BOUNDARY-SUM IRREDUCIBLE FINITE ORDER CORKS

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ABSTRACT. We prove for any positive integer n there exist boundary-sum irreducible $\mathbb{Z}_n$-corks with Stein structure. Here 'boundary-sum irreducible' means the manifold is indecomposable with respect to boundary-sum. We also verify that some of the finite order corks admit hyperbolic boundary by HIKMOT.

1. INTRODUCTION

1.1. **G-corks.** *Cork* (C, g) is a pair of a compact contractible (Stein[1]) 4-manifold C and a diffeomorphism g on the boundary ∂C that g cannot extend to the inside C as a smooth diffeomorphism. *Cork twist* means the 4-dimensional surgery by the following cut-and-paste

$$X' = (X - C) \cup_g C.$$

The manifold presented by the diagram as in FIGURE 1 becomes a cork. The map g is the $180°$ rotation about the horizontal line in the picture. In particular, $C(1)$ is the first cork which was used by Akbulut. Here a box with the integer x stands for the x-fold right handed full twist.

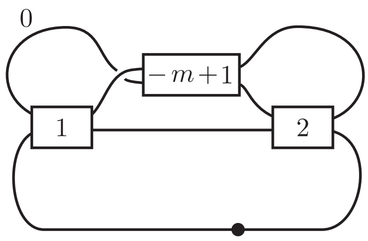

FIGURE 1. The handle diagram of $C(m)$.

In the definition of the original cork the condition $g^2 = \mathrm{id}_{\partial C}$ is included. Recently, in some papers the order of gluing map g is generalized to finite order ([11], [2]), infinite order [7] or generally any group G in [2]. In terms of

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[1]This condition is included in some original papers by Akbulut et.al., for example [1]

the view by Auckly, Kim, Melvin, and Ruberman [2], if a group G smoothly and effectively acts on the boundary of a contractible 4-manifold C and any non-trivial diffeomorphism $g \in G$ cannot smoothly extend to the inside C, then the pair (C, G) is called a *G-cork*.

As examples of finite order corks, the author [11] gave pairs of $(X_{n,m}, \tau_{n,m})$ for $X = C, D, E$ and or generally, $X = X(\mathbf{x})$ for $\{*, 0\}$-sequence $\mathbf{x} \neq (0, \cdots, 0)$ or $(*, \cdots, *)$, where we call such a sequence $\mathbf{x}$ *non-trivial*. The diffeomorphism $\tau_{n,m}$ is the $2\pi/n$-rotation with respect to the diagram. In the paper [11], we put the index X on $\tau_{n,m}$, like the notation $\tau^X_{n,m}$. We remove the indexes if it is understood from the context. We describe $C_{n,m}$ in FIGURE 2. $D_{n,m}$ is obtained by exchanging all the dots and 0-framings

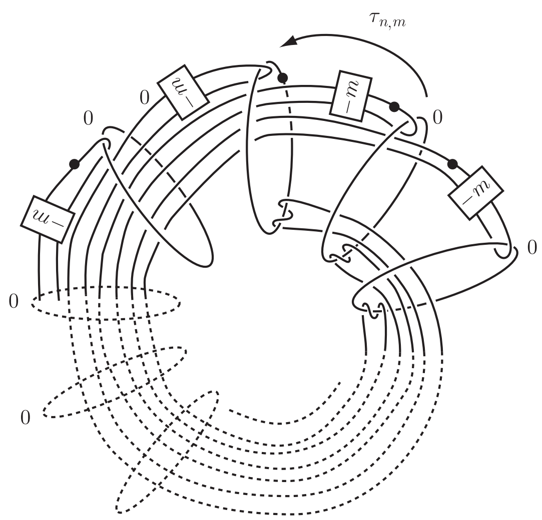

FIGURE 2. The handle decomposition of $C_{n,m}$.

in $C_{n,m}$. $E_{n,m}$ is obtained by modifying $C_{n,m}$ as in FIGURE 3. The concrete diagrams for these examples are described in [11].

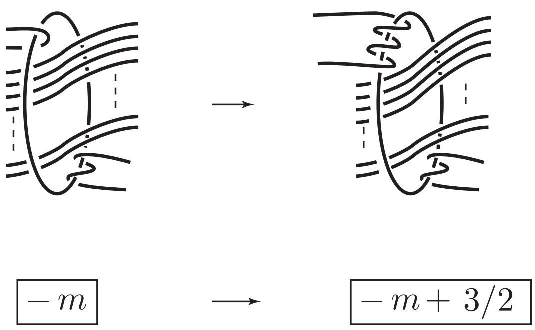

FIGURE 3. The modification.

Theorem 1.1 ([11])**.** *For $X = C, D, E$ or $X(\mathbf{x})$, for any non-trivial $\{*, 0\}$-sequence $\mathbf{x}$, $(X_{n,m}, \tau_{n,m})$ is a finite order cork. Furthermore, $C_{n,m}$ is a $\mathbb{Z}_n$-cork with Stein structure.*

Auckly, Kim, Melvin, and Ruberman [2] gave the examples of G-corks for any finite subgroup G of $SO(4)$.

Theorem 1.2 ([2])**.** *Let G be any finite subgroup in $SO(4)$. Then there exists a G-cork.*

Let Y_1, Y_2 be two n-manifolds with boundary. We call the surgery of attaching an n-dimensional 1-handle along two neighborhoods of $p_i \in \partial Y_i$ *boundary-sum* and the resulting manifold as $X_1 \natural X_2$. Their Stein corks in [2] were constructed by the boundary-sum of several copies of $C(1)$. They also announce the existence of finite order cork with hyperbolic boundary in [2].

We say that an n-manifold X with boundary is *boundary-sum irreducible* if $X = X_1 \natural X_2$, then X_1 or X_2 is homeomorphic to an n-disk. If X is not boundary-sum irreducible, then we call X *boundary-sum reducible*. Here a 4-manifold X is called *irreducible* if for any connected-sum decomposition $X = X_1 \# X_2$, X_1 or X_2 is a homotopy n-sphere. We call a 3-manifold Y *prime* if for any connected-sum decomposition $Y = Y_1 \# Y_2$, Y_1 or Y_2 is a 3-sphere. The following holds.

Lemma 1.3. *Let X be a 4-manifold. If X is irreducible and ∂X is prime, then X is boundary-sum irreducible.*

The problem of whether the examples $X_{n,m}$ in Theorem 1.1 are boundary-sum irreducible corks or not has remained. Our main theorem answers this question for the case of $X(\mathbf{x})_{n,m}$.

Theorem 1.4. *For any integer m and positive integer n. There exist boundary-sum irreducible $\mathbb{Z}_n$-corks $(C_{n,m}, \tau_{n,m})$ with Stein structure. For any non-trivial $\{*, 0\}$-sequence $\mathbf{x}$, $(X(\mathbf{x})_{n,m}, \tau_{n,m})$ are boundary-sum irreducible finite order corks.*

Another variation $(E_{n,1}, \tau_{n,1})$ is boundary-sum irreducible $\mathbb{Z}_n$-corks.

Indeed, $X(\mathbf{x})_{n,m}$ is irreducible and $\partial X(\mathbf{x})_{n,m}$ is prime. This result means that $(X(\mathbf{x})_{n,m}, \tau_{n,m})$ is a different finite order cork from the one used in Theorem A in [2]. We do not know whether our examples are different from their finite order corks with hyperbolic boundary.

We can show the following result which follows immediately from the proof of Theorem 1.4. We set $Y_{n,m} := \partial C_{n,m}$. Clearly this 3-manifold is diffeomorphic to any $\partial X_{n,m}(\mathbf{x})$, for any $\{*, 0\}$-sequence $\mathbf{x}$. We set $Y'_{n,m} := \partial E_{n,m}$.

Theorem 1.5. *Let n, m be integers as above. Then $Y_{n,m}$ and $Y'_{n,1}$ are prime homology spheres.*

Furthermore we can prove the following hyperbolicity. In [2] they suggested any $Y_{n,m}$ would be a hyperbolic 3-manifold.

Theorem 1.6. *Let n, m be integers with $0 \leq m \leq 2$ and $1 \leq n \leq 4$. $Y_{n,m}$ and $Y'_{n,m}$ are hyperbolic 3-manifolds.*

These are direct results by the computer software HIKMOT [5]. It is proven that $Y_{1,m} = Y'_{1,m} = \partial C(m)$ are hyperbolic 3-manifolds in [8], by using the fact that these are Dehn surgeries of the pretzel knot $Pr(-3, 3, -3)$. We put a question here.

Question 1.7. *Let X be $X(\mathbf{x})$ for non-trivial $\{*, 0\}$-sequence or E.*

- *Is $(X_{n,m}, \tau_{n,m})$ finite order cork with Stein structure?*
- *Is $(X_{n,m}, \tau_{n,m})$ finite order cork with hyperbolic boundary?*

Notice that it is not known at all what reflection for the exotic structures does a cork twist by a cork with hyperbolic boundary give. This theme is left up to a future study of exotic 4-manifolds.

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2. Primeness of $K_{n,m}$ and $K'_{n,m}$.

$Y_{n,m}$ and $Y'_{n,m}$ are n-fold cyclic branched covers of $Y(m) := \partial C(m)$ with the branch locus $K_{n,m}$ and $K'_{n,m}$ respectively. See Figure 4 for $K_{n,m}$. The picture of $K'_{n,m}$ is obtained by modifying the diagram of $K_{n,m} \subset Y(m)$ in Figure 4 according to Figure 3. In this picture, the slice disks of $K_{n,m}$ and $K'_{n,m}$ intersect with the 0-framed 2-handle at $2n$ points. Let $d(K)$ be the top degree of the symmetrized Alexander polynomial $\Delta_K(t)$.

Lemma 2.1. *For any integer m and positive integer n, the Alexander polynomials of $K_{n,m}$ and $K'_{n,m}$ are $\Delta_{K_{n,m}} \doteq 2t^n - 5 + 2t^{-n}$ and $\Delta_{K'_{n,m}} = 6t^n - 13 + 6t^{-n}$. Furthermore, the genera of $K_{n,m}$ and $K'_{n,m}$ are n.*

Note that the computation in the case of $K_{1,m}$ was done in [8].

Proof. $K_{n,m}$ has the genus n Seifert surface $\Sigma_{n,m}$ as in Figure 4. We compute the Seifert matrix for $\Sigma_{n,m}$. We take the generators $\{\lambda_i, \mu_i | i = 1, \cdots, n\}$ in $H_1(\Sigma_{n,m})$ as in Figure 5.

We define λ_i^+ and μ_i^+ to be the parallel transforms in the one side of the neighborhood of $\Sigma_{n,m}$. Consider the order of the generators as

$$\lambda_1, \lambda_2, \cdots, \lambda_n, \mu_1, \mu_2, \cdots, \mu_n.$$

The (r, s)-entry of the Seifert matrix $S_{n,m}$ is the linking number $lk(x_r^+, x_s)$, where x_i is the i-th generator above. Here we have the following:

$$lk(\lambda_i^+, \lambda_j) = 0, \; lk(\mu_i^+, \mu_j) = 0, \quad lk(\lambda_i^+, \mu_j) = \begin{cases} 2 & i \leq j \\ 1 & i > j \end{cases}$$

and

$$lk(\mu_i^+, \lambda_j) = \begin{cases} 1 & i \leq j \\ 2 & i > j. \end{cases}$$

These calculations are done by considering the linking indicated in FIGURE 6.

The Seifert matrix S_n is $\begin{pmatrix} O_n & A_n \\ B_n & O_n \end{pmatrix}$, where O_n is the $n \times n$ zero matrix, A_n and B_n are $n \times n$ matrices satisfying the following:

$$A_n = (a_{ij}),\ a_{ij} = \begin{cases} 2 & j \geq i \\ 1 & j < i. \end{cases} \text{ and } B_n = (b_{ij}),\ b_{ij} = \begin{cases} 1 & j \geq i \\ 2 & j < i. \end{cases}$$

Then we have

$$\begin{aligned} \Delta_{K_{n,m}} &= \det(tS_n - S_n^T) = \det \begin{pmatrix} O_n & tA_n - B_n^T \\ tB_n - A_n^T & O_n \end{pmatrix} \\ &= (-1)^n \det(tA_n - B_n^T) \det(tB_n - A_n^T) \\ &= \det(tA_n - B_n^T) \det(A_n - tB_n^T). \end{aligned}$$

We set $(\alpha_{ij}) = tA_n - B_n^T$, where $\alpha_{ij} = \begin{cases} 2t-2 & j > i \\ 2t-1 & i = j \\ t-1 & j < i. \end{cases}$ We define $\det(tA_n - B_n^T)$ to be α_n. By expanding α_n and deforming it, we have

$$\alpha_n = \det \begin{pmatrix} 1 & 2t-2 & \cdots & \cdots & 2t-2 \\ -t & 2t-1 & 2t-2 & \cdots & 2t-2 \\ 0 & t-1 & 2t-1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2t-2 \\ 0 & t-1 & \cdots & t-1 & 2t-1 \end{pmatrix} = \alpha_{n-1} + t\beta_{n-1},$$

where β_{n-1} is the $(n-1) \times (n-1)$ matrix satisfying the following:

$$\begin{aligned} \beta_{n-1} &= \det \begin{pmatrix} 2t-2 & 2t-2 & \cdots & \cdots & 2t-2 \\ t-1 & 2t-1 & 2t-2 & \cdots & 2t-2 \\ \vdots & t-1 & 2t-1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2t-2 \\ t-1 & t-1 & \cdots & t-1 & 2t-1 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & 2t-2 & \cdots & \cdots & 2t-2 \\ -t & 2t-1 & 2t-2 & \cdots & 2t-2 \\ 0 & t-1 & 2t-1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2t-2 \\ 0 & t-1 & \cdots & t-1 & 2t-1 \end{pmatrix} = t\beta_{n-2} \end{aligned}$$

$$\beta_2 = \det \begin{pmatrix} 2t-2 & 2t-2 \\ t-1 & 2t-1 \end{pmatrix} = 2t(t-1).$$

Thus $\beta_{n-1} = 2t^{n-2}(t-1)$, therefore, we have $\alpha_n = 2t^2 - 1 + \sum_{k=3}^{n} 2t^{k-1}(t-1) = 2t^n - 1$.

By using the following equality
$$\det(A_n - tB_n^T) = (-t)^n \det(1/tA_n - B_n^T)$$
we have $\det(A_n - tB_n^T) = (-t)^n(2t^{-n} - 1) = (-1)^n(2 - t^n)$. Therefore, we have
$$\Delta_{K_{n,m}}(t) = (2t^n - 1)(-1)^n(2 - t^n) \doteq 2t^n - 5 + 2t^{-n}.$$
Hence, since $d(K_{n,m})$ coincides with the genus of $\Sigma_{n,m}$, we can see that the surface is the minimal Seifert surface. Thus, we have $g(K_{n,m}) = n$.

In the case of $K'_{n,m}$, we can do the similar computation to above by taking the corresponding generators in the Seifert surface.

The Seifert matrix S'_n is $\begin{pmatrix} O_n & A'_n \\ B'_n & O_n \end{pmatrix}$, where O_n is the $n \times n$ zero matrix, A'_n and B'_n are $n \times n$ matrices satisfying the following:
$$A'_n = (a'_{ij}),\ a'_{ij} = \begin{cases} -2 & j \geq i \\ -3 & j < i. \end{cases} \text{ and } B_n = (b_{ij}),\ b_{ij} = \begin{cases} -3 & j \geq i \\ -2 & j < i. \end{cases}$$
Then we have
$$\begin{aligned} \Delta_{K'_{n,m}} &= \det(tS'_n - {S'}_n^T) = \det \begin{pmatrix} O_n & tA'_n - {B'}_n^T \\ tB'_n - {A'}_n^T & O_n \end{pmatrix} \\ &\doteq 6t^n - 13 + 6t^{-n}. \end{aligned}$$
□

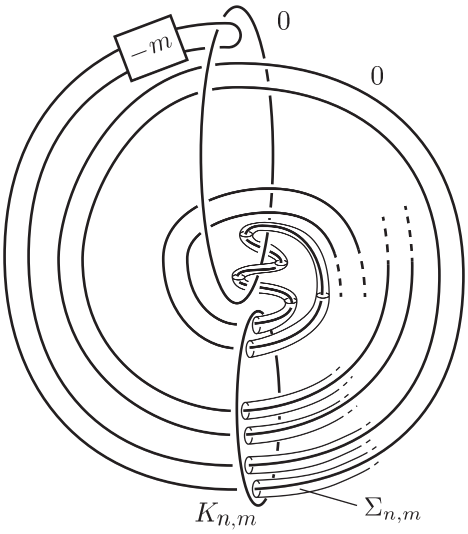

FIGURE 4. $K_{n,m}$ in $Y(m)$

Lemma 2.2. *Let K_1 and K_2 be two knots in two homology spheres Y_1, Y_2 respectively. Then*
$$g(K_1 \# K_2) = g(K_1) + g(K_2)$$
holds.

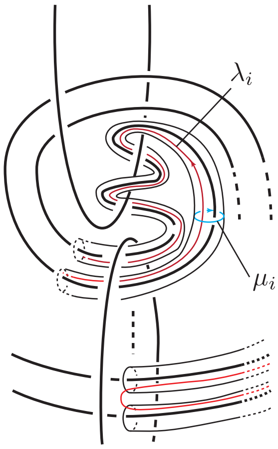

FIGURE 5. Generators of $H_1(\Sigma_{n,m})$.

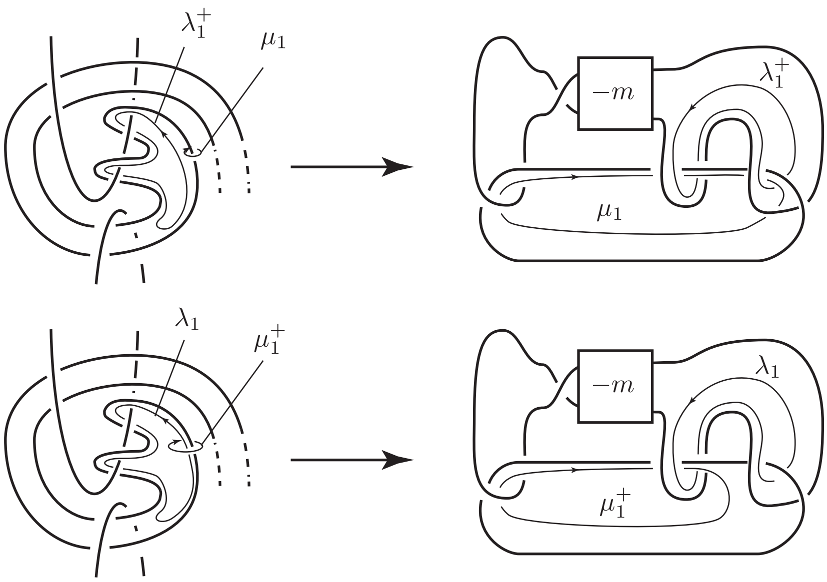

FIGURE 6. $lk(\lambda_i^+, \mu_j)$ and $lk(\lambda_i^+, \mu_j)$

This is a classical result, however, we prove it here again.

Proof. Let $S \subset Y_1 \# Y_2$ be the embedded separating sphere for Y_1 and Y_2. We suppose that S is separating $K_1 \# K_2$ i.e., $(K_1 \# K_2) \cap S$ are two points. Let Σ be the minimal genus Seifert surface of $K_1 \# K_2$. The set of the intersection $\Sigma \cap S$ consists of finite circles and single arc connecting the two points in the general position. We take the inner most circle C not including the arc in the interior. C bounds a disk in Σ because Σ is the minimal genus surface. Then by cutting the disk and capping two new disks, we can decrease the number of the intersection circles. We call the new embedded surface Σ again. This cut-and-past process preserves the genus of Σ. The isotopy class of Σ may be changed. By iterating this process we vanish all the intersection circles. Then $\Sigma = \Sigma_1 \natural \Sigma_2$ is obtained and

$$g(K_1 \# K_2) = g(\Sigma) = g(\Sigma_1) + g(\Sigma_2) \geq g(K_1) + g(K_2)$$

holds. Conversely, since $g(K_1)+g(K_2) \geq g(K_1\#K_2)$, we obtain $g(K_1\#K_2) = g(K_1) + g(K_2)$. □

Let K be a knot in a homology sphere. If $\Delta_K(t)$ cannot be decomposed as in $\Delta_K(t) = f_1(t)f_2(t)$ and $f_i(t)$ agrees with the Alexander polynomial of a knot in a homology sphere, then we call $\Delta_K(t)$ *A-irreducible.*

Lemma 2.3. *Let K be a knot in a homology sphere Y. If $\Delta_K(t)$ is A-irreducible and $g(K) = d(K)$, then K is prime.*

Proof. Suppose that K is not prime. Then K is isotopic to a composite knot $K_1\#K_2$. Then $\Delta_K(t) = \Delta_{K_1}(t)\Delta_{K_2}(t)$. Hence, $d(K) = d(K_1) + d(K_2)$ holds. Since $g(K) = d(K)$, we have $d(K) = g(K) = g(K_1) + g(K_2) \geq d(K_1) + d(K_2)$. Therefore, $g(K_1) + g(K_2) = d(K_1) + d(K_2)$ holds. From the inequalities $g(K_i) \geq d(K_i)$, $g(K_i) = d(K_i)$ holds for $i = 1, 2$.

On the other hand, since K is A-irreducible, $\Delta_{K_1}(t) = 1$ or $\Delta_{K_2}(t) = 1$. Thus $g(K_1) = 0$ or $g(K_2) = 0$ holds. This means that K is prime. □

We prove $K_{n,m}$ is a prime knot.

Lemma 2.4. *$K_{n,m}$ and $K'_{n,m}$ are prime knots in $Y_{n,m}$, $Y'_{n,m}$ respectively.*

Proof. The Alexander polynomials of $K_{n,m}$ and $K'_{n,m}$ are $2t^n - 5 + 2t^{-n}$ and $6t^n - 13 + 2t^{-n}$. These polynomials are A-irreducible. Because, the polynomials are completely decomposed as $\Delta_{K_{n,m}} \doteq 2t^n - 5 + 2t^{-n} = (2t^n - 1)(t^n - 2)$ and $\Delta_{K'_{n,m}} \doteq 6t^n - 13 + 6t^{-n} \doteq (2t^n - 3)(3t^n - 2)$ as a polynomial over $\mathbb{Z}$. Any factor of these decompositions is an irreducible polynomial by Eisenstein's criterion and is not an Alexander polynomial of a knot in a homology sphere. Since the genus of $K_{n,m}$ and $K'_{n,m}$ is n, from Lemma 2.3, $K_{n,m}$ and $K'_{n,m}$ are prime. □

3. Boundary-sum irreducibility of $X(\mathbf{x})_{n,m}$.

Before proving Theorem 1.4, we prove Lemma 1.3.

Proof. Suppose that X^4 is boundary-sum reducible. Then there exists a decomposition $X = X_1 \natural X_2$ and X_i is not homeomorphic to a 4-disk. Suppose that either ∂X_1 or ∂X_2 is diffeomorphic to S^3. We may assume $\partial X_1 \cong S^3$. Then X is connected-sum $\hat{X}_1 \# X_2$, where $\hat{X}_1$ is X_1 capped off by a 4-disk D^4 and $\hat{X}_1$ is not homeomorphic to S^4. Then X is irreducible. Therefore we get the desired assertion.

Here we prove Theorem 1.4.

Proof. For any $\{*, 0\}$-sequence we set $X = X(\mathbf{x})$. The irreducible decomposition of $X_{n,m}$ is already done and unique. Use Freedman's classification [3] for the double of X. Thus $X_{n,m}$ is irreducible.

We prove that $Y_{n,m}$ is a prime 3-manifold. $Y_{n,m}$ is the n-fold cyclic branched cover of $Y(m)$ along $K_{n,m}$. Namely, $Y_{n,m}/\langle\tau\rangle = Y(m)$, where

$\tau = \tau_{n,m}$. If $S \subset Y_{n,m}$ is an embedded 2-sphere, we assume that up to isotopy, S satisfies either of the following conditions for any $g \in \langle \tau \rangle$ due to [9] and [6]:

- $g(S) \cap S = \emptyset$
- $g(S) = S$.

Suppose that the first condition is satisfied. S does not intersect with the branch locus, namely, S is projected to a sphere in $Y(m)$. Since $Y(m)$ is a prime 3-manifold due to [8], then the sphere bounds a 3-ball in $Y(m)$. Hence, lifting the ball to $Y_{n,m}$, we can find a 3-ball in $Y_{n,m}$ with the boundary S.

Suppose that the second condition is satisfied. Then the action restricts on S. The action is orientation-preserving, because if the action on S is orientation-reversing, then the quotient space has a connected-sum component of $L(2,1)$. Then in the general position, S transversely intersects with the branch locus at finite points. By this argument, we can rule out the case where the branch locus is included in S.

This means that $\langle \tau \rangle$ acts on S with the fixed points discrete. The finite action of the 2-sphere is conjugate to the rotation in $SO(3)$ up to homotopy due to [10]. In particular, the fixed points are two points. Let S' be an image of S into $Y(m)$ and $S' \cap K_{n,m}$ are two points. Since $Y(m)$ is prime, S' bounds a 3-disk D in $Y(m)$. $D \cap K_{n,m}$ is the trivial arc, because $K_{n,m}$ is prime knot in $Y(m)$. Since the branched cover along the trivial arc is a 3-disk, S bounds a 3-disk in $Y_{n,m}$.

In each case, any embedded sphere in $Y_{n,m}$ bounds a 3-disk. This means $Y_{n,m}$ is prime and it follows that $X_{n,m}$ is boundary-sum irreducible.

$Y'_{n,1}$ is the n-fold branched cover over $K_{n,1}$ in $Y'(1) = C(1)$. This argument works for $Y'_{n,1}$. This means $Y'_{n,1}$ is prime. Thus, for any n, $(E_{n,1}, \tau_{n,1})$ is boundary-sum irreducible $\mathbb{Z}_n$-order cork. □

Here, we put a quick proof of Theorem 1.5.

Proof of Theorem 1.5. It immediately follows from the latter of the proof. □

For any integer m with $m \neq 1$, we do not know whether $E_{n,m}$ is boundary-sum irreducible or not. We need to prove the primeness of $Y'(m)$. The Dehn surgery diagram of $Y'(m)$ is drawn in Figure 7. This manifold is a Dehn surgery of $S^3_1(Pr(-3,3,-3))$.

4. Proof of hyperbolicity.

Finally, we prove Theorem 1.6.

Proof. The output "True" for the program HIKMOT means that the 3-manifold admits hyperbolic structure [4]. To get True-output, we need apply Algorithm 2 in [4]. The data after using the algorithm are updated in the site [12]. We can get "True" for these four examples by running the data by HIKMOT. □

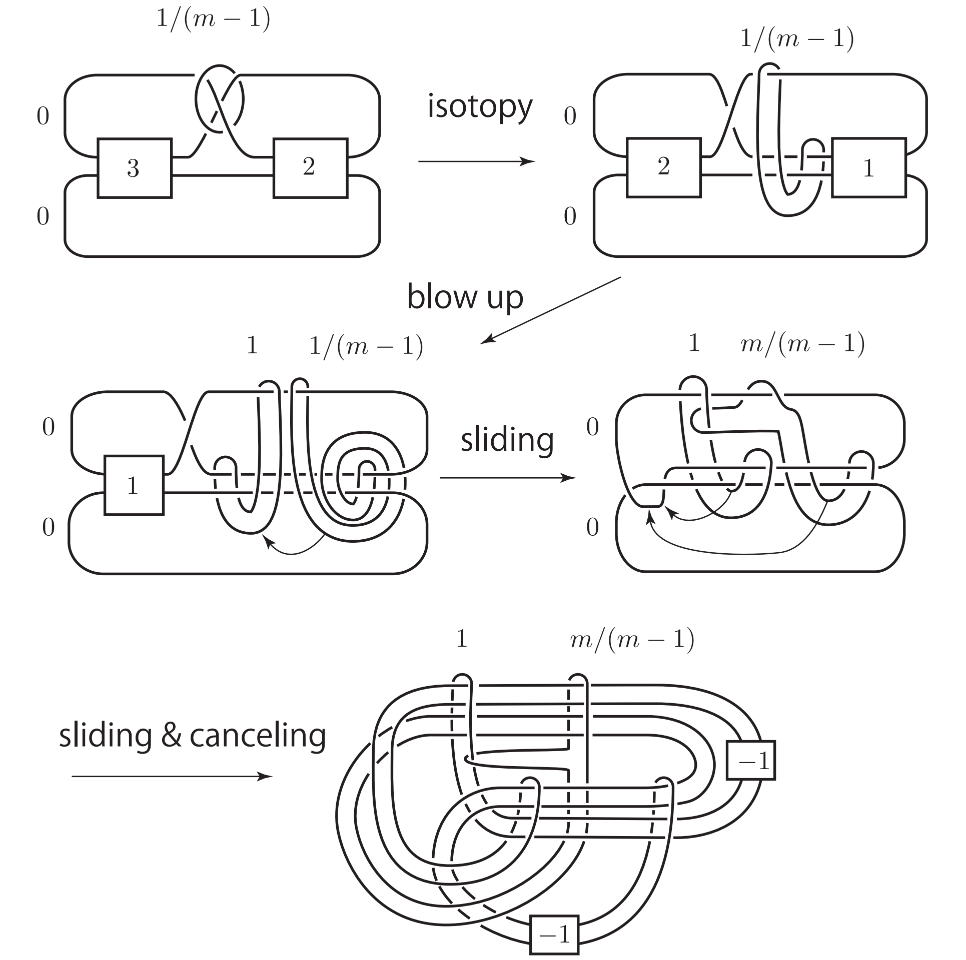

FIGURE 7. The Dehn surgery diagram of $Y'(m)$.

REFERENCES

[1] S. Akbulut and K. Yasui *Cork twisting exotic Stein 4-manifolds,* J. Differential Geom. 93 (2013), no. 1, 1–36.

[2] D. Auckly, H. J. Kim, P. Melvin, and D. Ruberman, *Equivariant corks*, Algebr. Geom. Topol. 17 (2017), no. 3, 1771–1783.

[3] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453

[4] N. Hoffman, K. Ichihara, M. Kashiwagi, H. Masai, S. Oishi, and A. Takayasu, *Verified computations for hyperbolic 3-manifolds,* Exp. Math. 25 (2016), no. 1, 66–78.

[5] HIKMOT, `http://www.oishi.info.waseda.ac.jp/~takayasu/hikmot/`

[6] M. J. Dunwoody, *An equivariant sphere theorem,* Bull. London Math. Soc. 17 (1985), no. 5, 437–448.

[7] R. Gompf, *Infinite order corks*, Geom. Topol. 21 (2017), no. 4, 2475–2484.

[8] C. Karakurt, T. Oba, and T. Ukida, *Planar Lefschetz fibrations and Stein structures with distinct Ozsváth-Szabó invariants on corks,* Topology Appl. 221 (2017), 630–637.

[9] W. Meeks, L. Simon, and S. T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2) 116 (1982), no. 3, 621–659.

[10] S. Smale, *Diffeomorphisms of the 2-Sphere*, Proceedings of the American Mathematical Society Vol. 10, No. 4 (Aug., 1959), pp. 621–626
[11] M. Tange, *Finite order corks*. Internat. J. Math. 28 (2017), no. 6, 1750034, 26. pp.
[12] Supplemental data for "Boundary-sum irreducible finite order cork". http://www.math.tsukuba.ac.jp/~tange/program.html

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