Pochette surgery on S^4

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Pochette

Definition 1 (Z.Iwase-Y.Matsumoto (East Asia Conf. in 2004))

We call

$$P = S^1 \times D^3 \natural S^2 \times D^2$$

a pochette.

$$P \simeq S^1 \vee S^2 \quad \text{(homotopy eq.)}$$

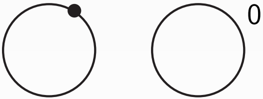


Figure: Pochette P

Submanifolds

$m, l, B, S \subset P$
$l = S^1 \times \{*\}$ (longitude)
$m = \{*\} \times \partial D^2$ (meridian)
$B = \{*\} \times \partial D^3$ (belt sphere)
$S = S^2 \times \{*\}$ (core sphere)

Homology
$H_1(\partial P) = \mathbb{Z}[m] \oplus \mathbb{Z}[l]$
$H_1(P) = \mathbb{Z}[l]$
$H_2(\partial P) = \mathbb{Z}[S] \oplus \mathbb{Z}[B]$
$H_2(P) = \mathbb{Z}[S]$

Another view of P

$$P = S^1 \times ST \cup H^2,$$

where $S^1 \times \{*\}$: attaching sphere of 2-handle H^2.

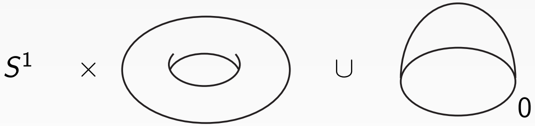


Figure: Pochette

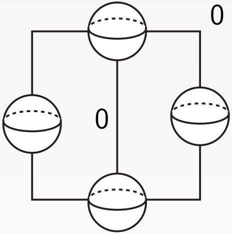


Figure: $P = S^1 \times ST \cup H$

Pochette surgery

Definition 2 (Iwase-Matsumoto)

M :a 4-manifold.
$e : P \hookrightarrow M$: an embedding $(P_e := e(P))$
$g : \partial P \to \partial(M \setminus P_e)$: a gluing map

$$M(e,g) := (M \setminus P_e) \cup_g P$$

It is called a pochette surgery.

$$M(e,g) = (M \setminus P_e) \cup_{g(m)} (\text{2-handle}) \cup (\text{3-handle}) \cup D^4$$

$g(m)$: the attaching sphere of the 2-handle

$g(m)$: the image of curve m

$m_e := e(m)$, $l_e := e(l)$.
$H_1(\partial(M \setminus P_e)) = \mathbb{Z}[m_e] \oplus \mathbb{Z}[l_e]$

$g : \partial P \to \partial(M \setminus P_e)$: gluing map

$$g_* : H_1(\partial P) \to H_1(\partial(M \setminus P_e))$$

$$g_*([m]) = p[m_e] + q[l_e],$$

$\gcd(p, q) = 1$

Lemma 3 (Iwase-Matsumoto)

$p/q \in \mathbb{Q}$ determines the isotopy type of $g(m)$.

Mod 2 framing

Lemma 4

The framing of $g(m)$ mod 2 determines the diffeomorphism type of $M(e, g)$.

Attaching with even framing can be extendable to P.
Recall Gluck surgery to understand mod 2 framing.

Definition 5 (Framing of $g(m)$)

We call the framing mod 2 of $g(m)$ mod 2 framing.

We write mod 2 framing by "ϵ".

Theorem 6 (Iwase-Matsumoto)

$M(e, g)$ is determined by the following data:

$$\begin{cases} e : P \hookrightarrow M \\ g_*([m]) \in H_1(\partial(M \setminus P_e)) \\ \textit{mod 2 framing of } g(m) \end{cases}$$

We denote the result of pochette surgery $M(e, g)$ by

$$M(e, p/q, \epsilon).$$

Examples

$p/q = 1/0$ case

$M(e, 1/0, 0) = M$ (trivial surgery)
$M(e, 1/0, 1)$: the Gluck surgery along S_e

$p/q = 0/1$ case

$M(e, 0/1, \epsilon)$: S^1-surgery (Cappell-Shaneson, Scharlemann used this.)

$\epsilon = 0$

$T^2 \subset M$: product nbd (with framing 0 vanishing cycle)
$M(e, p/q, 0) = M_{p/q}$: log-transformation along T^2
$T^2 \hookrightarrow S^1 \times ST \subset P \overset{e}{\hookrightarrow} M$

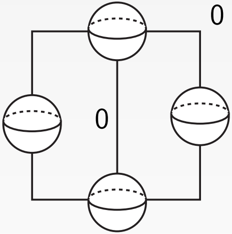


Figure: $P = S^1 \times ST \cup H$

Therefore, ...

Pochette surgery is a mixture of

- Gluck surgery
- log-transformation (near a framing 0 v.c.)
- S^1-surgery

There are so many intermediate surgeries.

Motivation

Our aim

To construct homotopy S^4's and investigate the diffeomorphism types.

Conjecture 7 (smooth Poincare conjecture in 4D)

Any smooth homotopy S^4 is diffeomorphic to S^4.

Our main results (Suzuki-T.)

- Computation of H_n.
- Determination of diffeomorphism types of several pochette surgeries yielding homotopy S^4's.

We will see main theorems later on.

The third view

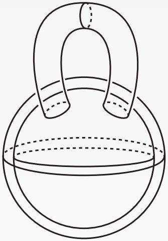


Figure: Pochette

$$P = S^1 \times D^3 \natural S^2 \times D^2 \cong S^2 \times D^2 \cup H^1$$

H^1: 1-handle

Core sphere & cord

$S := S^2 \times \{*\} \subset P$ (Core sphere)
$C \subset P$ (Cord).
$e : P \to S^4$ an embedding
$S_e := e(S),\ C_e := e(C),\ B_e = e(B),\ m_e := e(m)$

$$S^4 \setminus P_e = (S^4 \setminus N(S_e)) \setminus N(C_e)$$

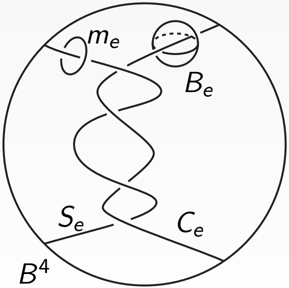


Figure: $(S^4 \setminus N(S_e)) \setminus N(C_e)$

Homology of $S^4(e, g, \epsilon)$

$g : \partial P \to \partial(S^4 \setminus P_e)$: slope p/q
$H_1(P) = \mathbb{Z}[l]$
$H_2(P) = \mathbb{Z}[S]$
$H_1(S^4 \setminus P_e) = \mathbb{Z}[m_e]$
$H_2(S^4 \setminus P_e) = \mathbb{Z}[B_e]$
$\ell = \ell k(C_e, S_e)$: linking number

H_1 of $S^4(e,g,\epsilon)$

$$H_1(\partial P) \xrightarrow{g_*} H_1(\partial(S^4 \setminus P_e))$$

$$g_*([m]) = p[m_e] + q[l_e], \quad g_*([l]) = p'[m_e] + q'[l_e]$$

$$H_1(\partial(S^4 \setminus P_e)) \xrightarrow{i_*} H_1(S^4 \setminus P_e)$$

$$i_*([m_e]) = [m_e], \quad i_*([l_e]) = \ell[m_e]$$

$$H_1(\partial P) \xrightarrow{j_1 \oplus j_2} H_1(S^4 \setminus P_e) \oplus H_1(P) \xrightarrow{k} H_1(S^4(e,g,\epsilon)) \quad \text{(M-V sequence)}$$

$$j_1 = i_* \circ g_*$$

$$j_1([m]) = (p+q\ell)[m_e], \; j_1([l]) = (p'+q'\ell)[m_e]$$

$$j_2([m]) = 0,\ j_2([l]) = [l]$$

$$A = \begin{pmatrix} p+q\ell & p'+q'\ell \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} H_1(S^4(e,g,\epsilon)) &= H_1(S^4 \setminus P_e) \oplus H_1(P)/\sim \\ &\cong \mathbb{Z}[m_e] \oplus \mathbb{Z}[l]/Im(A) \\ &\cong \mathbb{Z}[m_e]/(p+q\ell)[m_e]\ \ (p+q\ell \neq 0) \end{aligned}$$

Lemma 8

$$H_1(S^4(e,g,\epsilon)) \cong \begin{cases} \mathbb{Z}/(p+q\ell)\mathbb{Z} & p+q\ell \neq 0 \\ \mathbb{Z} & p+q\ell = 0. \end{cases}$$

H_2 of $S^4(e, g, \epsilon)$

$$H_2(S^4 \setminus P_e) = \mathbb{Z}[B_e],\ H_2(P) = \mathbb{Z}[S]$$

$$H_2(\partial P) \overset{g_*}{\to} H_2(\partial(S^4 \setminus P_e))$$

$$g_*([B]) = p[B_e] + q[S_e],\quad g_*([S]) = p'[B_e] + q'[S_e]$$

$$H_2(\partial(S^4 \setminus P_e)) \overset{i_*}{\to} H_2(S^4 \setminus P)$$

$$i_*([B_e]) = [B_e],\quad i_*([S_e]) = \ell[B_e]$$

$$H_2(\partial P) \overset{j_1 \oplus j_2}{\to} H_2(S^4 \setminus P_e) \oplus H_2(P) \overset{k}{\to} H_2(S^4(e, g, \epsilon)) \quad \text{(M-V sequence)}$$

$$j_1 = i_* \circ g_*$$

$$j_1([B]) = (p + q\ell)[B_e],\ j_1([S]) = (p' + q'\ell)[B_e]$$

$$j_2([B]) = 0,\ j_2([S]) = [S]$$

$$A = \begin{pmatrix} p + q\ell & p' + q'\ell \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} H_2(S^4(e,g,\epsilon)) &= H_2(S^4 \setminus P_e) \oplus H_2(P)/\sim \\ &\cong \mathbb{Z}[B_e] \oplus \mathbb{Z}[S]/Im(A) \\ &\cong \mathbb{Z}[B_e]/(p+q\ell)[B_e] \ \ (p+q\ell \neq 0) \end{aligned}$$

Lemma 9

$$H_2(S^4(e,g,\epsilon)) \cong \begin{cases} \mathbb{Z}/(p+q\ell)\mathbb{Z} & p+q\ell \neq 0 \\ \mathbb{Z} & p+q\ell = 0. \end{cases}$$

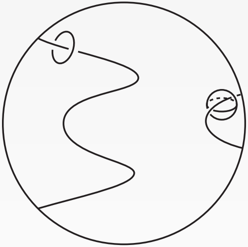
$$H_3(S^4(e,g,\epsilon)) \cong \begin{cases} 0 & p+q\ell \neq 0 \\ \mathbb{Z} & p+q\ell = 0. \end{cases}$$

Theorem 10 (Suzuki-T.)

$S^4(e,g,\epsilon)$ is a homotopy S^4, iff

$$\begin{cases} p+qh=\pm 1, \\ \pi_1(S^4(e,g,\epsilon))=e. \end{cases}$$

Trivial cord case



First, $\ell = 0$ holds.
$H_1 = \mathbb{Z}/p\mathbb{Z} \Rightarrow p = \pm 1$

Theorem 11 (Suzuki-T.)

Let $e : P \hookrightarrow S^4$ be an embedding of P with a trivial cord. Then if a pochette surgery $S^4(e, p/q, \epsilon)$ is a homotopy S^4, then it is diffeomorphic to

$$\begin{cases} \textit{standard } S^4 & \epsilon = 0 \\ \textit{the Gluck surgery along } S_e & \epsilon = 1 \end{cases}$$

Thus, if S_e is ribbon, then any pochette surgery with trivial cord is diffeomorphic to standard S^4.

(proof)
The pochette surgery.

$$S^4(e, 1/q, \epsilon)$$

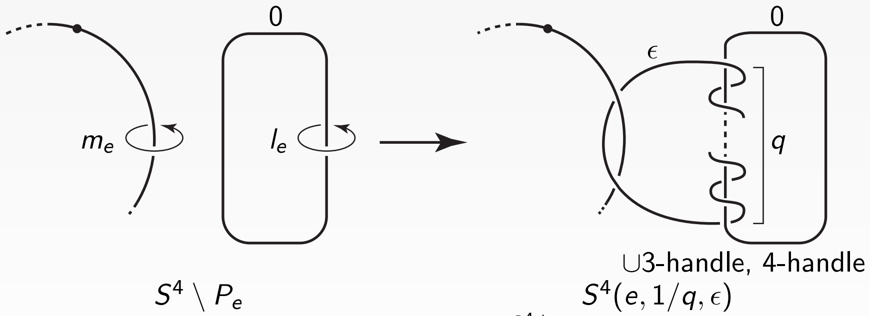


Figure: Attaching P on $S^4 \setminus P_e$.

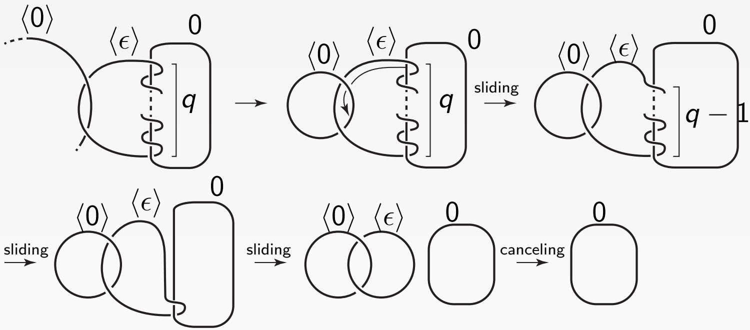


Figure: The isotopy type of the rightmost unknot component.

S_e: unknot case

Theorem 12 (Suzuki-T.)

Let e be an embedding of P into S^4 with core sphere S_e unknot. Then any $S^4(e, 1/q, \epsilon)$ is diffeomorphic to the standard S^4.

(proof)
$\pi_1(S^4 \setminus S_e, \partial)$: the set of isotopy classes of cord in $S^4 \setminus S_e$.

$$\pi_1(S^2 \times S^1) \xrightarrow{i} \pi_1(S^4 \setminus S_e) \to \pi_1(S^4 \setminus S_e, \partial)$$

$\pi_1(S^4 \setminus S_e) = \mathbb{Z}$
i: isomorphism
hence, $\pi_1(S^4 \setminus S_e, \partial) = e$
Any cord is trivial.
Therefore, from Theorem 11, $S^4(e, 1/q, \epsilon) = S^4$ $\square$

S_e: knotted & C_e: non-trivial case

Theorem 13 (Suzuki-T.)

Let S_e be the following 2-knot.

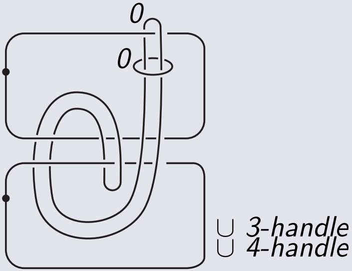


Figure: *A diagram of a ribbon 2-knot complement.*

Then $S^4(e, 1/2, \epsilon)$ is diffeomorphic to the standard S^4.

Proof

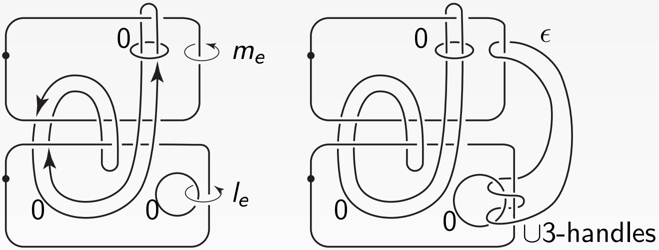
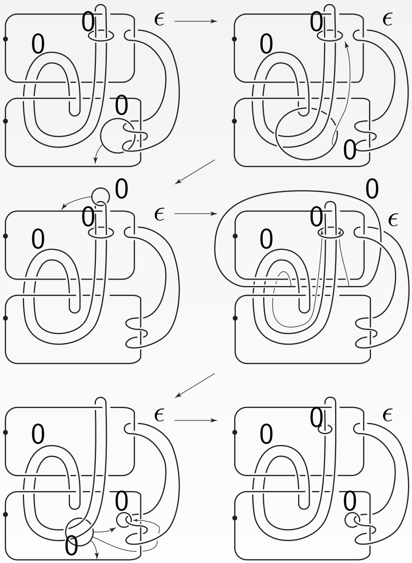


Figure: A pochette surgery for non-trivial 2-knot S_e with a nontrivial cord.



$S^4(e, 1/2, \epsilon) = C \cup (-C)$: C: a cont'ble 4-mfd with no 3-handles
A standard argument results in $C \cup (-C) = S^4$. $\square$

Corollary 14

If S_e is a ribbon with C_e a cord with $\ell = 1$ and if $S^4(e, 1/2, \epsilon)$ is simply-connected, then $S^4(e, 1/2, \epsilon)$ is diffeomorphic to S^4.

Question 15

If $S^4(e, p/q, \epsilon)$ is a homotopy sphere, is $S^4(e, p/q, \epsilon)$ always standard S^4?