Figure 14 with the intersection form isomorphic to $-E_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The direct sum $-E_8 \oplus H$ implies the existence of a homology 3-sphere Y separating $-E_8$ and H. The H-summand corresponds to a 4-manifold X with $Q_X \cong H$ and $\partial X = Y$. Can such a homology 3-sphere Y be taken as one satisfying $[Y] = 0$ in the homology cobordism group $\Theta^3_{\mathbb{Z}}$? We post the following question, which is equivalent to $\mathfrak{ds}(Y^-_{2k+1}) = 1$ for any k.

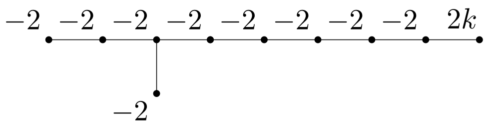


FIGURE 14. The plumbing graph for S_k.

Question 3.1. *Let k be a positive number. Can any homology 3-sphere Y in S_k separating the intersection form $Q_{S_k} = -E_8 \oplus H$ (i.e., $\partial X = Y$ and $Q_X \cong H$) bound an acyclic 4-manifold?*

3.5. **The embedding of Y^-_{2k+1} in $E(1)$.** Question 3.1 is unknown, although, we can give several negative E_8 boundings for Y^-_{2k+1}.

In the case of $n = 0$, it is well-known that $Y^-_1 = \Sigma(2,3,5)$ is the boundary of the E_8-plumbing. In the case of $n = 1$, since $Y^+_3 = \Sigma(2,3,13)$ bounds a contractible 4-manifold, we have a homology cobordism

$$Y^-_3 \simeq_h Y^-_3 \# (-Y^+_3) = M_3,$$

where M_n is defined in Section 3.1. By use of Theorem 1.3, we can give a negative E_8-bounding of Y_3^- with $g_8 = 1$.

Proposition 3.6. *For some integer k with $0 \leq k \leq 12, 14$, $E(1)$ can be decomposed along Y_{2k+1}^- so that $E(1) = W_k \cup_{Y_{2k+1}^-} N_{2k+1}$. Here W_k is a simply-connected, E_8-bounding of Y_{2k+1}^- with $g_8 = 1$ and $\epsilon = -1$.*

Proof. We start from a well-known decomposition $E(1) = M(2,3,5) \cup N_1$, where N_1 is the nuclei, which is defined in [11]. FIGURE 15 (Figure 16 in [1]) is the handle diagram for the decomposition. In the following, we

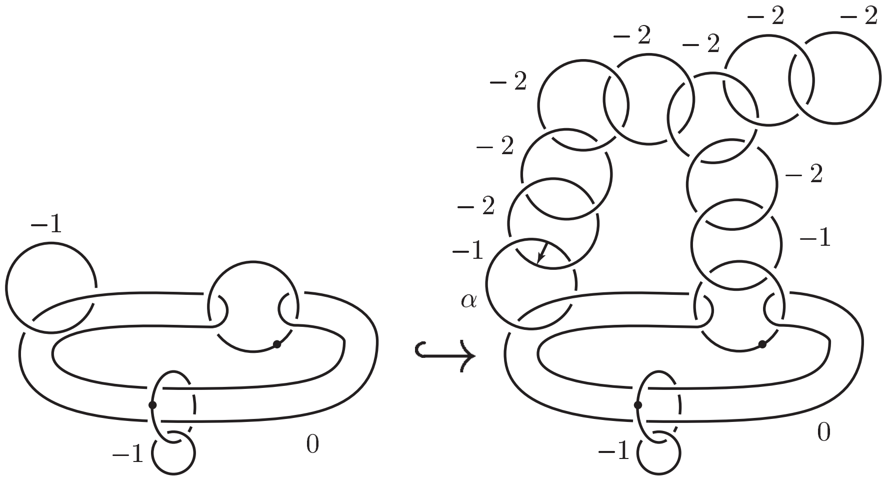


FIGURE 15. Figure16 in [1] and the embedding of N_1.

deform the decomposition into other ones via the following 2-handle slide of α in FIGURE 16. The handle slide by a straight band keeps the framing (the left picture in FIGURE 16). On the other hand, the handle slide by a twisting band (the right picture in FIGURE 16) decreases the framing by 4. Therefore, the framings of α become -1 and -5 respectively. We iterate this process to the linear 7-component link connecting the -2-framed 2-handle except the -2-framed 2-handle adjacent to another -1-framed 2-handle. We can realize 2-handle α with the framings of $-1, -5, -9, -13, -17, -21, -25$, and -29. These attaching spheres are all unknots. The 2-handles with framing $-3, -7, -11, -15, -19$, and -23 are obtained by sliding linear sub-k-chain ($0 \leq k \leq 5$) and the unknot in the 7-component link. For example, FIGURE 17 realizes a -7-framed unknot by sliding -5-framed 2-handle to an un-connecting -2-framed 2-handle.

This process gives other decomposition $E(1) = W_k \cup_{Y_{2k+1}^-} N_{2k+1}$, where k is $0 \leq k \leq 12$ or 14. In fact W_k is a 4-manifold with intersection form $-E_8$ and the boundary is Y_{2k+1}^-. The process above preserves the intersection form of the complement. As a result, W_k is a simply-connected 4-manifold with intersection form $-E_8$ whose boundary is Y_{2k+1}^- ($0 \leq k \leq 12$ or 14). The complement is the nuclei N_{2k+1}. See [11] for the definition of the nuclei. $\square$

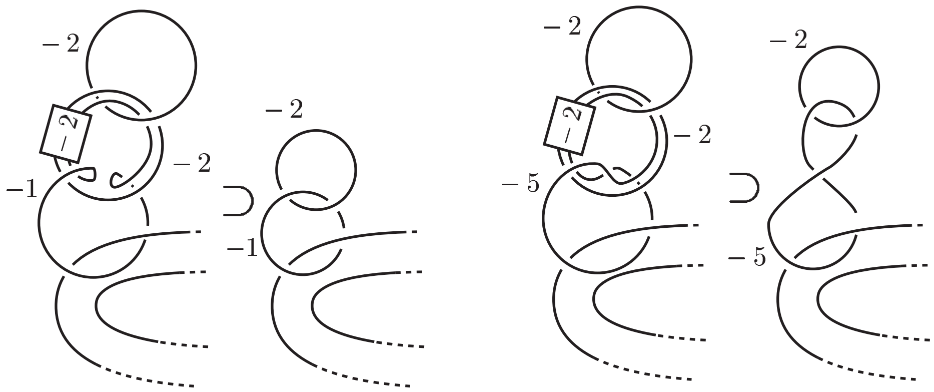


FIGURE 16. The straight handle slide and twisting handle slide.

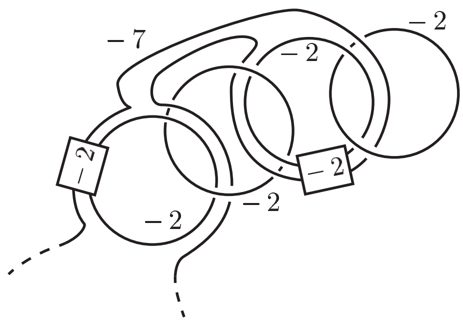


FIGURE 17. A realization of −7-framed 2-handle.

Proof of Theorem 1.7. The 4-manifold W_k is a negative E_8-bounding of Y^-_{2k+1} for $0 \le k \le 12$ or 14. Thus, for the integer k, we have $g_8(Y^-_{2k+1}) = 1$ and $\epsilon(Y^-_{2k+1}) = -1$. $\square$

There exists a homology cobordism $Y^+_{2k+1} \simeq_h (-M_{2k+1})\#Y^-_{2k+1}$, where $\simeq_h$ stands for homology cobordant. The homology 3-sphere $Y^+_{2k+1}\#(-Y^-_{2k+1}) = -M_{2k+1}$ has a 'positive' E_8-bounding with $g_8 = 1$ by Theorem 1.3. Even if Y^-_{2k+1} has a 'negative' E_8-bounding with $g_8 = 1$, we do not know whether Y^+_{2k+1} bounds a contractible 4-manifold or not. In general, what condition for homology spheres Y_1, Y_2 with $\epsilon(Y_1) + \epsilon(Y_2) = 0$ and $g_8(Y_i) = 1$ can cancel out the intersection form $E_8 \oplus (-E_8)$ into $\emptyset$? We pose a more general question (Question 5.8) in the final section.

4. The several sphere classes in $E(1)$.

Theorem 4.1. *The classes $k[f] - [s]$ ($1 \le k \le 13$ or $k = 15$) in $H_*(E(1))$ are represented by embedded spheres, where f is the general fiber and s is the section in the elliptic fibration.*

Proof. Let Q_l be the quadratic form by representing by the matrix $\begin{pmatrix} -l & 1 \\ 1 & 0 \end{pmatrix}$. The decomposition $W_k \cup_{Y_{2k+1}} N_{2k+1}$ in Proposition 3.6 gives $Q_{E(1)} = Q_{W_k} \oplus Q_{N_{2k+1}} \cong -E_8 \oplus Q_{2k+1}$. Let α denote the same class as the one in Proposition 3.6. This class is the first class $(1,0)$ in the N_{2k+1}-part.

By applying an isomorphism $(\mathbb{Z}^2, Q_{2k+1}) \to (\mathbb{Z}^2, Q_1)$ that the class $(1,0) \in \mathbb{Z}^2$ is mapped to $(1,-k)$, we obtain an isomorphism

$$(\mathbb{Z}^{10}, -E_8 \oplus Q_{2k+1}) \cong (\mathbb{Z}^{10}, -E_8 \oplus Q_1) \cong (\mathbb{Z}^{10}, \langle 1 \rangle \oplus 9\langle -1 \rangle).$$

Here the element for α is mapped to $(-3k, k, \cdots, k, k+1)$ by this isomorphism, because $(0,1) \in (\mathbb{Z}^2, Q_1)$ is the class for the general fiber of $N_1 \subset E(1)$.

By using the result in [19], this isomorphism induces a diffeomorphism $E(1) \cong \mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2}$. The element α is mapped to

$$-3k \cdot [h] + k \sum_{i=1}^{9} [e_i] + [e_9] = -k[f] + [s],$$

where $\{[h], [e_i] | 1 \le i \le 9\}$ is the generator in $H_2(\mathbb{C}P^2 \#^9 \overline{\mathbb{C}P^2})$. The classes $[f]$ and $[s]$ correspond to the fiber and the section of $E(1)$ respectively. In the case of $0 \le k \le 12, 14$, α, that is, $-k[f] + [s]$ can be represented as a sphere. $\square$

5. Some questions and problems.

Here we post several questions and problems.

Question 5.1. *Let Y be a homology 3-sphere.*

(1) *When does Y have a definite spin bounding?*
(2) *If $\mathfrak{ds}(Y) < \infty$, then does Y have an E_8-bounding?*
(3) *When the equality $m(-Y)/2 = \mathfrak{ds}(Y)$ or $\overline{m}(-Y)/2 = \underline{\mathfrak{ds}}(Y)$ hold?*

Question 5.2. *Does there exist any homology 3-sphere Y with $g_8(Y) > \underline{g_8}(Y)$, $\mathfrak{ds}(Y) \neq g_8(Y)$ or $\underline{\mathfrak{ds}}(Y) \neq \underline{g_8}(Y)$?*

Question 5.3. *Let Y be a Brieskorn homology 3-sphere. If $4d(Y) = -8\overline{\mu}(Y) > 0$, then is $\mathfrak{ds}(Y) = 4d(Y)$ true?*

Question 5.4. *Let Y be a Brieskorn homology 3-sphere with finite E_8-genus. Then is $g_8(Y) = \underline{g_8}(Y)$ true?*

We post some inequalities for bounding genus which are presumed by (11/8)-conjecture and Theorem 1.5.

Question 5.5. *For positive integer n, do the inequalities:*

$$|\Sigma(8n-2, 8n-1, 16n-3)|,\ |\Sigma(8n-1, 8n, 16n-1)| \ge 3n$$

$$|\Sigma(8n-2, 8n-1, 32n^2-8n+1)|,\ |\Sigma(8n-1, 8n, 32n^2-1)| \ge 3n$$

hold?

If one of these inequalities does not hold, then the (11/8)-conjecture does not hold.

Question 5.6. *Do these homology 3-spheres above give some E_8-boundings with $g_8 = 2n$?*

Question 5.7. *Let a_k denote the 2nd homology class $k[f] - [s]$ in $E(n)$, where f is the general fiber and s is a section. Does there exist an upper bound of k for a_k to be represented by an embedded S^2?*

Question 5.8. *For two homology 3-spheres with $\mathfrak{ds}(X_i) < \infty$ $(i = 1, 2)$, let us denote $\tilde{\mathfrak{ds}}(Y) = \epsilon(Y)\mathfrak{ds}(Y)$. Then when does the equality*

$$\tilde{\mathfrak{ds}}(X_1) + \tilde{\mathfrak{ds}}(X_2) = \tilde{\mathfrak{ds}}(X_1 \# X_2)$$

hold?

Although, in the case of $X_1 = \Sigma(2,3,17)$ and $X_2 = \Sigma(2,3,13)\#(-\Sigma(2,3,17))$, the equality holds, this equality seems unlikely, in general. In order to satisfy this equality, some geometrically special condition would be necessary.

Finally, we post future's direction for this paper's topic.

Problem 5.1. *Find more general constructions of positive (or negative) E_8-boundings for many homology 3-spheres.*

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