The link surgery of $S^2 \times S^2$ and Scharlemann's manifolds

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Abstract

Fintushel-Stern's knot surgery gave many exotic pairs, which are homeomorphic but non-diffeomorphic. We show that if an elliptic fibration has two, parallel, oppositely-oriented vanishing circles (for example $S^2 \times S^2$ or Matsumoto's S^4), then the knot surgery gives rise to the standard manifold. The diffeomorphism can give an alternative proof that Scharlemann's manifold is standard (originally as proven by Akbulut [Ak1]).

1 Introduction.

1.1 Knot surgery.

For a 4-manifold X containing the cusp neighborhood C the *knot surgery* X_K is defined by R. Fintushel and R. Stern [FS], where K is a knot in S^3, see definitions in the next section. It is easy to see that X_K is homeomorphic to X by Freedman's celebrated result if X is simply connected and closed. We have a natural question: When is (X, X_K) an exotic pair?

The Seiberg-Witten (SW-invariant) formula in [FS] by R. Fintushel and R. Stern

$$SW_{X_K} = SW_X \cdot \Delta_K, \tag{1}$$

showed that many pairs (X, X_K) are exotic. Here Δ_K is the Alexander polynomial of K. In the case where $\Delta_K(t) = 1$ or $SW_X = 0$, it is unknown whether the pair is exotic or not in general.

It is well-known that $S^2 \times S^2$ is diffeomorphic to the double $\overline{C} \cup C$ of C, namely $S^2 \times S^2$ admits achiral elliptic fibration, where the overbar notation stands for the reversed orientation of the manifold. The diagram is drawn in Figure 1.

Definition 1.1. *We denote by A_K the knot surgery $\overline{C} \cup C_K$ of $S^2 \times S^2 = \overline{C} \cup C$.*

The 4-manifold A_K is homotopy equivalent to $S^2 \times S^2$, while SW-invariant cannot distinguish whether A_K is exotic or not, since $SW_{S^2 \times S^2} = 0$ holds.

In [Ak2] S. Akbulut showed that A_{3_1} is diffeomorphic to $S^2 \times S^2$. The diffeomorphism is essentially due to his another result [Ak1]. Our first main theorem is:

Theorem 1.1. *A_K is diffeomorphic to $S^2 \times S^2$ for any knot K.*

We will prove this theorem in Section 3. The theorem shows the existence of exotic embedding of C into $S^2 \times S^2$.

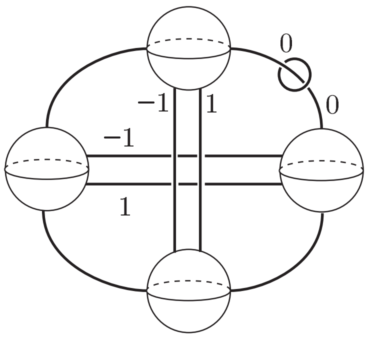

Figure 1: Two parallel, oppositely-oriented cusp fibers in $S^2 \times S^2$.

1.2 Link surgery.

We extend the knot surgery for links (according to Fintushel and Stern's definition in [FS]) as *link surgery (operation)*. This is regarded as a variation of fiber-sum operation connecting some manifolds rather than surgery. Our link surgery operation is defined for an n-tuple $(X_1, X_2, \cdots, X_n)$ of 4-manifolds each of which contains (specified) C and an n-component link L in S^3.

When every X_i is a copy of $S^2 \times S^2$, we denote the link surgery operation by A_L. We will give a generalization of Theorem 1.1.

Theorem 1.2. *For an n-component link L, A_L is diffeomorphic to*

$$A_L = \begin{cases} \#^{2n-1} S^2 \times S^2 & \text{if } L \text{ is a proper link} \\ \#^{2n-1}\mathbb{C}P^2 \#^{2n-1}\overline{\mathbb{C}P^2} & \text{otherwise.} \end{cases}$$

For the proof we give a way to draw handle pictures of link surgery operation $X(C, \cdots, C; L)$ for a split link $L = K_1 \cup K_2$ or Hopf link $L = H$

1.3 Scharlemann's manifolds.

Let $S^3_p(K)$ be the p-surgery along K in S^3 and $\gamma(\epsilon)$ an embedded framed curve in $S^3_p(K)$. Here γ is a simple closed curve in $S^3 - K \subset S^3_p(K)$ and ϵ is a framing. This framing is defined in Section 5 (Definition 5.1).

We use the obvious diagram of $S^3_p(K)$ to assign an integer to the framing. We concern with the free homotopy class of the framed curve and denote the homotopy class by the same notation $\gamma(\epsilon)$. In particular the framing is (mod2)-framing. Any homotopy class of the framed curve in $S^3_p(K)$ can be represented in the form of $\gamma(\epsilon)$.

For an embedded framed curve $\gamma(\epsilon)$ in $S^3_p(K)$ we define a 4-manifold $B_{K,p}(\gamma(\epsilon))$ (*Scharlemann's manifold*) to be the result of a surgery (see Definition 5.2) along $S^1 \xrightarrow{\gamma} S^3_p(K) \hookrightarrow S^3_p(K) \times S^1$ using the framing ϵ (see Definition 5.2). The diffeomorphism type depends only on (K, p) and the free isotopy type of the image of the framed curve $\gamma(\epsilon)$ in $S^3_p(K) \times S^1$. Note that if two framed curves $\gamma(\epsilon)$ and $\gamma'(\epsilon')$ are free homotopic in $S^3_p(K)$, then the two framed curves give the same isotopy class in $S^3_p(K) \times S^1$. Hence the homotopy class of a framed curve $\gamma(\epsilon)$ definitely determines the diffeomorphism type $B_{K,p}(\gamma(\epsilon))$. This is the reason why we consider the homotopy class of $\gamma(\epsilon)$. If γ is a normal generator of $\pi_1(S^3_p(K))$, then each manifold $B_{K,p}(\gamma(\epsilon))$ is homeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ or $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ as can be seen from results presented in [FQ]. In the case of $p = -1$ we drop the suffix p from $B_{K,p}(\gamma(\epsilon))$ as $B_K(\gamma(\epsilon))$.

Scharlemann in [Sc] studied the case where $(K, p) = (3_1, -1)$ and $\gamma = \gamma_0$ (the meridian of 3_1) and showed that $B_{3_1}(\gamma_0(1))$ has a fake self-homotopy structure on $S^3 \times S^1 \# S^2 \times S^2$. At that time the diffeomorphism type of $B_K(\gamma(\epsilon))$ was not determined. After that Akbulut [Ak1] showed the following theorem using an amazingly difficult handle calculus.

Theorem 1.3 ([Ak1])**.** *$B_{3_1}(\gamma_0(1))$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.*

Since Akbulut's result it had been unknown for a long time whether in general $B_K(\gamma(\epsilon))$ is diffeomorphic to the standard manifold or not. Here we show the following as the third main theorem.

Theorem 1.4. *For any knot K in S^3 and $\gamma_0 \subset S^3_{-1}(K)$ the meridian of K in the diagram $B_K(\gamma_0(1))$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.*

In the second half of Section 5.2 we will consider the diffeomorphism type of $B_{3_1}(\gamma(\epsilon))$ for other homotopy classes than $\gamma_0(\epsilon)$.

Theorem 1.1 and 1.4 are proven by S. Akbulut in [Ak4] independently. Our proofs are based on Lemma 3.3 regarding knot surgery in some achiral elliptic fibration. The essence in the lemma becomes important to extend knot surgery case to any link surgery operation case.

Acknowledgments.

The author originally was taught the candidates A_K of exotic $S^2 \times S^2$ by Professor Manabu Akaho ([Aka]). This paper is the negative but complete answer for his question. I thank him for telling me about attractive 4-dimensional world.

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2 Preliminaries

2.1 Several singular fiber neighborhoods and knot surgery.

First we recall the cusp neighborhood C and fishtail neighborhood F, see [GS] as the explanation about the topology of elliptic fibrations and singular fibers. We define two more neighborhoods with some singular fibers as well.

Definition 2.1 (Fishtail (or Cusp) neighborhood.)**.** *A fishtail (or cusp) neighborhood F (or C) is an elliptic fibration over D^2 with one fishtail (or cusp) singular fiber. The handle picture is the top-left (or top-right) in Figure 2. The neighborhood C (or F) includes self-intersection 0 torus as the general fiber.*

Definition 2.2 (Symmetric fishtail (or cusp) neighborhood.)**.** *We denote a fiber-sum of two parallel oppositely-oriented fishtail (or cusp) fibers over D^2 by SyF (or SyC). The handle picture is the bottom-left (or bottom-right) in Figure 2. The neighborhood SyF (or SyC) includes self-intersection 0 torus as the general fiber. We call SyC (or SyF) symmetric cusp (or fishtail) neighborhood.*

By the diagrams in Figure 2 SyF and SyC have the obvious embeddings $F \hookrightarrow SyF$ and $C \hookrightarrow SyC$ respectively.

Let X be a 4-manifold that contains C or F, and K a knot in S^3. The symbol ν (and $\overline{\nu}$) represents the open neighborhood (and its closure).

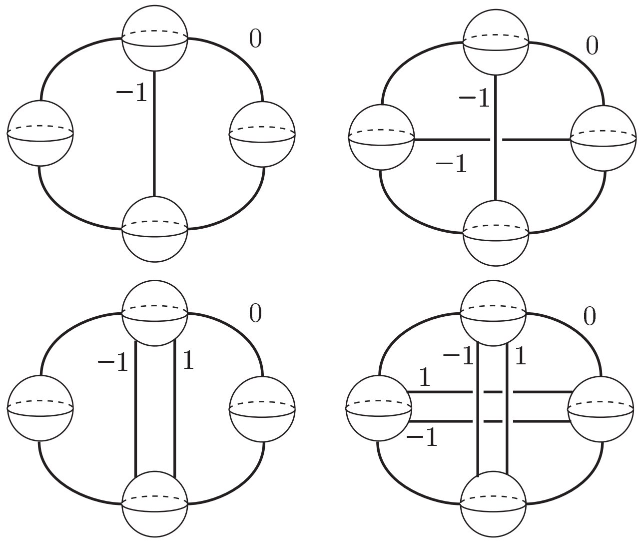

Figure 2: F, C, SyF, and SyC.

Definition 2.3. *We define (Fintushel-Stern's) knot surgery $X_{K,n}$ as*

$$X_{K,n} := [X - \nu(T)] \cup_{\varphi_n} [(S^3 - \nu(K)) \times S^1].$$

Here the gluing map is

$$\varphi_n : \partial\overline{\nu}(K) \times S^1 \to \partial\overline{\nu}(T) = T^2 \times \partial D^2$$

such that the map φ_n induces the following on the 1st homology:

$$[\{\textit{the meridian of } K\} \times \{pt\}],\ [\{pt\} \times S^1] \to \alpha,\ \beta$$

$$[\{\textit{the longitude of } K\} \times \{pt\}] + n[\{\textit{the meridian of } K\} \times \{pt\}] \to [\{pt\} \times \partial D^2]$$

where α, β are generators of $H_1(T^2)$. When X contains F, we assume that α is the class of the the vanishing circle. In the case of $n = 0$, we denote the result of the knot surgery by simply X_K.

2.2 The logarithmic transformation.

We define the logarithmic transformation. Let X be an oriented 4-manifold and $T \subset X$ an embedded torus with self-intersection 0.

Definition 2.4. *Let γ be an essential simple closed curve in T and φ a homeomorphism $\partial D^2 \times T^2 \to \partial\nu(T)$ satisfying $\varphi(\partial D^2 \times \{pt\}) = q(\{pt\} \times \gamma) + p(\partial D^2 \times \{pt\})$. Removing $\nu(T)$ from X and attaching $D^2 \times T^2$ by use of φ, we get the following surgery*

$$[X - \nu(T)] \cup_\varphi D^2 \times T^2.$$

We say this surgery logarithmic transformation and denote it by $X_{T,p,q,\gamma}$.

It is well-known that the diffeomorphism type of logarithmic transformation depends only on the data (T, p, q, γ). The integer p is *multiplicity* of the logarithmic transformation, γ the *direction* and q the *auxiliary multiplicity.*

If $p = 1$, then we call $X_{T,1,q,\gamma}$ a *q-fold Dehn twist* of $\partial\nu(T)$ along T parallel to γ.

Lemma 2.1 (Lemma 2.2 in [G1]). *Suppose $N = D^2 \times S^1 \times S^1$ is embedded in a 4-manifold X. Suppose there is a disk $D \subset X$ intersecting N precisely in $\partial D = \{q\} \times S^1$ for some $q \in \partial D^2 \times S^1$, and that the normal framing of D in X differs from the product framing on $\partial D \subset \partial N$ by ± 1 twist. Then the diffeomorphism type of X does not change if we remove N and reglue it by a k-fold Dehn twist of ∂N along $S^1 \times S^1$ parallel to $\gamma = \{q\} \times S^1$.*

The submanifold $N \cup \nu(D)$ in Lemma 2.1 is diffeomorphic to a fishtail neighborhood F. Lemma 2.1 implies the following.

Lemma 2.2. *Let X be a 4-manifold containing F. Then a k-fold Dehn twist of a neighborhood of the general fiber parallel to the vanishing circle of the fishtail fiber does not change the differential structure.*

3 Knot surgery case.

3.1 1-strand twist.

Let X be a 4-manifold containing C, K_1 any knot in S^3, and K_2 the meridian of K_1. The torus $T_2 := K_2 \times S^1 \subset [S^3 - \nu(K_1)] \times S^1 \subset X_{K_1}$ is self-intersection 0. The subset $N_2 := \nu(K_2) \times S^1$ is the trivial normal bundle over T_2.

Definition 3.1 (1-strand twist). *We call an (n-fold) Dehn twist along $T_2 \subset X_{K_1}$ parallel to K_2 (n-fold) 1-strand twist of X_{K_1} along K_2.*

Lemma 3.1. *The n-fold 1-strand twist of X_{K_1} along K_2 does not change the differential structure.*

Proof. Any parallel copy $K_2' \subset \partial N_2$ of K_2 moved through the use of obvious trivialization of N_2 is isotopic to one of vanishing circles of C_{K_1}. Thus there exists a disk $D \subset C_{K_1}$ with $\partial D = K_2'$ whose framing of ∂D coming from the trivialization of $\nu(D)$ differs from the normal framing of the trivialization of N_2 by -1. Hence $N_2 \cup \nu(D)$ is the fishtail neighborhood.

Therefore Lemma 2.2 gives the following:

$$X_{K_1,n} \cong X_{K_1,0} = X_{K_1}.$$

□

This diffeomorphism can be also understood using handle calculus as in Figure 3, which was pointed out by S. Akbulut in [Ak1]. The left in Figure 3 is the 4_1 surgery of the cusp neighborhood. Sliding the top -1-framed 2-handle over one of two 0-framed 2-handles below, we get the right-top one in Figure 3. Sliding upper 0-framed 2-handle over the -1-framed 2-handle, we have the right-bottom picture. This process increases the framing of the knot by 1. Iterating the process or the inverse one, we can change the framing to the arbitrary integer.

3.2 3-strand twist.

Finding a hidden fishtail neighborhood in SyF_K or SyC_K, we shall prove a diffeomorphism using 3-strand twist.

Let L be a 2-component link as in Figure 4. The left box is some tangle which presents K_1. Let X be a 4-manifold containing SyC or SyF. Along the general torus fiber in the fibration we perform a knot-surgery X_K. The torus $T_2 = K_2 \times S^1 \subset [S^3 - \nu(K_1)] \times S^1$ has the trivial neighborhood in X_{K_1}. We denote the neighborhood of the torus by N_2.

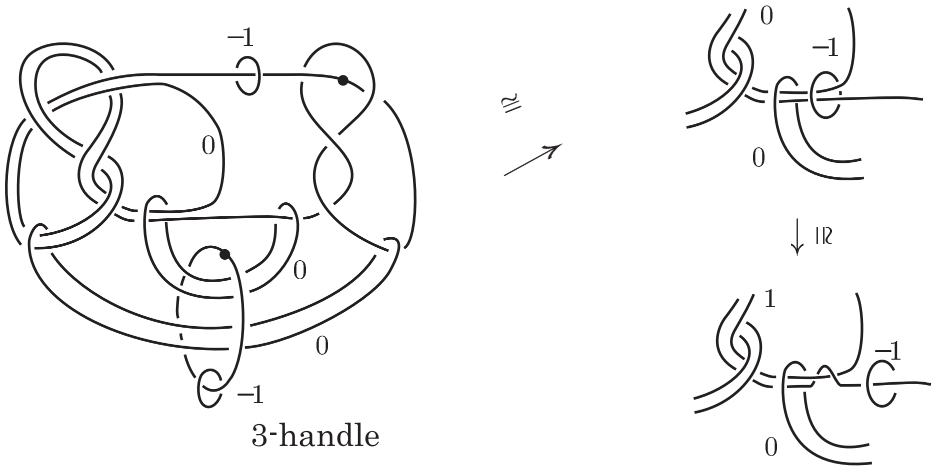

Figure 3: Diagram C_K (ex. $K = 4_1$) and framing change.

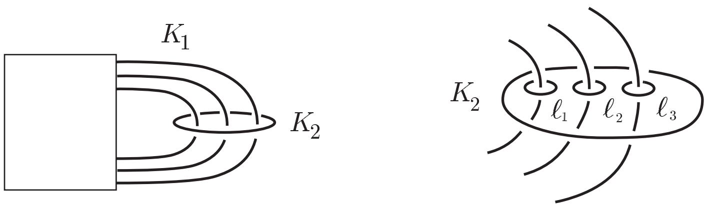

Figure 4: $L = K_1 \cup K_2$ and ℓ_1, ℓ_2, ℓ_3.

Definition 3.2 (3-strand twist)**.** *Let X be a 4-manifold containing SyC or SyF. We call any 1-fold Dehn twist along $T_2 \subset X_{K_1}$ parallel to K_2 3-strand twist along K_2. Here in the case of SyF K_2 is parallel to the vanishing circles.*

Lemma 3.2. *For a manifold X containing SyC or SyF, the 3-strand twist of X_{K_1} along K_2 does not change the differential structure.*

Proof. Our main strategy here is to construct a fishtail neighborhood in which $K_2 \times S^1$ is a general fiber. Here we can find an obvious three-punctured disk P whose boundaries are K_2, ℓ_1, ℓ_2, and ℓ_3 as indicated in Figure 4. Here each meridian ℓ_i lies in the boundary of N_1 which is the neighborhood of K_1. Figure 5 is the following modification of either of the handle pictures of SyF or SyC in Figure 2. We take the middle 1-handle, two 2-handles running the 1-handle in Figure 2, and an additional 1-framed 2-handle (canceling with a 3-handle by one slide to another 1-framed 2-handle). Each image $\varphi_0(\ell_i)$ is parallel to two vanishing circles in X_{K_1} as in Figure 5.

We construct mutually disjoint three annuli A_1, A_2 and A_3 such that one component of each ∂A_i is $\varphi_0(\ell_i)$. In addition these annuli and P are also disjoint because P is embedded in the $[S^3 - \nu(K_1)] \times S^1$ part. A_1 is indicated in Figure 6 and the right side of ∂A_1 is $\varphi_0(\ell_1)$. A_2 and A_3 are indicated in the left and right in Figure 7 respectively. A_3 runs through the carved 2-handle (the dotted 1-handle) once. The right sides of ∂A_2 and ∂A_3 are $\varphi_0(\ell_2)$ and $\varphi_0(\ell_3)$. From the pictures obviously A_1, A_2 and A_3 are disjoint annuli in A_{K_1}.

The other sides of ∂A_i coincide with the boundaries of 2-disks parallel to the cores of the 2-handles in Figure 5. The three 2-disks are disjoint from $P \cup A_1 \cup A_2 \cup A_3$ since these 2-handles are disjoint from P and A_i. Capping the 2-disks C_1, C_2 and C_3 to three components of $\partial(P \cup A_1 \cup A_2 \cup A_3) - K_2$, we obtain an embedded disk

$$D := P \cup A_1 \cup A_2 \cup A_3 \cup C_1 \cup C_2 \cup C_3$$

in A_{K_1} whose boundary is K_2.

The framing in $\partial\nu(D)$ inducing from the trivialization of $\nu(D)$ differs from the framing of K_2 inducing from the normal bundle of N_2 by $-1+1+1=1$. Therefore $N_2 \cup \nu(D)$ is diffeomorphic to $\overline{F}$.

Instead, sliding the canceling 0-framed 2-handle to the -1-framed 2-handle, we can prove the existence of three embedded disks whose boundaries are $\varphi_0(\ell_i)$. As a result the disk D obtained in the similar way has -1-framing on the boundary. The -1-framed 2-disk and N_2 construct a fishtail neighborhood F whose general fiber is T_2.

Applying Lemma 2.2 to this situation, we obtain the assertion of Lemma 3.2. □

For a 4-manifold X satisfying the same assumption as Lemma 3.2, we can also prove that any odd-strand twist does not change the differential structure.

3.3 Proof of Theorem 1.1.

Since $\overline{C} \cup C$ includes SyC as in Figure 1, 3-strand twist of A_{K_1} along K_2 gives rise to the same manifold, namely we have $A_{K_1} \cong \overline{C} \cup C_{K_3,n}$. The integer n is one of ∓ 1, ∓ 9. K_3 is the knot obtained by the ± 1-Dehn surgery along K_2 as in Figure 8. By using 1-strand twist in Section 3.1 we have $A_{K_3} \cong \overline{C} \cup C_{K_3,n} \cong A_{K_1}$.

Y. Ohyama in [Oh] has proven that local 3-strand twist of knots is an unknotting operation of knots. Therefore for any knot K there exists a finite sequence of local 3-strand twists: $K = k_0 \to k_1 \to \cdots \to k_n =$ unknot. The sequence implies a sequence of diffeomorphisms:

$$A_K = A_{k_0} \cong A_{k_1} \cong \cdots \cong A_{k_n} = S^2 \times S^2.$$

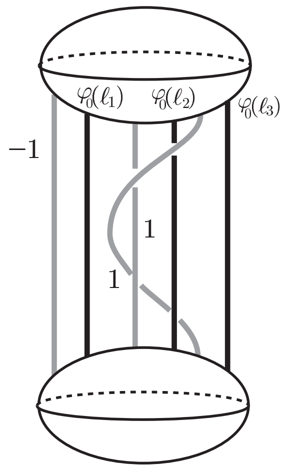

Figure 5: An isotopy of $\varphi_0(\ell_i)$.

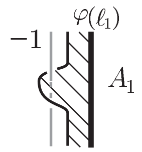

Figure 6: A_1.

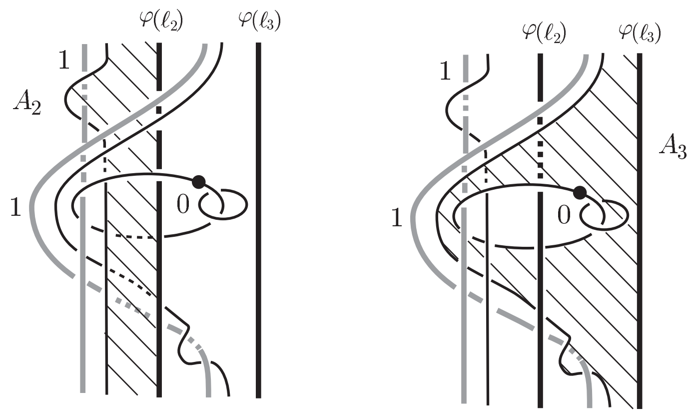

Figure 7: Two embedded annuli A_2, A_3.

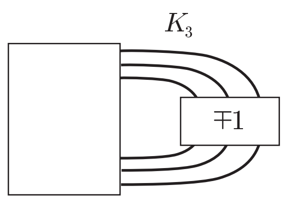

Figure 8: K_3: ±1-Dehn surgery along K_2 of K_1. The right box is the ∓1 full twist.

□

The argument in the proof of Theorem 1.1 can be summarized as follows:

Lemma 3.3. *Any knot surgery of any achiral elliptic fibration containing SyF (or SyC) does not change the differential structure.*

Y. Matsumoto's achiral elliptic fibration on S^4 in [M] includes SyF. The handle picture can be seen in Figure 8.38 in [GS].

Corollary 3.1. *Any knot surgery along a general fiber in Matsumoto's elliptic fibration on S^4 (such that the meridian of the knot is isotopic to the vanishing circle) is diffeomorphic to standard S^4.*

3.4 Infinitely many exotic embeddings.

Using the diffeomorphism, we obtain infinitely many embeddings:

$$C \hookrightarrow C \cup \overline{C_K} = S^2 \times S^2. \tag{2}$$

Thus we can give the following:

Corollary 3.2. *There exist infinitely many homeomorphic but mutually non-diffeomorphic embeddings $C \hookrightarrow S^2 \times S^2$. Namely the embeddings give infinitely many exotic complements.*

Proof. We show that the complements $\overline{C_K}$ of the embeddings (2) give mutually non-diffeomorphic infinite exotic 4-manifolds. The cusp neighborhood C is embedded in K3 surface $E(2)$ as a neighborhood of a singular fiber of the elliptic surface. The group of self-diffeomorphisms up to isotopy on $\partial C \cong \Sigma(2,3,6)$ is $\mathbb{Z}/2\mathbb{Z}$ in the same way as the proofs of Lemma 8.3.10 in [GS] and Lemma 3.7 in [G2]. The nontrivial self-diffeomorphism is a $180°$ rotation of ∂C about the horizontal line in the top-right picture in Figure 2. Since the diffeomorphism is caused by a symmetry on 0-framed trefoil, this diffeomorphism extends to $E(2)$ (see also the proof of Theorem 0.1 in [Ak2]). Thus, if $E(2)_{K_1}$ and $E(2)_{K_2}$ are non-diffeomorphic for some knots K_1, K_2, then C_{K_1} and C_{K_2} are non-diffeomorphic. The formula (1) and $SW_{E(2)} = 1$ give infinitely many differential structures in $\{C_K|K\text{:knot}\}$. The homeomorphism $C \approx C_K$ for any knot K is due to the fact $C \cup \overline{C_K} \cong S^2 \times S^2$ (spin) and the result (0.8) Proposition-(iii) in [B]. Therefore $\{C_K|K\text{:knot}\}$ includes infinitely many exotic structures. □

4 Link surgery case.

In this section we give how to draw a handle picture of the link surgery operation $X(C, \cdots, C; L)$ in the cases where L is a split link and is a Hopf link. Finally we will prove A_L is the standard manifold (Theorem 1.2).

Let $L = K_1 \cup \cdots \cup K_n$ be an n-component link and X_i $(i = 1, \cdots, n)$ oriented 4-manifolds which contain the cusp neighborhood C_i. Let T_i be a general fiber of C_i. By the gluing maps

$$\varphi_i : \partial\bar{\nu}(K_i) \times S^1 \to \partial\bar{\nu}(T_i) = T_i \times \partial D^2$$

satisfying

$$\varphi_i(l_i \times \{\text{pt}\}) = \{\text{pt}\} \times \partial D^2$$
$$\varphi_i(m_i \times \{\text{pt}\}) = \alpha_i,\ \varphi_i(\{\text{pt}\} \times S^1) = \beta_i,$$

where l_i and m_i are the longitude and meridian of K_i and α_i, β_i are two circles in $\partial\bar{\nu}(T_i)$ corresponding to a basis in $H_1(T_i)$.

Definition 4.1. *For any i removing $\nu(T_i)$ from X_i, gluing K_i-component in the boundary of $[S^3 - \nu(L)] \times S^1$ using φ_i, we define the link surgery operation as*

$$\coprod_{i=1}^{n} X_i \to [X_i - \nu(T_i)] \cup_{\varphi_i} [S^3 - \nu(L)] \times S^1.$$

We write the link surgery operation of $(X_1, \cdots, X_n)$ along a link L by $X(X_1, \cdots, X_n; L)$.

The SW-invariant of $X(X_1, \cdots, X_n; L)$ is computed as follows:

$$SW_{X(X_1,\cdots,X_n;L)} = \Delta_L(t_1, \cdots, t_n) \cdot \prod_{i}^{n} SW_{E(1)\#_{T=T_i} X_i},$$

where $\Delta_L(t_1, \cdots, t_n)$ is the n variable Alexander polynomial of L and $E(1)\#_{T=T_i} X_i$ is the fiber sum of the elliptic fibration $E(1)$ and X_i along general fibers T and T_i respectively. The definition of the fiber sum can be seen in [FS].

Here we consider the link surgery operation of $\coprod_{i=1}^{n} S^2 \times S^2$ along any n-component link L. We denote the operation by A_L. The following diffeomorphism

$$E(1)\#_{T=T_i} S^2 \times S^2 \cong E(1)\#^2 S^2 \times S^2 = \#^3 \mathbb{C}P^2 \#^{11} \overline{\mathbb{C}P^2} \tag{3}$$

holds. The first diffeomorphism is due to Figure 9. The leftmost figure is a picture of $E(1)\#_{T=T_i} S^2 \times S^2$, where the handle decomposition of $E(1)$ uses the diagram of Figure 8.10 in [GS]. Sliding handles several times, we find a separated Hopf link in the rightmost figure. The second equality in (3) is due to some blow ups and downs. Thus the vanishing theorem of SW-invariant implies $SW_{A_L} = 0$.

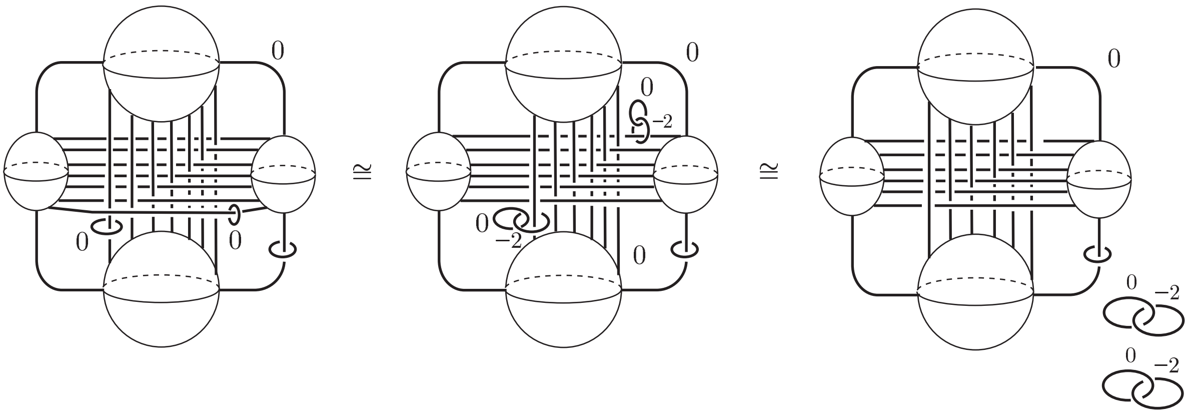

Figure 9: $E(1)\#_{T=T_i} S^2 \times S^2 = E(1)\#^2 S^2 \times S^2$

We prepare several lemmas to prove Theorem 1.2.

Lemma 4.1. *Let $L = U_1 \cup U_2$ be a 2-component unlink. Then the handle picture of $X(C, C; L)$ is Figure 11.*

Suppose that $L = L_1 \cup L_2$ is any split link. Then the handle picture of $X(C, C; L)$ is obtained by replacing the two dotted 1-handles in Figure 11 with the slice 1-handles corresponding to L_1 and L_2.

In particular, in the case where $L = L' \cup U$ is an n-component link and U is a split unknot,

$$A_{L' \cup U} \cong A_{L'} \#^2 S^2 \times S^2.$$

Proof. Let $L = K_1 \cup K_2$ be a split link. First we consider the case where K_1, K_2 are both unknots. Let D_1 and D_2 be 3-disks splitting U_1 and U_2 respectively. In other words $D_1 \cup D_2 = S^3$, $D_1 \cap D_2 = S^2$, and $K_i \subset \mathrm{int}(D_i)$. Then we get a decomposition $[S^3 - \nu(L)] \times S^1 = [(D_1 - \nu(U_1)) \cup (D_2 - \nu(U_2))] \times S^1$. Each component $[D_i - \nu(U_i)] \times S^1$ is diffeomorphic to $D^2 \times S^1 \times S^1 - \nu(\beta_i)$ (see Figure 10), where β_i is $\{p_i\} \times S^1$ and p_i is a point in $D^2 \times S^1$.

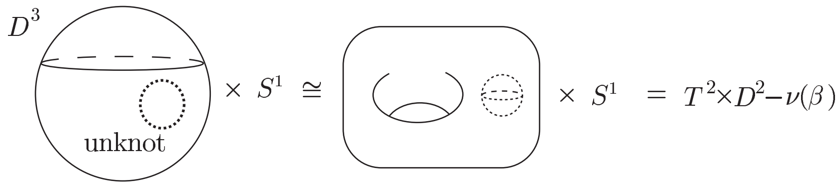

Figure 10: $[D^3 - \nu(\text{unknot})] \times S^1 \cong D^2 \times T^2 - \nu(\beta)$

The handle picture of $D^2 \times T^2 - \nu(\beta_1)$ is the left in Figure 13. The $S^2 \times S^1$ boundary component $\partial\nu(\beta_1) \cong S^2 \times S^1$ corresponds to the cylinder in the picture. The gluing of $D^2 \times T^2 - \nu(\beta_1)$ and $D^2 \times T^2 - \nu(\beta_2)$ along the $S^2 \times S^1$ component using the identity map has the handle picture of the right in Figure 13. With the dotted 1-handles description, the handle picture of $X(C, C; L)$ is Figure 11. Since $X(C, C; L)$ has two boundary components, we must draw a 3-handle as can be seen in Figure 11.

In the case where K_1, K_2 are any link, the handle picture of $X(C, C; L)$ can be drawn replacing the solid torus in Figure 10 with the knot complement $D^3 - \nu(K_i)$. The replacement of handle pictures can be viewed as in [Ak2]. For example in the case of $K_1 = 3_1$ and $K_2 = 4_1$, the handle picture is Figure 12.

In particular if K_2 is the unknot, then A_L gives rise to two connected-sum components of $S^2 \times S^2$ as can be seen in Figure 14. Here we apply handle calculus in Figure 15. We denote 0-framed 2-handles by unlabeled links. Therefore $A_{L' \cup U} \cong A_{L'} \#^2 S^2 \times S^2$ holds. $\Box$

Next we draw a handle picture of the link surgery operation $X(C, C; H)$ along a Hopf link H and we compute A_H.

Lemma 4.2. *Let H be a Hopf link. Then A_H is diffeomorphic to $\#^3(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$.*

Proof. The complement $[S^3 - \nu(H)] \times S^1$ is the diffeomorphic to $T^2 \times S^1 \times I$ (the left in Figure 16), where I is the interval $[0, 1]$ and some unlabeled links are 0-framed 2-handles.

Attaching two -1-framed 2-handles to $T^2 \times S^1 \times \{0\}$ in $T^2 \times I$ and the other two -1-framed 2-handles to $T^2 \times S^1 \times \{1\}$, we get the picture of $X(C, C; H)$ (the right in Figure 16). We denote the attached circles in $T^2 \times S^1 \times \partial I$ are $\alpha := \bar{\alpha} \times (p_2, 0), \beta := \bar{\beta} \times (p_2, 0)$ and $\alpha \times (p_2, 1), \eta := \{p_1\} \times S^1 \times \{1\}$, where $p_1 \in T^2, p_2 \in S^1$ and $\bar{\alpha}, \bar{\beta}$ are two 1-cycle generators in T^2. Next attaching on two boundaries of $X(C, C; H)$ four vanishing circles with opposite orientation (four meridional 0-framed 2-handles), and two sections (two 0-framed 2-handles), we get the top-left handle decomposition in Figure 17. The decomposition can be modified into the top-right picture by two handle slides as indicated in the top-left picture. The resulting picture can be modified into the bottom-left picture by two handle slides indicated by the two arrows in the top-right picture. Two (unlinked) 0-framed 2-handles obtained by this modification are canceled with 2 3-handles. By applying Figure 15 and easy handle calculus, the bottom-left picture can be modified into the bottom-middle picture in Figure 17. This picture is the diagram of $\#^3(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$ using handle calculus. $\Box$

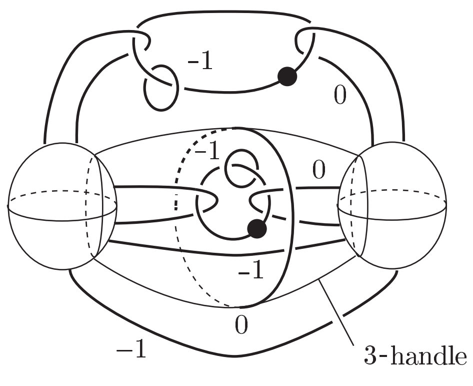

Figure 11: The handle picture of $X(C, C; U_0 \cup U_1)$.

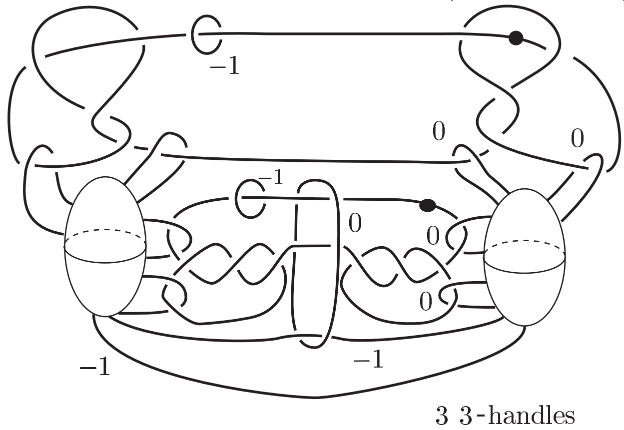

Figure 12: $X(C, C; 3_1 \coprod 4_1)$

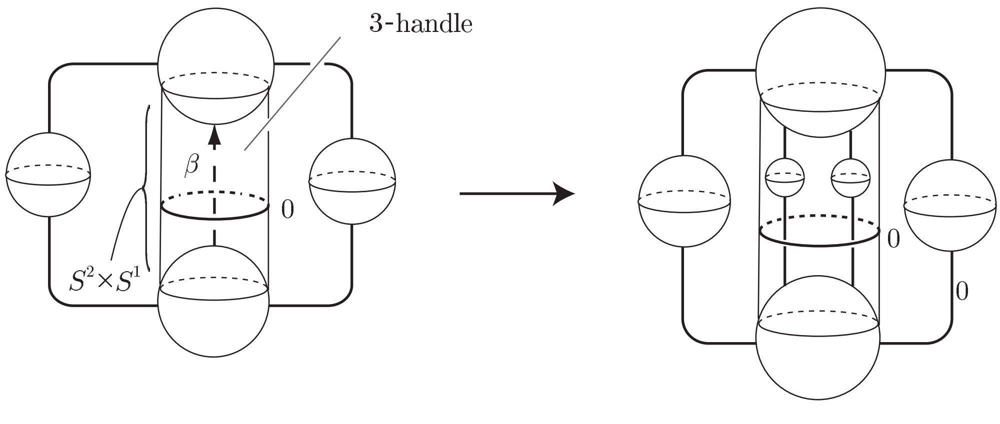

Figure 13: $T^2 \times D^2 - \nu(\beta) \to (T^2 \times D^2 - \nu(\beta_1)) \cup (T^2 \times D^2 - \nu(\beta_2))$.

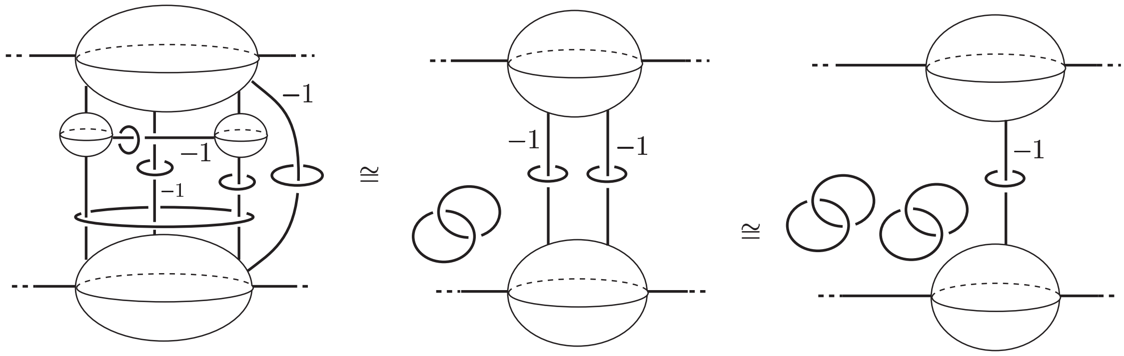

Figure 14: The handle picture of $A_{L'\cup U} = A_{L'}\#^2 S^2 \times S^2$.

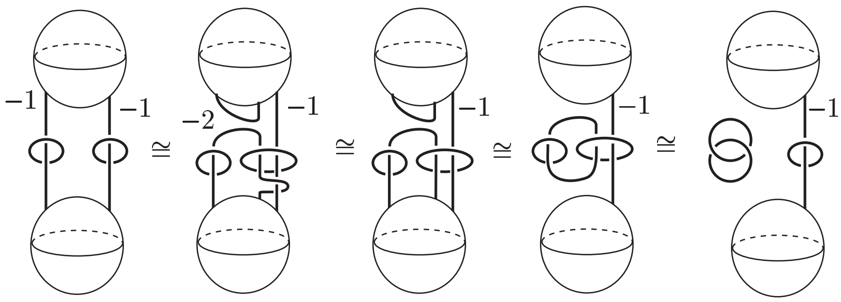

Figure 15: To make an $S^2 \times S^2$-component from two parallel -1-framed 2-handles.

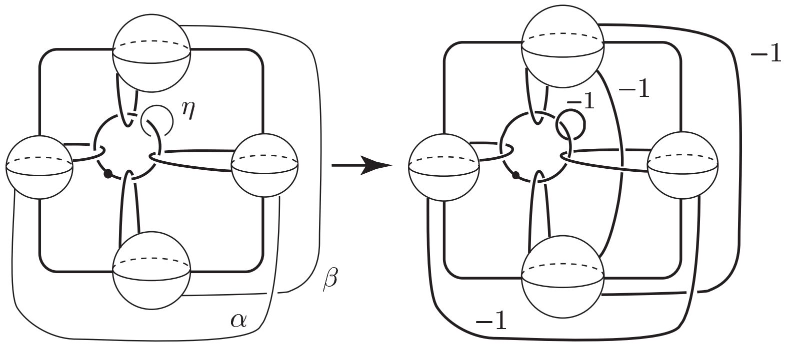

Figure 16: $T^2 \times S^1 \times I \to X(C, C; H)$.

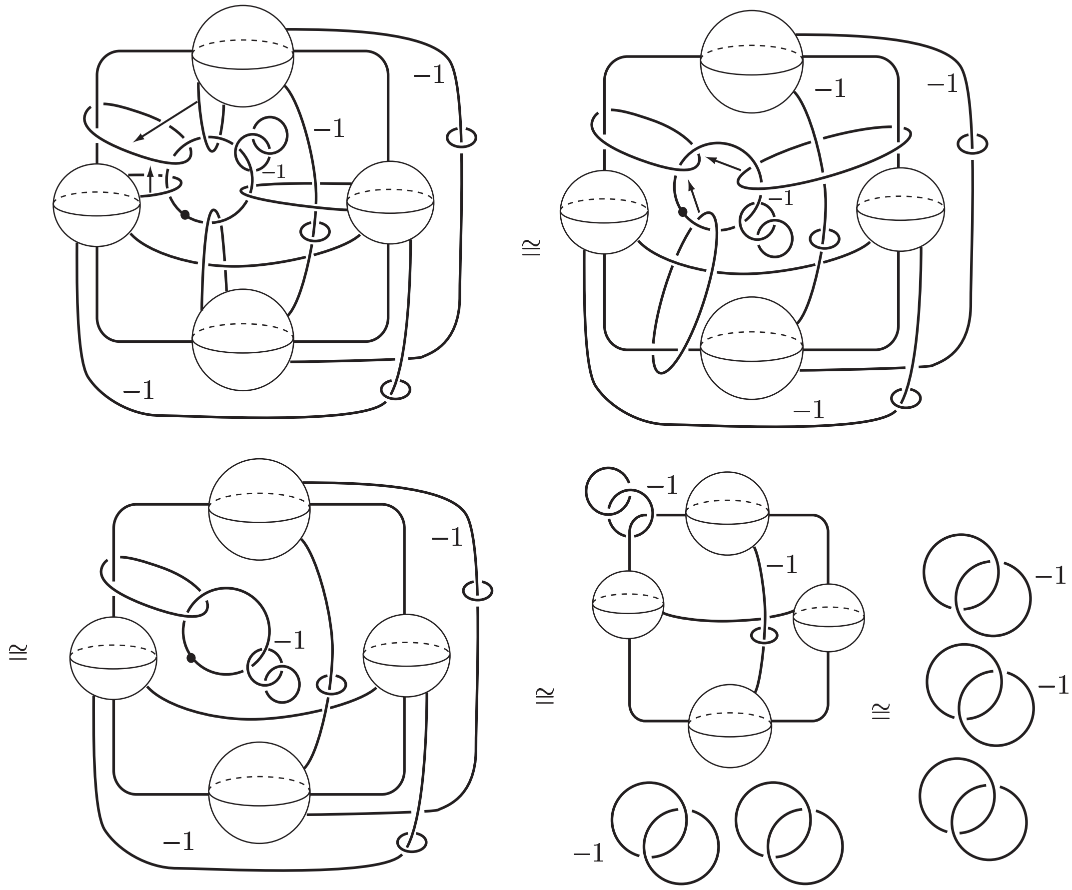

Figure 17: The handle picture of $A_H = \#^3(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$.

At this point we can prove Theorem 1.2.

Proof of Theorem 1.2. Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be any n-component link. The set $\tilde{\mathcal{L}}_n$ of all n-component links up to local 3-strand twist consists of 2^{n-1} classes due to Nakanishi and Ohyama's results [Oh, Na]. Forgetting the ordering of the components of any link in $\tilde{\mathcal{L}}_n$ we get a set $\mathcal{L}_n$. The set $\mathcal{L}_n$ has n classes. A standard representative in each class is a link $L_{n,\ell}$ ($\ell = 0, 1, \cdots, n-1$) as presented in Figure 18. Applying 3-

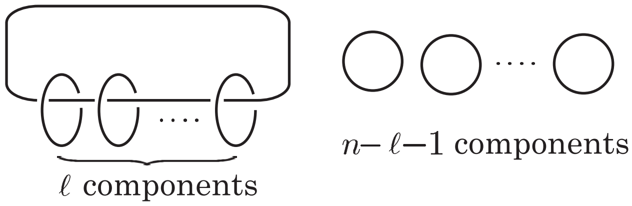

Figure 18: The representation $L_{n,\ell}$ of $\mathcal{L}_n$

strand twist to link surgery operation A_L we have only to consider the diffeomorphism type of $A_{L_{n,\ell}}$ for some ℓ.

Notice that $L_{n,0}$ is the representative of all proper links ($\overset{\text{def}}{\Leftrightarrow} \sum_{i \neq j} lk(K_i, K_j) \equiv 0 \pmod 2\ {}^\forall i$) and $L_{n,\ell}$ ($\ell > 0$) are the representatives of improper link ($\overset{\text{def}}{\Leftrightarrow}$ not proper link).

Now suppose that $1 \leq \ell \leq n-2$. Applying Lemma 4.1 to the $n-\ell-1$ component unlink, we have

$$A_{L_{n,\ell}} = A_{L_{\ell+1,\ell}} \#^{2(n-\ell-1)} S^2 \times S^2.$$

In addition since ℓ parallel meridians in the remaining components construct a fibersum of ℓ-copied symmetric cusp neighborhoods, by using the handle calculus in Figure 15 we have

$$A_{L_{\ell+1,\ell}} = A_H \#^{2(\ell-1)} S^2 \times S^2.$$

Using Lemma 4.2, therefore

$$\begin{aligned} A_{L_{n,\ell}} &= \#^3(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \#^{2(\ell-1)} S^2 \times S^2 \#^{2(n-\ell-1)} S^2 \times S^2 \\ &= \#^{2n-1}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}). \end{aligned}$$

Suppose that $\ell = 0$. The link $L_{n,0}$ is n-component unlink. Thus using Lemma 4.1 we have

$$A_{L_{n,0}} = S^2 \times S^2 \#^{2(n-1)} S^2 \times S^2 \cong \#^{2n-1} S^2 \times S^2.$$

Suppose that $\ell = n-1$. Since the link $L_{n,n-1}$ does not have unlink component,

$$A_{L_{n,n-1}} = \#^3(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \#^{2(n-2)} S^2 \times S^2 \cong \#^{2n-1}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}).$$

Therefore

$$A_L \cong \begin{cases} A_{L_{n,0}} \cong \#^{2n-1} S^2 \times S^2 & L \text{ is proper} \\ A_{L_{n,\ell}} \cong \#^{2n-1}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) & \text{otherwise.} \end{cases}$$

$\square$

5 Scharlemann's manifolds.

Let K be a knot in S^3 and $\gamma(\epsilon)$ an embedded framed curve in $S^3_p(K)$, where γ is the embedding of the curve and ϵ is the framing.

Definition 5.1. *The 0-framing is defined as the Seifert framing of the curve embedded in the surgery diagram (p-surgery along K).*

Embedding the framed curve in $S^3_p(K) \times S^1$ in the obvious way, we can find a framed curve $\tilde{\gamma}$ in $S^3_p(K) \times S^1$.

The isotopy type of the framed curve $\tilde{\gamma}$ depends only on the homotopy type of the framed curve $\gamma(\epsilon)$ as mentioned in Subsection 1.3. Therefore the framing ϵ is (mod2)-framing. Figure 19 is an example of framed curve presentation.

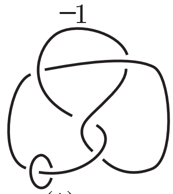

Figure 19: A curve γ_0 with (mod2)-framing.

The 0-framing is designated by Definition 5.1 in Subsection 1.3.

Definition 5.2. *We fix a diagram of γ in the surgery presentation of $S^3_p(K)$. Let $\gamma(\epsilon)$ be an embedded framed curve in $S^3_p(K)$. Namely the induced framing on $\tilde{\gamma}$ gives a trivialization $t_\epsilon : \bar{\nu}(\tilde{\gamma}) \cong D^3 \times S^1$.*

We define the (ϵ)-surgery along γ as

$$B_{K,p}(\gamma(\epsilon)) := [S^3_p(K) \times S^1 - \nu(\tilde{\gamma})] \cup_{\psi_\epsilon} S^2 \times D^2.$$

The gluing map ψ_ϵ is the composition of the identity map $S^2 \times \partial D^2 \to \partial D^3 \times S^1$ and the boundary restriction of t_ϵ^{-1}. We call $B_{K,p}(\gamma(\epsilon))$ Scharlemann's manifold. In the case of $p = -1$ we drop the suffix p. This manifold depends only on the homotopy type of $\gamma(\epsilon)$ in $S^3_p(K)$.

In other words this operation coincides with taking the boundary after attaching a 5-dimensional 2-handle along $\tilde{\gamma}$ with the framing.

5.1 Scharlemann's manifolds along the meridian curves.

In this subsection we consider Scharlemann's manifolds with respect to the meridian γ_0 of K. When we choose the presentation of γ_0, we only use the meridian of K in the surgery presentation of $S^3_{-1}(K)$ as in Figure 19. We remark the following.

Remark 5.1. Let γ_0 be the meridian circle in $S^3_{-1}(K)$. All Scharlemann's manifolds $B_K(\gamma_0(0))$ are diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In the case of $\epsilon = 1$, we note the relationship between $B_K(\gamma_0(1))$ and the knot surgery of the fishtail neighborhood.

Lemma 5.1. *$B_K(\gamma_0(1))$ is diffeomorphic to $\overline{F} \cup F_K$.*

Proof. Performing knot surgery of the fishtail neighborhood as in Definition 2.3 to $\overline{F} \cup F$, we have

$$\overline{F} \cup F_K = \overline{F} \cup [F - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1].$$

The handle picture is Figure 20 (the case of $K = 4_1$).

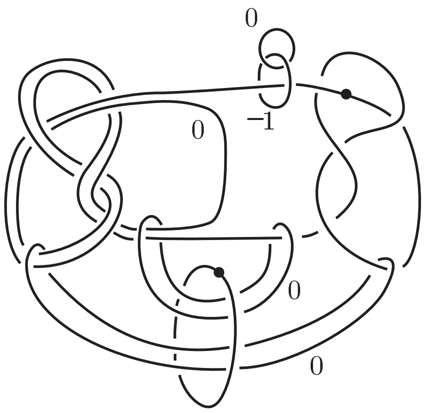

Figure 20: $\overline{F} \cup [F - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1]$.

The surgery along $\tilde{\gamma}_0$ in $S^3_{-1}(K) \times S^1$ is the right in Figure 21. Hence we get the following diffeomorphisms.

$$\begin{aligned} B_K(\gamma_0(1)) &= [S^3_{-1}(K) \times S^1 - \nu(\tilde{\gamma}_0)] \cup_{\psi_1} S^2 \times D^2 \\ &\cong \overline{F} \cup (F - \nu(T)) \cup_{\varphi_{-1}} [S^3 - \nu(K)] \times S^1 \quad \text{(See Figure 3 and 21.)} \\ &\cong \overline{F} \cup (F - \nu(T)) \cup_{\varphi_0} [S^3 - \nu(K)] \times S^1 \quad \text{(Lemma 3.1)} \\ &= \overline{F} \cup F_K \end{aligned}$$

□

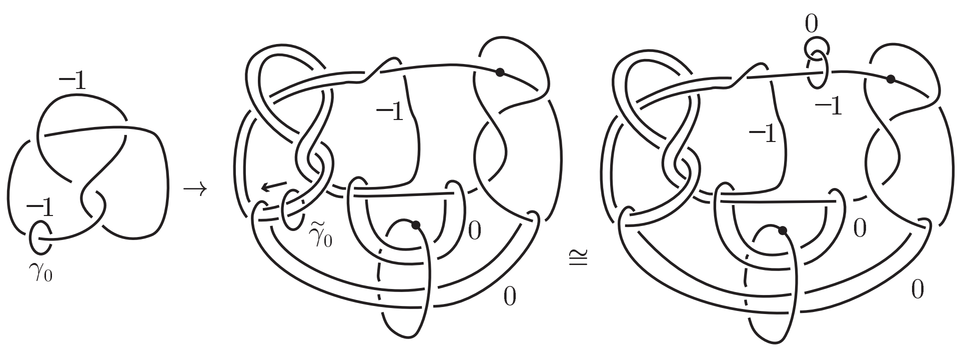

Figure 21: The surgery along $\tilde{\gamma}_0$ with the framing 1.

Proof of Theorem 1.4. Since $\overline{F} \cup F$ contains SyF, applying Lemma 3.3 to this situation, we have

$$\overline{F} \cup F_K = \overline{F} \cup F \cong S^3 \times S^1 \# S^2 \times S^2,$$

where the last diffeomorphism is due to Figure 22. □

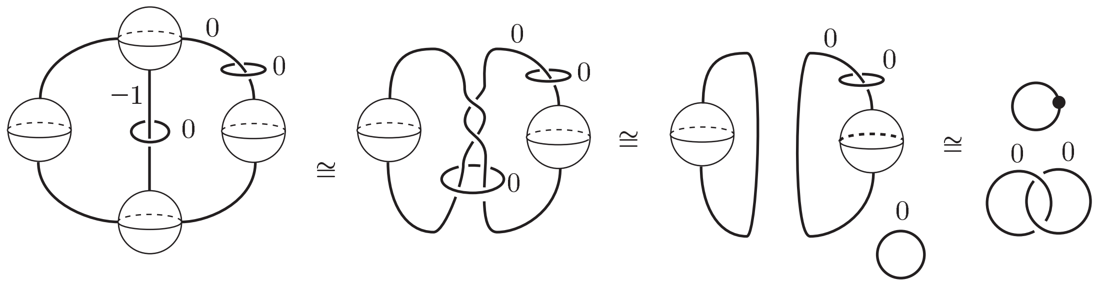

Figure 22: $F \cup \overline{F} = S^3 \times S^1 \# S^2 \times S^2$.

Corollary 5.1. *Let γ_0 be a meridian of K in the surgery presentation of $S^3_p(K)$. $B_{K,p}(\gamma_0(\epsilon))$ is classified as follows*

$$B_{K,p}(\gamma_0(\epsilon)) = \begin{cases} S^3 \times S^1 \# S^2 \times S^2 & (\epsilon - 1)p \equiv 0 \ (2) \\ S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & (\epsilon - 1)p \equiv 1 \ (2). \end{cases}$$

Here we fix γ_0 as the meridian of K in the surgery presentation of $S^3_p(K)$.

Proof. In the case of $\epsilon = 1$, using 1-strand twist

$$B_{K,p}(\gamma_0(1)) \cong B_K(\gamma_0(1)) \cong S^3 \times S^1 \# S^2 \times S^2.$$

In the case of $\epsilon = 0$ for the same reason as Remark 5.1 we obtain

$$B_{K,p}(\gamma_0(0)) \cong \begin{cases} S^3 \times S^1 \# S^2 \times S^2 & p \equiv 0 \ (2) \\ S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & p \equiv 1 \ (2) \end{cases}$$

(see Figure 23).

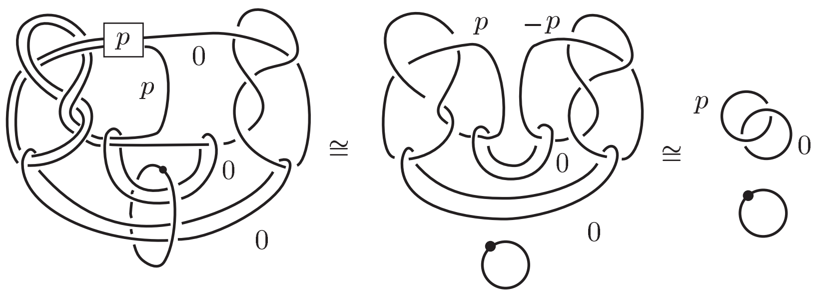

Figure 23: $B_{K,p}(\gamma_0(0))$.

Remark 5.2. $B_K(\gamma_0(1))$ is obtained from A_K as a surgery along an embedded S^2. The neighborhood of the sphere Σ is the union of the bottom 0-framed 2-handle and the 4-handle (the left of Figure 24). Attaching the 3-handle and 4-handle to the complement gets $B_K(\gamma_0(1))$ (the right of Figure 24). The circle δ in Figure 24 is the core circle of $S^1 \times D^3$ attached.

Remark 5.3. In [Ak3] Akbulut got a plug twisting $(W_{1,2}, f)$ satisfying $E(1) = N \cup_{\mathrm{id}} W_{1,2}$ and $E(1)_{2,3} = N \cup_f W_{1,2}$. The definitions of plug, N and $W_{1,2}$ are written

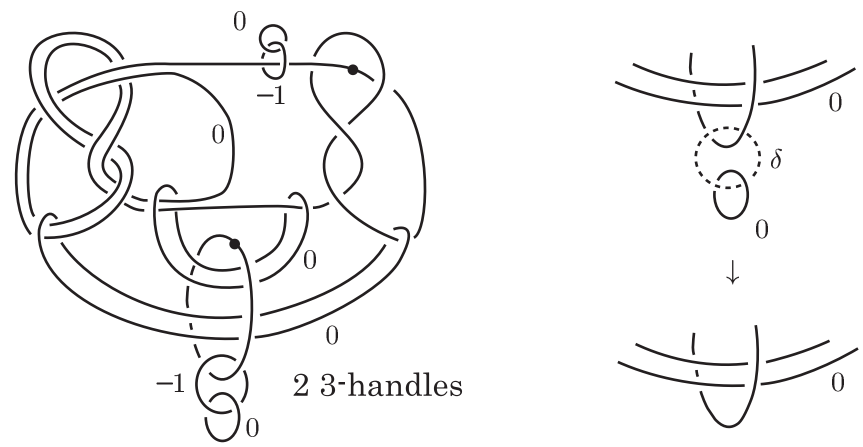

Figure 24: The left: A_K. The right: surgery $B_K(\gamma_0(-1)) \cong [A_K - \nu(\Sigma)] \cup S^1 \times D^3$.

down in [Ak3]. In the same way as [Ak3] we can also show that there exist infinitely many plug twistings $(W_{1,2}, f_K)$ of $E(1)$ with the same plug $W_{1,2}$. As a result any plug twisting satisfies $E(1) = M \cup_{\mathrm{id}} W_{1,2}$ and $E(1)_K = M \cup_{f_K} W_{1,2}$. Infinite variations of Alexander polynomial $\Delta_K(t)$ of knot imply the existence of infinite embeddings $W_{1,2} \hookrightarrow M \cup_{id} W_{1,2}$.

5.2 Scharlemann's manifold along non-meridian curves.

In this subsection we consider $B_{3_1}(\gamma(\epsilon))$ in the case where γ are some non-meridian curves.

The fundamental group of $S^3_{-1}(3_1)$ is homomorphic to

$$\pi = \pi_1(S^3_{-1}(3_1)) = \langle x, y | x^5 = (xy)^3 = (xyx)^2 \rangle \cong \tilde{A}_5.$$

These elements x, y are two generators as in Figure 25.

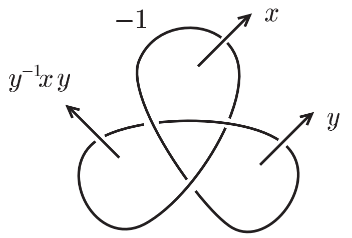

Figure 25: The generators x, y of $\pi_1(\Sigma)$.

The set

$$[S^1, S^3_{-1}(3_1)] = \pi/\mathrm{conj}. \tag{4}$$

of free homotopy classes of maps $S^1 \to S^3_{-1}(3_1)$ possesses 9 classes as follows.

Classes	$[e]$	$[x^5]$	$[xyx]$	$[x]$	$[x^2]$	$[x^3]$	$[x^4]$	$[xy]$	$[(xy)^2]$
Orders	1	2	4	10	5	10	5	6	3

Each of the classes is a normal generator of the fundamental group except $[e], [x^5]$. When we consider non-meridian curve γ, we fix the concrete presentation of γ in $S^3_{-1}(3_1)$.

In the case of $z = x$, according to Corollary 5.1 (originally Akbulut's result Theorem 1.3 and Remark 5.1) the surgeries along a curve γ_0 with $[\gamma_0] = x$ completely gives the diffeomorphism types as following:

$$B_{3_1}(\gamma_0(\epsilon)) = \begin{cases} S^3 \times S^1 \# S^2 \times S^2 & \epsilon = 1 \\ S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \epsilon = 0. \end{cases}$$

We consider for another conjugacy class we will prove the following.

Proposition 5.1. *Let γ_{xy} be a presentation in Figure 26, where $[\gamma_{xy}] = xy$. $B_{3_1}(\gamma_{xy}(1))$*

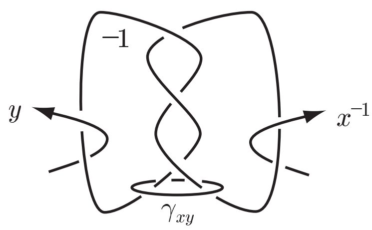

Figure 26: γ_{xy}

is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In the handle pictures in this subsection $\sim$ and $\sim^1$ stand for 3-manifold homeomorphism and 1-strand twist, respectively. They correspond to some 4-dimensional diffeomorphism.

By using 3-dimensional diffeomorphism and 1-strand twist we can prove a diffeomorphism as in Figure 27. We show that this diffeomorphism can be extended to any twist along $\gamma(1)$.

Lemma 5.2 (1-strand twist along $\gamma(1)$)**.** *A full-twist of any number of strand along $\gamma(1)$ does not change the diffeomorphism type of the 4-manifold: If a framed link $(K'; p')$ is obtained from $(K; p)$ by a full-twist along $\gamma(\epsilon)$, then $B_{K',p'}(\gamma(\epsilon))$ is diffeomorphic to $B_{K,p}(\gamma(\epsilon))$. We call such a deformation 1-strand twist as well.*

Proof. A Dehn twist (that is, 1-strand twist as in Lemma 3.1) along a curve parallel to γ does not change the differential structure because $\gamma(1)$ plays a role in the vanishing circle in a fishtail neighborhood. □

Remark 5.4. To avoid reader's confusion, we must note that 1-strand twist in Lemma 5.2 is 1-strand twist along $\gamma(1)$, thus it does *not* mean that we can generalize Lemma 3.2 to any even-strand twist case. Any odd-strand twist is interpreted as 'a kind of 1-strand twist' given by a summation of odd 1-strand twists as in Figure 28 ((odd number)$\times 1 \equiv 1(2)$). This summation is due to the proof of Theorem 1.1. At any rate for a twist to give a 4-dimensional diffeomorphism we require an odd situation.

We use the same notation $\sim^1$ for any 1-strand twist along $\gamma(1)$ in Lemma 5.2. Through Figure 27-34 curves with (1) or (0) mean the (1) or (0)-surgery along the curves.

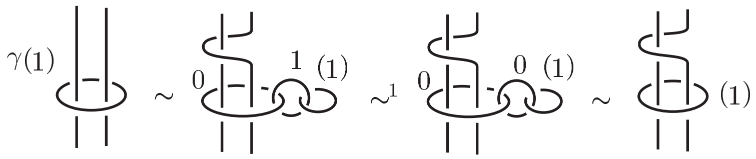

Figure 27: A 1-strand twist along $\gamma(1)$.

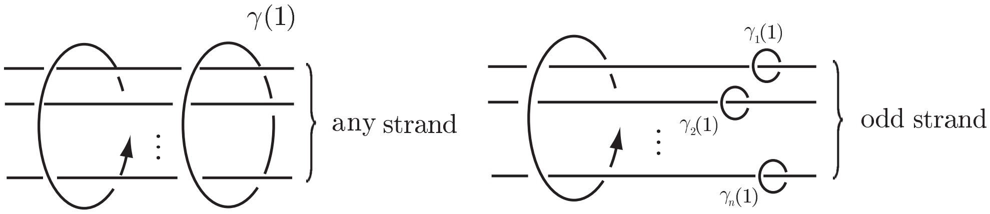

Figure 28: 1-strand twist along $\gamma(1)$ and odd-strand twist.

Proof of Proposition 5.1. By using Figure 29 and Corollary 5.1 we have

$$B_{3_1}(\gamma_{xy}(1)) \cong B_{\mathrm{unknot},3}(\gamma_0(0)) \cong S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}.$$

□

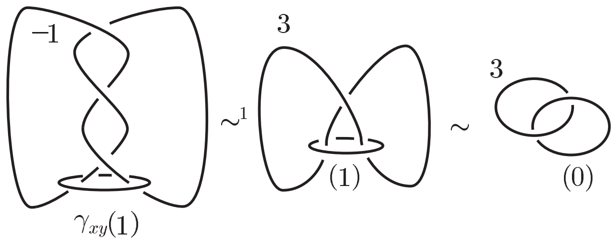

Figure 29: $B_{3_1}(\gamma_{xy}(1)) \cong B_{\mathrm{unknot},3}(\gamma_0(0))$.

Here we will argue several other cases.

Proposition 5.2. *We fix diagrams γ_{x^2}, γ_{x^3} and γ_{x^4} as in the leftmost pictures in Figure 30, Figure 31, and Figure 32 respectively.* $B_{3_1}(\gamma_{x^2}(1))$, $B_{3_1}(\gamma_{x^3}(0))$ *and* $B_{3_1}(\gamma_{x^4}(1))$ *are diffeomorphic to* $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Proof. In the case of $B_{3_1}(\gamma_{x^2}(1))$, by using Figure 30 and Corollary 5.1 we have $B_{3_1}(\gamma_{x^2}(1)) \cong S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In the case of $B_{3_1}(\gamma_{x^3}(0))$ the last picture in Figure 31 represents a knot γ (the positive (2,7)-torus knot) in S^3 with odd framing. It is obviously homotopic to the unknot. Namely the manifold $B_{3_1}(\gamma_{x^3}(1))$ is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In the case of $B_{3_1}(\gamma_{x^4}(1))$, the last picture in Figure 32 gives $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the similar way. Here $Pr(-2,3,7)$ is the $(-2,3,7)$-pretzel knot. □

Proposition 5.3. *We fix diagrams γ_{xyx} and $\gamma_{(xy)^2}$ as in the leftmost pictures in Figure 33 and Figure 34 respectively.* $B_{3_1}(\gamma_{xyx}(0))$ *and* $B_{3_1}(\gamma_{(xy)^2}(1))$ *are diffeomorphic to* $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

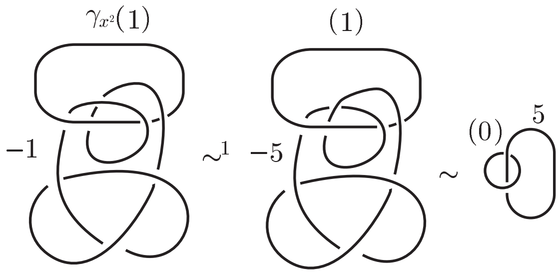

Figure 30: $B_{3_1}(\gamma_{x^2}(1)) \cong B_{\mathrm{unknot},5}(\gamma_0(0))$.

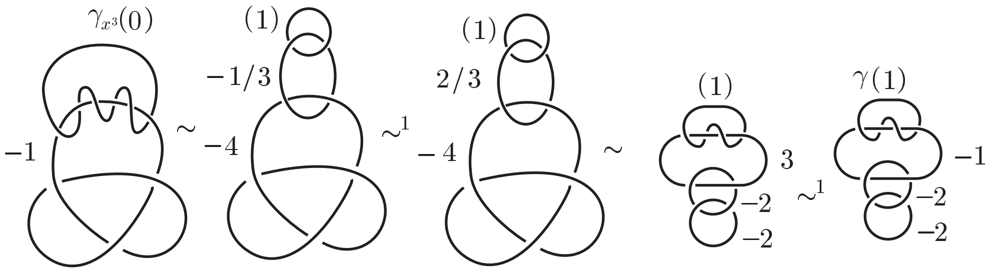

Figure 31: The diffeomorphism for $B_{3_1}(\gamma_{x^3}(0))$.

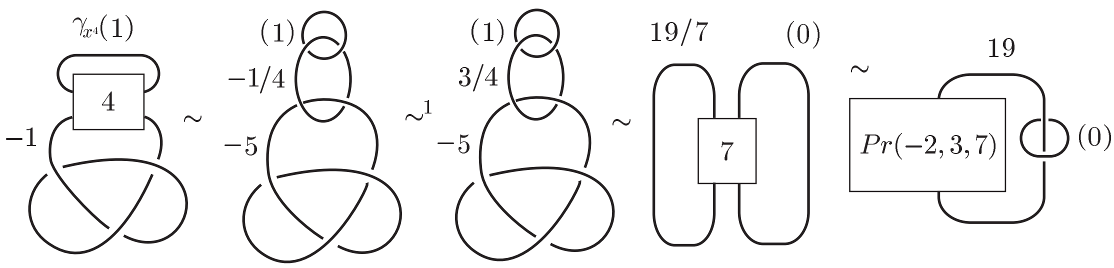

Figure 32: The diffeomorphism for $B_{3_1}(\gamma_{x^4}(1))$.

Proof. In the case of γ_{xyx}, the homotopy class of the curve γ_{xyx} is $xyy^{-1}xy \sim x^2y \sim xyx$. Thus by Figure 33, we get $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

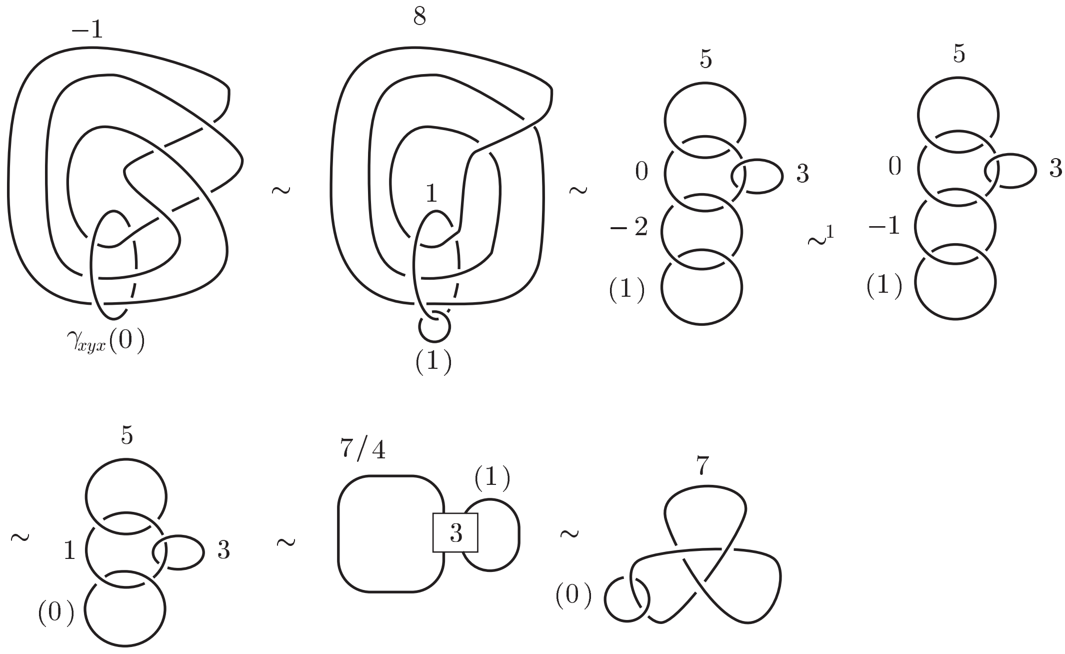

Figure 33: The diffeomorphism for $B_{3_1}(\gamma_{xyx}(0))$.

In the case of $\gamma_{(xy)^2}$ the deformation as in Figure 34 gets $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Here $T_{2,-7}$ is the negative $(2,7)$-torus knot $\square$.

In the end of paper we raise a question.

Question 5.1. *In the following manifolds*

$$B_{3_1}(\gamma_{xy}(0)),\ B_{3_1}(\gamma_{x^2}(0)),\ B_{3_1}(\gamma_{x^3}(1)),\ B_{3_1}(\gamma_{x^4}(0)),\ B_{3_1}(\gamma_{xyx}(1)),\ B_{3_1}(\gamma_{(xy)^2}(0)),$$

does there exist any non-standard manifold?

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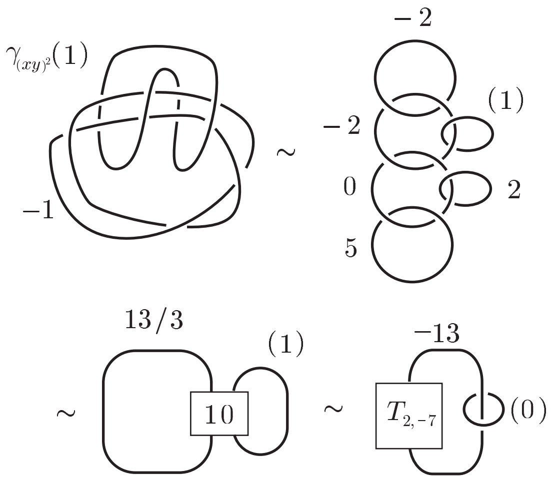

Figure 34: The diffeomorphism for $B_{3_1}(\gamma_{(xy)^2}(1))$.

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