UPSILON INVARIANTS OF L-SPACE CABLE KNOTS

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Abstract. We compute the Upsilon invariant of L-space cable knots $K_{p,q}$ in terms of p, Υ_K and $\Upsilon_{T_{p,q}}$. The integral value of the Upsilon invariant gives a $\mathbb{Q}$-valued knot concordance invariant. We also compute the integral values of the Upsilon of L-space cable knots.

1. Introduction

1.1. **Υ-invariant.** In [14], Ozsváth, Stipsicz and Szabó defined a knot concordance invariant $\Upsilon : \mathcal{C} \to C([0,2])$ where $C([0,2])$ is the set of continuous functions over the closed interval $[0,2]$. After [14], Livingston in [12] gave a simpler definition of Υ_K.

This invariant is defined by extracting a "τ-like" information coming from the knot filtration of the (whole) knot Floer chain complex $CFK^\infty(S^3, K)$. Recall that Ozsváth-Szabó's τ-invariant is defined by using the knot filtration over the subcomplex $CFK^\infty(S^3,K)\{i=0\} \subset CFK^\infty(S^3,K)$. Naturally, this invariant Υ_K is a refinement of the τ-invariant and has properties analogous to τ. In fact, τ_K is computed by the formula $\tau_K = -\Upsilon'_K(0)$.

K is called an *L-space knot* if a positive surgery of K is an L-space, which is defined to be a rational homology sphere with the same Heegaard Floer homology as S^3 for any spin^c structure of the rational homology sphere. Borodzik and Livingston wrote down a Υ-invariant formula for any L-space knot K by use of the formal semigroup S_K for any L-space knot K. The formal semigroup is explained in Section 2.2.

Proposition 1 ([2]). *Let K be an L-space knot with genus g. Then for any $t \in [0,2]$ we have*

$$\Upsilon_K(t) = \max_{m\in\{0,\cdots,2g\}} \{-2\#(S_K \cap [0,m)) - t(g-m)\}.$$

In this paper we consider the following invariant $\widetilde{\Upsilon}_K(t,m) = -2\#(S_K \cap [0,m)) - t(g-m)$ for the formal semigroup S_K of an L-space knot K. Hence the Υ-invariant is written as $\Upsilon_K(t) = \max_{m\in\{0,\cdots,2g\}} \widetilde{\Upsilon}_K(t,m)$.

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1.2. **A cabling formula of the Υ-invariant.** Let K be a knot in S^3. Let V be a tubular neighborhood of K. For integers p, q, the (p, q)-cable $K_{p,q}$ of K is defined to be the simple closed curves on ∂V whose homology class is $p \cdot \mathbf{l} + q \cdot \mathbf{m}$ in $H_1(\partial V)$, where $\mathbf{m}$ and $\mathbf{l}$ are represented by a meridian and longitude curves on ∂V. If p, q are coprime integers, then $K_{p,q}$ is a knot and we call it the (p, q)-*cable knot*.

We consider the Υ-invariant of the cable knot. W. Chen in [3] gives an inequality for the Υ-invariant of any cable knot. Our purpose of this article is to give a cabling formula for any L-space knots.

Here we recall, to compare our cabling formula of $\Upsilon_{K_{p,q}}$ in this paper, two cabling formulas for two invariants: Alexander polynomial and Tristram-Levine signature.

The Alexander polynomial of the (p, q)-cable knot is computed as follows:

$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p)\Delta_{T_{p,q}}(t). \tag{1}$$

The Tristram-Levine signature $\sigma_K(\omega)$ is defined as the signature of the matrix

$$(1-\omega)S + (1-\bar{\omega})^T S,$$

where S is the Seifert matrix of K and ω is any unit complex number. Due to [11], the Tristram-Levine signature of the (p, q)-cable knot is computed as follows:

$$\sigma_{K_{p,q}}(\omega) = \sigma_K(\omega^p) + \sigma_{T_{p,q}}(\omega). \tag{2}$$

Here we recall Hedden and Hom's necessary and sufficient condition for a cable knot $K_{p,q}$ to be an L-space knot.

Theorem 2 (Hedden [6] and Hom [9])**.** *Let K be a knot with the Seifert genus g. $K_{p,q}$ is an L-space knot if and only if K is an L-space knot and $(2g-1)p \leq q$.*

1.3. **The L-space cabling formula of Υ.** The first main theorem is the following.

Theorem 3 (The case of $2gp \leq q$)**.** *Let K be an L-space knot with the Seifert genus g. Let p, q be relatively prime positive integers with $2gp \leq q$. Then the Υ-invariant of $K_{p,q}$ is computed as follows:*

$$\Upsilon_{K_{p,q}}(t) = \Upsilon_K(pt) + \Upsilon_{T_{p,q}}(t). \tag{3}$$

Here $\Upsilon_K(pt)$ means the p-fold amalgamated function of $\Upsilon_K(t)$ in the sense of the deformation as in Figure 1. In other words, the p-fold amalgamated function is presented by $\Upsilon_K(s)$ for $2i/p \leq t \leq 2(i+1)/p$ $(i = 0, 1, \cdots, p-1)$ and $2i + s = pt$.

This formula (3) is similar to the cabling formula (2), however (3) does not always hold even L-space cable knots. In fact, in the case of $(2g-1)p \leq q < 2gp$, which is the remaining one, a different formula holds as mentioned in the below.

We set

$$\mu_K := \min_{0<m<2g} \frac{2\#(S_K \cap [0, m))}{m}$$

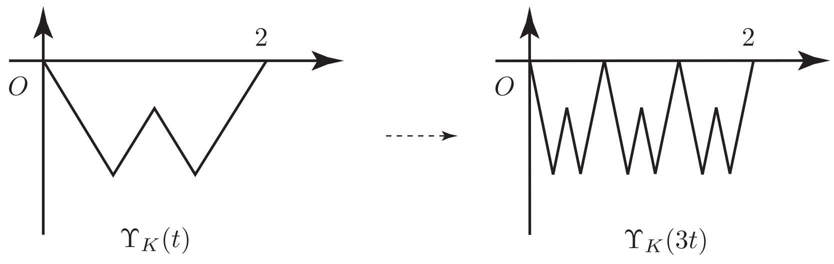

FIGURE 1. The amalgamated function of 3-copies of $\Upsilon_K(t)$.

and

$$\delta := q - (2g-1)p.$$

Let $g_{p,q}$ denote $g(T_{p,q}) = (p-1)(q-1)/2$. The following theorem gives the region of t that $\Upsilon_{K_{p,q}}(t)$ satisfies the same formula as the one in the first case.

Theorem 4 (The case of $(2g-1)p < q < 2gp$)**.** *Let K be an L-space knot with the Seifert genus g. Let p, q be relatively prime integers with $(2g-1)p < p \leq 2gp$.*

Let t be a real number with $0 \leq t \leq 2$. Let i be an integer with $2i/p \leq t \leq 2(i+1)/p$ and $0 \leq i < p$. We set the real number s satisfying $2i + s = pt$. Suppose that s satisfies either of the following conditions:

$$\begin{cases} 0 \leq s \leq 2 - \mu_K & i = 0 \\ \mu_K \leq s \leq 2 - \mu_K & 1 \leq i \leq p-2 \\ \mu_K \leq s \leq 2 & i = p-1. \end{cases}$$

Then

$$\Upsilon_{K_{p,q}}(t) = \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t)$$

holds.

In the region of t other than for the condition in Theorem 4, the formula $\Upsilon_{K_{p,q}}(t) = \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t)$ fails. Here we observe this failure by an example.

1.4. **Example** $(T_{3,7})_{3,35}$**.** Consider the $(3,35)$-cable knot of $K = T_{3,7}$. Then $p = 3$, $q = 35$, $g(K) = 6$ and $(2g-1)p \leq q < 2gp$ hold. Therefore $K_{3,35}$ is an L-space knot from the Hedden and Hom's criterion. Then the value μ_K defined above is $2/3$. We compare the functions $\Upsilon_{K_{3,35}}(t) - \Upsilon_{T_{3,35}}(t)$ and $\Upsilon_K(3t)$. See FIGURE 2. Let i, s and t be $i = 0, 1, 2$, $2i + s = 3t$ and

$$\begin{cases} 0 \leq s \leq 4/3 & i = 0 \\ 2/3 \leq s \leq 4/3 & i = 1 \\ 2/3 \leq s \leq 2 & i = 2. \end{cases}$$

Then $\Upsilon_{K_{3,35}}(t) = \Upsilon_{T_{3,35}}(t) + \Upsilon_K(s)$ holds, as indicated in FIGURE 2.

On the other hands, for the remaining regions, e.g., $i = 1$ and $0 < s < 2/3$ or $4/3 < s < 2$, the $\Upsilon_{K_{3,35}}(t)$ violates the formula (3). Theorem 5 gives a cabling formula on the such regions. As an example, in Section 6, we try to compute some of the actual functions of $\Upsilon_{K_{3,35}}(t)$ over the following regions: $i=1$ and $0<s<2/3$ and $i=0$ and $4/3<s<2$.

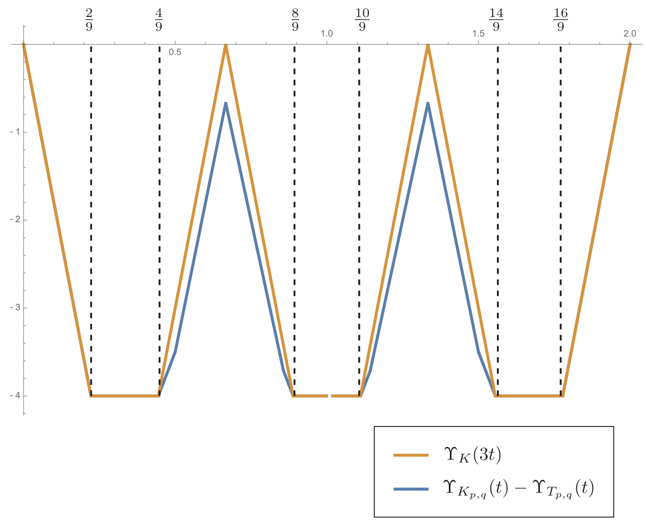

FIGURE 2. The red graph is $\Upsilon_K(3t)$. The blue graph is the different part of $\Upsilon_{K_{3,35}}(t)-\Upsilon_{T_{3,35}}(t)$ from $\Upsilon_K(3t)$.

1.5. **An L-space cabling formula of Υ over $0<s<\mu_K$ or $2-\mu_K<s<2$.** The behaviors of $\Upsilon_{K_{p,q}}(t)$ over the region of $0<s<\mu_K$ $(0<i<p-1)$ or $2-\mu_K<s<2$ $(0\le i<p-1)$ are more complicated.

Here for any real number t with $2i/p\le t\le 2(i+1)/p$ we define $\Upsilon^{\delta,1}_{p,q}(t)$ and $\Upsilon^{\delta,2}_{p,q}(t)$ to be

$$\Upsilon^{\delta,1}_{p,q}(t)=\max_{iq-\delta<m\le iq}\widetilde{\Upsilon}_{T_{p,q}}(t,m),\ \Upsilon^{\delta,2}_{p,q}(t)=\max_{iq-p<m\le iq-\delta}\widetilde{\Upsilon}_{T_{p,q}}(t,m).$$

Then,

$$\max\left\{\Upsilon^{\delta,1}_{p,q}(t),\Upsilon^{\delta,2}_{p,q}(t)\right\}=\Upsilon_{T_{p,q}}(t)$$

holds. In general, we have $\Upsilon_{T_{p,q}}(t)=\max_{iq-p<m\le iq}\widetilde{\Upsilon}_{p,q}(t,m)$, due to Proposition 12. For any L-space knot K we define the *truncated* Υ*-invariant* as follows:

$$\Upsilon^{tr}_K(s)=\max_{\nu\in\{1,2,\cdots,2g-1\}}\widetilde{\Upsilon}_K(s,\nu).$$

Actually, this invariant satisfies $\Upsilon^{tr}_K(s)=\Upsilon_K(s)$ for $\mu_k\le s\le 2-\mu_K$ (Lemma 14).

Theorem 5. *Let K be an L-space knot with the Seifert genus g. Let p,q be relatively prime integers with $(2g-1)p<q<2gp$. Let t be a real number with $0\le t\le 2$. We assume that i and $s\in\mathbb{R}$ satisfy $2i/p\le t\le 2(i+1)/p$ and $2i+s=tp$.*

Suppose that $0 < i < p$. *If* $0 < s < \mu_K$*, then* $\Upsilon_{K_{p,q}}(t)$ *is computed as follows:*

$$\Upsilon_{K_{p,q}}(t) = \max\left\{\Upsilon_K(s) + \Upsilon^{\delta,1}_{p,q}(t), \Upsilon^{tr}_K(s) + \Upsilon^{\delta,2}_{p,q}(t)\right\}. \tag{4}$$

Suppose that $0 \leq i < p-1$. *If* $2-\mu_K < s < 2$*, then* $\Upsilon_{K_{p,q}}(t)$ *is computed as follows:*

$$\Upsilon_{K_{p,q}}(t) = \max\left\{\Upsilon_K(s) + \Upsilon^{\delta,1}_{p,q}(2-t), \Upsilon^{tr}_K(s) + \Upsilon^{\delta,2}_{p,q}(2-t)\right\}. \tag{5}$$

We note that the equalities (4) and (5) hold for $0 < s < 1$ and $1 < s < 2$ respectively. Because, since $\Upsilon^{tr}_K(s) = \Upsilon_K(s)$, these equalities (4) and (5) become (3).

Corollary 6. *Let* K *be an L-space knot. We assume that* $(2g(K)-1)p < q < 2g(K)p$. *For* $0 \leq t \leq 2$*, let* i *and* s *be an integer and a real number with* $2i/p \leq t \leq 2(i+1)/p$*,* $2i + s = pt$ *and* $0 \leq s \leq 2$*. Then*

$$\Upsilon_{T_{p,q}}(t) + \Upsilon_K(s) \geq \Upsilon_{K_{p,q}}(t) \geq \Upsilon_{T_{p,q}}(t) + \Upsilon^{tr}_K(s)$$

holds.

In particular if $\mu_K \leq s \leq 2-\mu_K$, then the inequalities become the corresponding equalities. This means Theorem 4.

1.6. **The integral value of** $\Upsilon_K(t)$**.** We compute the integral value of $\Upsilon_K(t)$ over $[0,2]$, which is also a concordance knot invariant:

$$I(K) = \int_0^2 \Upsilon_K(t)dt.$$

Then the values of torus knots are computed as follows:

Proposition 7. *Let* p, q *be relatively prime positive integers. Let* a_i *be the* i*-th non-negative continued fraction of* q/p*:*

$$q/p = a_1 + \cfrac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}} =: [a_1, \cdots, a_n]. \tag{6}$$

Then we have

$$2I(T_{p,q}) = -\frac{1}{3}(pq - \sum_{i=1}^n a_i).$$

We can compare $I(T_{p,q})$ with the S^1-integral $\int_{S^1} \sigma_{T_{p,q}}(\omega)$ of the Tristram-Levine signature as follows:

$$\int_{S^1} \sigma_{T_{p,q}}(\omega) = -\frac{1}{3}\left(pq - \frac{p}{q} - \frac{q}{p} + \frac{1}{pq}\right) = 4(s(q,p) + s(p,q) - s(1,pq)),$$

where the function s is the Dedekind sum. This computation has been done by many topologists for example [10], [13], [1] and [4].

Here we give a formula of $I(L)$ with $L = (\cdots(K_{p_1,q_1})_{p_2,q_2}\cdots)_{p_n,q_n}$ of any iterated cable L-space knot. We denote the iterated cable L-space knot $(\cdots(K_{p_1,q_1})_{p_2,q_2}\cdots)_{p_i,q_i}$ by L_i.

Theorem 8. *Let (p_i, q_i) be positive coprime integers. Let K be an L-space knot. Let denote $L := L_n$. If (p_i, q_i) satisfies $q_i \geq 2g(L_i)p_i$ for any i, then the integral $I(L)$ is computed as follows:*

$$I(L) = I(K) + \sum_{i=1}^{n} I(T_{p_i,q_i}).$$

The similar formula to the S^1 integral of $\sigma_L(\omega)$ for iterated torus knot L is given in [1].

Acknowledgements

This work was started by computing the integral values of the Υ-invariants of any torus knots. The author thanks for Min Hoon Kim in terms of the below. He told me the Υ-invariant formula for the torus knots and the reference [5]. This became my motivation to compute the Υ-invariants of the L-space cable knots. Furthermore, he gave me many useful comments for my earlier manuscript.

2. Preliminaries

In this section we introduce the tools to prove our main theorem (Theorem 3).

2.1. **L-space cable knot.** We skip all the definitions relating to the Heegaard Floer homology, e.g., $\widehat{HF}$ and $\widehat{HFK}$. The set of L-space knots, whose definition is given in the previous section, forms a class of the most simple knots in terms of the property that $\widehat{HFK}(S^3, K, j)$ is at most 1-dimensional at each j, see [18]. To study the definitions we recommend the papers [15], [16] and [17].

Recall Theorem 2 in the previous section, proven by Hedden and Hom. These results give the necessary and sufficient condition for the cable knot $K_{p,q}$ to be an L-space knot as follows:

$K_{p,q}$ is an L-space knot $\Leftrightarrow$ K is an L-space knot and $(2g(K) - 1)p \leq q$.

2.2. **Formal semigroup.** Suppose that K is an L-space knot. Then due to [18], the Alexander polynomial $\Delta_K(t)$ of K is flat and has an alternating condition on the non-zero coefficients. Here a polynomial is called *flat* if any coefficient a_i of the polynomial satisfies $|a_i| \leq 1$.

Expanding the following rational function $\Delta_K(t)/(1-t)$ as follows:

$$\frac{\Delta_K(t)}{1-t} = \sum_{s \in S_K} t^s,$$

we obtain a subset $S_K \subset \mathbb{Z}_{\geq 0}$. This subset S_K is called the *formal semigroup of* K. According to [20], if K is an algebraic knot, then S_K is a semigroup. In particular, if K is a right-handed torus knot $T_{p,q}$, then $S_{T_{p,q}}$ is the semigroup generated by the positive integers p, q, namely, $S_{T_{p,q}} = \langle p, q \rangle = \{pa + qb \in \mathbb{Z} \mid a, b \in \mathbb{Z}_{\geq 0}\}$ holds. If K is an L-space knot, the knot is not always an algebraic knot because there exists an L-space knot K whose formal semigroup S_K is not semigroup. For example, the formal semigroup of the $(-2,3,2n+1)$ pretzel knot K_n for $n\ge 1$ is an L-space knot and the formal semigroup is as follows:

$$S_{K_n}=\{0,3,5,7,\cdots,2n-1,2n+1,2n+2\}\cup\mathbb{Z}_{n\ge 2n+4}.$$

Furthermore $K_1=T_{3,4}, K_2=T_{3,5}$ hold. It can be easily seen that if $n\ge 3$, then the S_{K_n} is not a semigroup. The Alexander polynomials of $(-2,3,2n+1)$-pretzel knots can be found, for example, in [8].

Wang, in [21], proved that the cabling formula of the formal subgroup of any L-space knot as follows:

Proposition 9 (A cabling formula of formal semigroup [21])**.** *Let K be a nontrivial L-space knot. Suppose $p\ge 2$ and $q\ge p(2g(K)-1)$. Then* $S_{K_{p,q}}=pS_K+q\mathbb{Z}_{\ge 0}:=\{pa+qb\mid a\in S_K, b\in\mathbb{Z}_{\ge 0}\}$.

Here we prove the following lemma.

Lemma 10. *Let S_K be a formal semigroup coming from non-trivial L-space knot. Then $1\notin S_K$ holds.*

Proof. If $1\in S_K$, then the Alexander polynomial of the L-space knot is computed as follows:

$$\Delta_K(t)=(1-t)(1+t+t^sf(t))=1-t^2+t^s(1-t)f(t),$$

where $s\ge 2$ and $f(t)$ is a series. Thus the coefficient of t in $\Delta_K(t)$ vanishes. The coefficient of t of the Alexander polynomial of a non-trivial L-space knot is -1 due to [7]. Thus K must be the trivial knot. □

In the case of lens space knots, there would be some restrictions to S_K. The results in [19] can give some restrictions.

3. Proofs of Proposition 7 and Theorem 8.

In [5] Feller and Krcatovich proved that the recurrence formula $\Upsilon_{T_{p,q}}(t)=\Upsilon_{T_{p,q-p}}(t)+\Upsilon_{T_{p,p+1}}(t)$. By using this formula, they proved the following Υ-invariant formula of torus knots.

Proposition 11 (Proposition 2.2 in [5])**.** *Let a_i be the same coefficient defined in (6) and p_i the denominator of $[a_i,a_{i+1},\cdots,a_n]$. Then we have*

$$\Upsilon_{T_{p,q}}(t)=\sum_{i=1}^{n}a_i\Upsilon_{T_{p_i,p_i+1}}(t). \tag{7}$$

Note that the formula depends on the way of taking the continued fraction in general, but it does not depend on the way to take the non-negative integral continued fraction expansions $q/p=[a_i,\cdots,a_n]$, i.e., $a_i\ge 0$ for any i. Here we prove Proposition 7 by using the formula (7).

Proof. From the torus knot formula, we immediately have

$$I(T_{p,q})=\sum_{i=1}^{n}a_iI(T_{p_i,p_i+1}).$$

Comparing the first derivative of (7) at $t = 0$, we have

$$(p-1)(q-1) = \sum_{i=1}^{n} a_i p_i (p_i - 1). \tag{8}$$

The direct computation for $T_{p,p+1}$ implies the following:

$$I(T_{p,p+1}) = -\frac{p^2-1}{6}.$$

Thus, we have

$$2I(T_{p,q}) = -\frac{1}{3}\sum_{i=1}^{n} a_i(p_i^2 - 1) = -\frac{1}{3}\sum_{i=1}^{n}(a_i p_i(p_i-1) - a_i + a_i p_i).$$

Since $p_{i-1} = a_i p_i + p_{i+1}$,

$$\sum_{i=1}^{n} a_i p_i = \sum_{i=1}^{n}(p_{i-1} - p_{i+1}) = q + p_1 - p_n = q + p - 1.$$

Thus using (8) we get the following:

$$2I(T_{p,q}) = -\frac{1}{3}\left((p-1)(q-1) - \sum_{i=1}^{n} a_i + q + p - 1\right) = -\frac{1}{3}\left(pq - \sum_{i=1}^{n} a_i\right).$$

□

Next, we prove Theorem 8 using Theorem 3.

Proof. Let denote $L' = L_{n-1}$. First we obtain the equality:

$$\int_0^2 \Upsilon_{L'}(pt)dt = \int_0^{2p} \Upsilon_{L'}(s)\frac{1}{p}ds = p\int_0^2 \Upsilon_{L'}(s)\frac{1}{p}ds = I(L').$$

This equality can be justified by regarding $\Upsilon_K(pt)$ as a function which is naturally expanded to the periodic function over $\mathbb{R}$ with the period $2/p$. Using Theorem 3 and this computation we have

$$I(L) = \int_0^2 (\Upsilon_{L'}(pt) + \Upsilon_{T_{p_n,q_n}}(t))dt = I(L') + I(T_{p_n,q_n}).$$

By iterating this relationship we have

$$I(L) = I(K) + \sum_{i=1}^{n} I(T_{p_i,q_i}).$$

□

4. Proof of Theorem 3.

Let K be an L-space knot with the Seifert genus g. Throughout this section we assume that the relatively prime positive integers p, q satisfy $2gp \leq p$. In particular, $K_{p,q}$ is also an L-space knot.

For any L-space knot K we denote $\#(S_K \cap [0,m))$ by $\varphi_K(m)$. Let $\Phi_K(t,m)$ denote $\varphi_K(m) - tm/2$. According to Proposition 1, the Υ-invariant of an L-space knot K is rewritten as follows:

$$\begin{aligned}\Upsilon_K(t) &= -2 \min_{m\in\{0,1,\cdots,2g\}} \{\varphi_K(m) - tm/2\} - tg(K). \\ &= -2 \min_{m\in\{0,1,\cdots,2g\}} \Phi_K(t,m) - tg(K). \qquad (9)\end{aligned}$$

Extending the function $\varphi_K(m)$ as $\varphi_K(m) \equiv 0$ if $m < 0$, we can define $\Phi_K(t,m)$ over $m \in \mathbb{Z}$. We note that the function $\Phi_K(t,m)$ satisfies the following:

$$\begin{cases} -tm/2 & m < 0 \\ (1-t/2)m - g & m > 2g. \end{cases}$$

Thus if a subset $S \subset \mathbb{Z}$ includes $\{0,1,\cdots,2g\}$ then we have

$$\min_{m\in S} \Phi_K(t,m) = \min_{m\in\{0,1,\cdots 2g\}} \Phi_K(t,m).$$

The genus $g(K_{p,q})$ coincides with the degree of $\Delta_{K_{p,q}}(t)$ and K is an L-space knot. Thus from the cabling formula (1), we have

$$g(K_{p,q}) = pg + g_{p,q}.$$

We denote $\varphi_{K_{p,q}}(m)$ by $\varphi(m)$. Let $\Phi(t,m)$ denote $\Phi_{K_{p,q}}(t,m)$.

Lemma 12. *Let K be an L-space knot with $g = g(K)$. Let p,q be relatively prime integers with $2gp \leq q$. Let i be an integer with $0 \leq i < p$. Suppose that t is any real number with $2i/p \leq t \leq 2(i+1)/p$. Then we have*

$$\min_{0\leq m\leq 2g(K_{p,q})} \Phi(t,m) = \min_{iq-p<m\leq iq+2gp} \Phi(t,m).$$

Proof. We can extend the range $0 \leq m \leq 2g(K_{p,q})$ in the minimality to all integers. We fix a real number t with $0 \leq t \leq 2$. Let i be an integer with $2i/p \leq t \leq 2(i+1)/p$ and $0 \leq i < p$. Suppose that m is any integer with $m \leq iq - p$.

$$\begin{aligned}\Phi(t,m+p) - \Phi(t,m) &= \varphi(m+p) - t(m+p)/2 - \varphi(m) + tm/2 \\ &= \#(S_{K_{p,q}} \cap [m,m+p)) - tp/2\end{aligned}$$

Since $\#(S_{K_{p,q}} \cap [m,m+p)) \leq i$, we have $\Phi(t,m+p) - \Phi(t,m) \leq i - tp/2 \leq 0$. Thus the minimal value of $\Phi(t,m)$ over $m \in [0,iq]$ is the same as the minimal value over $m \in (iq-p,iq]$. See FIGURE 3 for the aid of our argument. This graph stands for elements in $S_{K_{p,q}}$ with $pS_K + \{0,1,2,\cdots i-2\}\mathbb{Z}_{\geq 0}$ omitted. All the circles mean the elements in $S_{T_{p,q}}$, the black circles mean the elements in $S_{K_{p,q}}$ and white circles mean the elements not in $S_{K_{p,q}}$.

Suppose that m is an integer with $iq + (2g-1)p < m \leq 2g(K_{p,q})$. Since $\varphi(m+p) - \varphi(m) = \#(S_{K_{p,q}} \cap [m,m+p)) \geq i+1$ holds, we have $\Phi(t,m+p) - \Phi(t,m) \geq i + 1 - tp/2 \geq 0$. Thus the minimal value of $\Phi(t,m)$ over $(iq+(2g-1)p, 2g(K_{p,q})]$ coincides with the minimal over $(iq+(2g-1)p, iq+2gp]$.

Therefore the minimal value of $\Phi(t,m)$ over $0 < m \leq 2g(K_{p,q})$ attains over $iq - p < m \leq iq + 2gp$. □

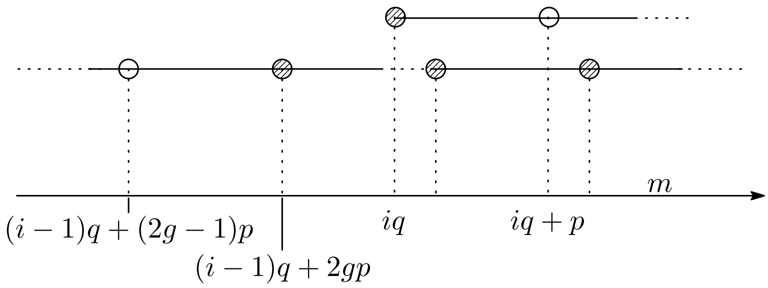

FIGURE 3. S_K without $pS_K + jq$ with $j < i - 1$.

As a corollary of this lemma, if K is the unknot, then we have

$$\min_{0 \le m \le 2g_{p,q}} \Phi_{T_{p,q}}(t, m) = \min_{iq-p<m\le iq} \Phi_{T_{p,q}}(t, m).$$

Next, we investigate the minimal values of $\Phi(t, m)$ in the region

$$I_i = \{m \in \mathbb{Z} \mid iq - p < m \le iq + 2gp\}.$$

The minimal value of $\Phi(t, m)$ over I_i coincides with

$$\min_{\nu \in S_K, \nu-1 \notin S_K} \left\{ \min_{iq+(\nu-1)p<m\le iq+\nu p} \Phi(t, m) \right\}.$$

This minimal value can be rewritten as follows:

$$\sum_{l=0}^{m} p \left(\frac{i + \epsilon(l) + 1}{p} - \frac{t}{2} \right) + \mu_i \quad (m = -1, 0, 1, 2, \cdots, 2g - 1), \tag{10}$$

where μ_i is the minimal value of $\Phi(t, m)$ over $(iq - p, iq]$. The function $\epsilon(l)$ is defined as follows:

$$\epsilon(\nu) = \begin{cases} 0 & \nu \in S_K \\ -1 & \nu \notin S_K \end{cases}$$

Here in the case of $m = -1$ the sum means 0. Since $\sum_{l=0}^{m}(\epsilon(l) + 1) = \#(S_K \cap [0, m + 1))$ holds, the summation in (10) is computed as follows:

$$\begin{aligned} & \min_{-1\le m\le 2g-1} \left\{ \#(S_K \cap [0, m + 1)) - \left(\frac{tp}{2} - i \right) (m + 1) \right\} \\ = & \min_{0\le m\le 2g} \{\#(S_K \cap [0, m)) - sm/2\} = \min_{0\le m\le 2g} \Phi_K(s, m). \end{aligned}$$

Then we have

$$\min_{m\in I_i} \Phi(t, m) = \min_{0\le m\le 2g} \Phi_K(s, m) + \mu_i. \tag{11}$$

Hence, we obtain

$$\begin{aligned} \Upsilon_{K_{p,q}}(t) &= -2 \min_{0\le m\le 2g} \Phi_K(s, m) - 2\mu_i - tg(K_{p,q}) \\ &= \Upsilon_K(s) + sg - 2\mu_i - t(pg + g_{p,q}). \end{aligned}$$

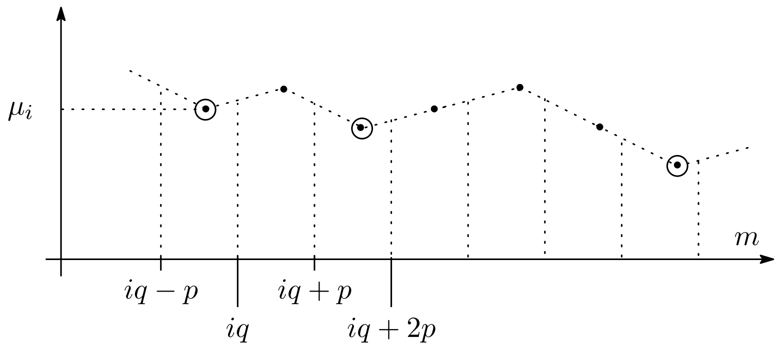

FIGURE 4. The places of local minimal points of $\Phi(t,m)$ over $m \in I_i$.

$S_{K_{p,q}}$ is the semigroup obtained by removing several copies of $[0,2g]\cap\mathbb{Z}-S_K$ from $S_{T_{p,q}}$. Taking K as the unknot in Lemma 12, we have the following:

$$\begin{aligned} \mu_i &= \min_{iq-p<m\leq iq} \Phi_{p,q}(t,m) - ig \\ &= \min_{0\leq m\leq 2g_{p,q}} \Phi_{p,q}(t,m) - ig. \end{aligned} \tag{12}$$

Here $\Phi_{p,q}(t,m)$ means $\Phi_{T_{p,q}}(t,m)$.

Therefore the formula (9) for L-space knots, we obtain the following:

$$\begin{aligned} \Upsilon_{K_{p,q}}(t) &= \Upsilon_K(s) + sg - 2\min_{0\leq m\leq 2g_{p,q}} \Phi_{p,q}(t,m) + 2ig - (pg+g_{p,q})t \\ &= \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t). \end{aligned}$$

□

5. THE CASE OF $(2g-1)p < q < 2gp$.

5.1. **The minimal value of $\Phi(t,m)$.** Let K be an L-space knot with the Seifert genus g. Throughout this section we assume that the relatively prime positive integers p,q satisfy $(2g-1)p < q < 2gp$. In particular, $K_{p,q}$ is an L-space knot. We consider $\Phi(t,m) = \#(S_{K_{p,q}} \cap [0,m)) - tm/2$. We set the difference $q-(2g-1)p$ as δ.

We denote $\{m \in \mathbb{Z} \mid iq-\delta < m \leq (i+1)q\}$ by I_i^δ. Here we prove the following lemma.

Lemma 13. *Suppose that t is any real number with $2i/p \leq t \leq 2(i+1)/p$ for $0 \leq i < p$. Then we have*

$$\min_{0\leq m\leq 2g(K_{p,q})} \Phi(t,m) = \min_{m\in I_i^\delta} \Phi(t,m).$$

Proof. We consider the following difference in the same way as Lemma 12

$$\Phi(t,m+p) - \Phi(t,m) = \varphi(t,m+p) - \varphi(t,m) - tp/2.$$

If $m \leq iq-\delta$, then the difference $\varphi(t,m+p)-\varphi(t,m) = \#(S_{K_{p,q}} \cap [m,m+p)) \leq i$ holds. Hence we have $\Phi(t,m+p)-\Phi(t,m) \leq i - tp/2 \leq 0$.

Thus the minimal value of $\Phi(t,m)$ over $(-\infty, iq-\delta+p]$ coincides with the minimal value over $(iq-\delta, iq-\delta+p]$.

In the case of $(i+1)q-p<m\le 2g(K_{p,q})-p$ the difference is computed as follows: $\varphi(m+p)-\varphi(m)=\#(S_{K_{p,q}}\cap[m,m+p))\ge i+1$. Hence we have $\Phi(t,m+p)-\Phi(t,m)\ge i+1-tp/2\ge 0$. Thus the minimal value of $\Phi(t,m)$ coincides with the minimal value over I_i^δ. □

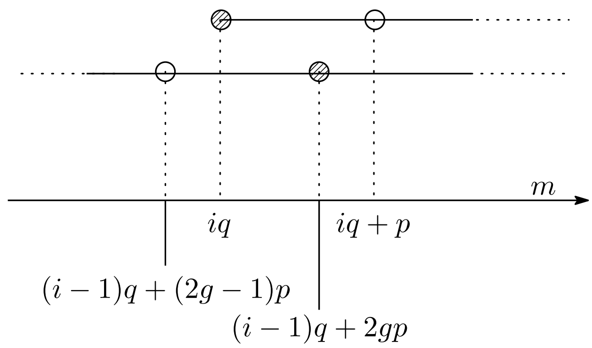

FIGURE 5. $S_{K_{p,q}}$ with pS_K+jq with $j=i-1,i$.

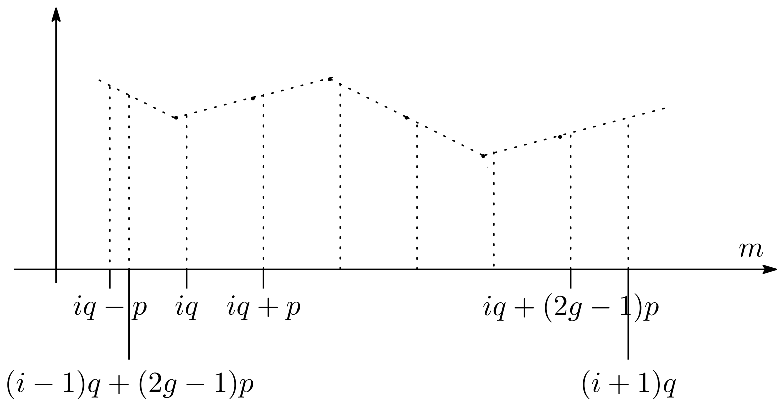

FIGURE 6. The places of local minimal points of $\Phi(t,m)$ over $m\in[(i-1)q+(2g-1)p, iq+(2g-1)p]$.

5.2. **Proof of Theorem 4.** Let p,q be relatively prime positive integers with $(2g-1)p\le q<2gp$. For a real number t with $0\le t\le 2$, let i be an integer with $2i/p\le t<2(i+1)/p$ for some integer $0\le i<p$. We set s as a real number with $2i+s=pt$.

A real number t satisfies $\Upsilon_{K_{p,q}}(t)=\Upsilon_K(s)+\Upsilon_{T_{p,q}}(t)$ if and only if

$$\min_{0\le m\le 2g(K_{p,q})}\Phi(t,m)=\min_{0\le m\le 2g}\Phi_K(t,m)+\mu_i$$

holds, where we recall $\mu_i=\min_{iq-p<m\le iq}\Phi_{p,q}(t,m)-ig$. In other words, such t satisfies either of the following conditions. Let $S^i_{K_{p,q}}$ be

$$\big(S_{K_{p,q}}\cup\{iq-\delta\}-\{(i+1)q\}\big)\cap[0,2g(K_{p,q}))\cup\mathbb{Z}_{\ge 2g(K_{p,q})}.$$

Let $\Phi^i(t,m)$ be

$$\min_{m\in I_i}(\#(S^i_{K_{p,q}}\cap[0,m))-tm/2)+\begin{cases}0 & i=0\\ -1 & 0<i\le p-1.\end{cases}$$

The functions $\Phi^i(t,m)$ and $\Phi(t,m)$ coincide on $iq-\delta<m\le(i+1)q$, while $\Phi^i(t,m)$ is the shift of $\Phi(t,m)$ by -1 on the regions $0\le m\le iq-\delta$ and $(i+1)q<m$ as in Figure 7.

Condition 1. *The minimal value of $\Phi^i(t,m)$ over $iq\le m<iq+(2g-1)p$ is not greater than the minimal value of $\Phi(t,m)$ over $iq-p<m\le iq$. This is equivalent to the condition*

$$\min_{iq-\delta<m}\Phi(t,m)=\min_{iq-p<m}\Phi^i(t,m).$$

Condition 2. *The minimal value of $\Phi^i(t,m)$ over $iq\le m<iq+(2g-1)p$ is not greater than the minimal value of $\Phi(t,m)$ over $iq+(2g-1)p<m\le iq+2gp$. This is equivalent to the condition*

$$\min_{m\le(i+1)q}\Phi(t,m)=\min_{m\le iq+2gp}\Phi^i(t,m).$$

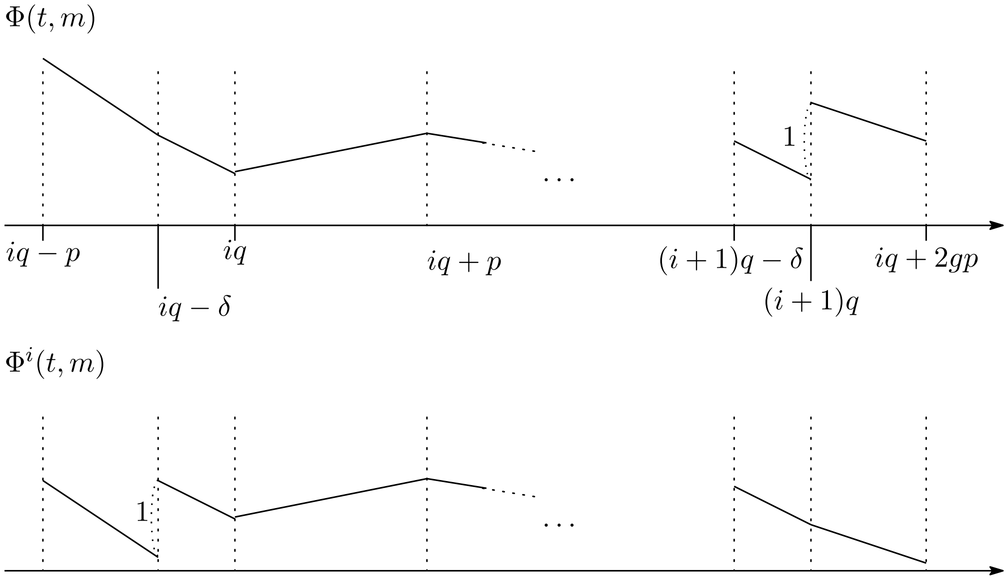

Figure 7. The functions $\Phi(t,m)$ and $\Phi^i(t,m)$ (in case of $0<i<p-1$).

If m is an integer with $iq-p<m\le iq-\delta$, then we have

$$\begin{aligned}\Phi^i(t,m+\nu p)-\Phi^i(t,m) &= \#(S^i_{K_{p,q}}\cap[m,m+\nu p))-\nu pt/2\\ &= i\nu+\#(S_K\cap[0,\nu))-\nu(i+\frac{s}{2})\\ &= \#(S_K\cap[0,\nu))-\nu s/2=\Phi_K(s,\nu).\end{aligned}$$

Condition 1 is satisfied if and only if there exists ν satisfying $\Phi_K(s,\nu) \le 0$ $(1 \le \nu \le 2g-1)$. This is equivalent to

$$s \ge \min_{1\le\nu\le 2g-1} \frac{2\varphi_K(\nu)}{\nu} =: \mu_K.$$

If m is an integer with $(i+1)q < m \le (i+1)q - \delta + p$, then we have

$$\begin{aligned}\Phi^i(t,m) - \Phi^i(t,m-\nu p) &= \#(S^i_{K_{p,q}} \cap [m-\nu p, m)) - \nu pt/2 \\ &= i\nu + \#(\bar{S}_K \cap [0,\nu)) - \nu(i + \frac{s}{2}) \\ &= \nu - \#(S_K \cap [0,\nu)) - \nu s/2 = -\Phi_K(2-s,\nu).\end{aligned}$$

Here $\bar{S}_K$ is the complement of S_K in $\mathbb{Z}$. Condition 2 is satisfied if and only if there exists ν satisfying $-\Phi_K(2-s,\nu) \ge 0$ $(1 \le \nu \le 2g-1)$. This is equivalent to

$$2 - s \ge \min_{1\le m\le 2g-1} \frac{2\varphi_K(\nu)}{\nu} = \mu_K.$$

Suppose that $0 < i < p-1$. The region $\mu_K \le s \le 2-\mu_K$ holds if and only if there exist $1 \le \nu, \nu' \le 2g-1$ such that $\Phi_K(s,\nu) < 0$ and $\Phi_K(2-s,\nu') < 0$ hold. Namely, this means that

$$\min_{m\in I_i^\delta} \Phi(t,m) = \min_{m\in I_i} \Phi^i(t,m).$$

Thus for such an s we have

$$\Upsilon_{K_{p,q}}(t) = \Upsilon_{T_{p,q}}(t) + \Upsilon_K(s).$$

Suppose that $i = 0$. Then we have

$$\min_{m\in I_0^\delta} \Phi(t,m) = \min_{-p\le m\le q} \Phi(t,m).$$

Furthermore if $s \le 2-\mu_K$, then $\min_{-p\le m\le q} \Phi(t,m) = \min_{-\delta<m\le 2gp} \Phi^0(t,m) = \min_{m\in I_0} \Phi^0(t,m)$ holds. Thus $s \le 2-\mu_K$ means that

$$\Upsilon_{K_{p,q}}(t) = \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t).$$

Suppose that $i = p-1$. Then we have

$$\min_{m\in I_{p-1}^\delta} \Phi(t,m) = \min_{(p-1)q-\delta\le m\le(p-1)q+2gp} \Phi(t,m).$$

Furthermore if $\mu_K \le s$, then

$$\begin{aligned}\min_{(p-1)q-\delta\le m\le(p-1)q+2gp} \Phi(t,m) &= \min_{(p-1)q-p\le m\le(p-1)q+2gp} \Phi^{p-1}(t,m) \\ = \min_{m\in I_{p-1}} \Phi^{p-1}(t,m)\end{aligned}$$

holds. Thus if $\mu_K \le s$, then we have

$$\Upsilon_{K_{p,q}}(t) = \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t).$$

□

5.3. **The variations Υ^{tr}_K, $\Upsilon^{\delta,i}_{p,q}$ $(i=1,2,3,4)$.** Recall the definitions of $\Upsilon^{\delta,i}_{p,q}$ $(i=1,2)$ in Section 1.5. Here we also define $\Upsilon^{\delta,3}_{p,q}(t)$ and $\Upsilon^{\delta,4}_{p,q}(t)$ for $0<\delta<p$ as follows. Let t be a real number with For $2i/p \le t \le 2(i+1)/p$ $0\le t\le 2$. Then we define $\Upsilon^{\delta,3}_{p,q}$ and $\Upsilon^{\delta,4}_{p,q}$ to be

$$\Upsilon^{\delta,3}_{p,q}(t)=\max_{iq-p<m\le iq-p+\delta}\widetilde{\Upsilon}_{T_{p,q}}(t,m),\ \Upsilon^{\delta,4}_{T_{p,q}}(t)=\max_{iq-p+\delta<m\le iq}\widetilde{\Upsilon}_{p,q}(t,m)$$

Here we prove properties of $\Upsilon^{tr}_K(s)$, $\Upsilon^{\delta,i}_{p,q}(t)$.

Lemma 14. *Let K be an L-space knot. Then,*

$$\Upsilon^{tr}_K(s)=\Upsilon^{tr}_K(2-s),\ \Upsilon^{\delta,3}_{p,q}(t)=\Upsilon^{\delta,1}_{p,q}(2-t),\textit{ and }\Upsilon^{\delta,4}_{p,q}(t)=\Upsilon^{\delta,2}_{p,q}(2-t)$$

hold.

Proof. By using the equality

$\varphi_K(2g-\nu)=g-\#(S_K\cap[2g-\nu,2g)=g-\#(\bar{S}_K\cap[0,\nu)=g-\nu+\varphi_K(\nu)$,

we have

$$\begin{aligned}\Upsilon^{tr}_K(s) &= \max_{\nu=1,2,\cdots,2g-1}\widetilde{\Upsilon}_K(s,\nu)=\max_{\nu=1,2,\cdots,2g-1}\widetilde{\Upsilon}_K(s,2g-\nu)\\ &= -2\min_{\nu=1,2,\cdots,2g-1}(g-\nu+\varphi_K(\nu)-s(2g-\nu)/2)-sg\\ &= -2\min_{\nu=1,2,\cdots,2g-1}(\varphi_K(\nu)-(2-s)\nu/2)-(2-s)g=\Upsilon^{tr}_K(2-s).\end{aligned}$$

We assume that $2i+s=pt$ and $0\le s\le 2$.

$$\begin{aligned}&\Upsilon^{\delta,3}_{p,q}(t)=-2\min_{iq-p<m\le iq-p+\delta}\Phi_{p,q}(t,m)-tg_{p,q}\\ =&\ -2\min_{iq-p<m\le iq-p+\delta}(\varphi_{T_{p,q}}(m)-tm/2)-tg_{p,q}\\ =&\ -2\min_{(p-1-i)q-\delta<m\le(p-1-i)q}(\varphi_{T_{p,q}}(2g_{p,q}-m)-t(2g_{p,q}-m)/2)-tg_{p,q}\\ =&\ -2\min_{(p-1-i)q-\delta<m\le(p-1-i)q}\big(g_{p,q}-m+\varphi_{T_{p,q}}(m)-t(2g_{p,q}-m)/2\big)-tg_{p,q}\\ =&\ -2\min_{(p-1-i)q-\delta<m\le(p-1-i)q}(\varphi_{T_{p,q}}(m)-(2-t)m/2)-(2-t)g_{p,q}.\\ =&\ \Upsilon^{\delta,1}_{p,q}(2-t)\end{aligned}$$

In the same way, we have

$$\Upsilon^{\delta,4}_{p,q}(t)=\Upsilon^{\delta,2}_{p,q}(2-t).$$

□

5.4. **Theorem 5.** Here we give a proof of Theorem 5.

Proof. Suppose that $0<i<p-1$ and $2i/p\le t\le 2(i+1)/p$. By applying the equalities (11) and (12) in the case of $2gp\le q$ we obtain the below computation.

We suppose $0<s<\mu_K$. We consider the minimal value of $(i+1)q-\delta<m\le(i+1)q$. Since $s<2-\mu_K$, if $1\le\nu\le 2g-1$ and $(i+1)q-\delta<m\le(i+1)q-\delta+p$ then there exists $1\le\nu\le 2g-1$ such that

$$\Phi^i(t,m)-\Phi^i(t,m-\nu p)=\#(\bar{S}_K\cap[0,\nu))-\nu s/2=-\Phi(2-s,\nu)>0.$$

Then

$$\begin{aligned}
&\min_{iq-\delta<m\leq(i+1)q}\Phi(t,m)=\min_{iq-\delta<m\leq(i+1)q-\delta}\Phi^i(t,m)\\
=\ &\min\left\{\min_{\nu=0,1,\cdots,2g-1}\left(\min_{iq-\delta+\nu p<m\leq iq+\nu p}\Phi^i(t,m)\right),\right.\\
&\left.\min_{\nu=1,2,\cdots,2g-1}\left(\min_{iq+(\nu-1)p<m\leq iq-\delta+\nu p}\Phi^i(t,m)\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\min_{\nu=0,1,\cdots,2g-1}\left(\min_{iq-\delta+\nu p<m\leq iq+\nu p}\Phi^i(t,m)\right) &= \min_{\nu=0,1,\cdots,2g-1}\left(\Phi_K(s,\nu)+\mu_i^1(t)\right)\\
&= \min_{\nu=0,1,\cdots,2g}\Phi_K(s,\nu)+\mu_i^1(t),
\end{aligned}$$

and

$$\begin{aligned}
\min_{\nu=1,2,\cdots,2g-1}\left(\min_{iq+(\nu-1)p<m\leq iq-\delta+\nu p}\Phi^i(t,m)\right) &= \min_{\nu=1,\cdots,2g-1}(\Phi_K(s,\nu)+\mu_i^2(t))\\
&= \min_{\nu=1,2,\cdots,2g-1}\Phi_K(s,\nu)+\mu_i^2(t).
\end{aligned}$$

Here $\mu_i^1(t)$ $\mu_i^2(t)$ are the minimal values of $\Phi^i(t,m)$ over $iq-\delta<m\leq iq$ and $iq-p<m\leq iq-\delta$ respectively.

$$\begin{aligned}
-2\mu_i^1(t) &= -2\min_{iq-\delta<m\leq iq}\Phi^i(t,m)=-2\min_{iq-\delta<m\leq iq}\Phi_{p,q}(t,m)+2ig\\
&= \Upsilon_{p,q}^{\delta,1}(t)+2ig+tg_{p,q}
\end{aligned}$$

$$\begin{aligned}
-2\mu_i^2(t) &= -2\min_{iq-p<m\leq iq-\delta}\Phi^i(t,m)=-2\min_{iq-p<m\leq iq-\delta}\Phi_{p,q}(t,m)+2ig\\
&= \Upsilon_{p,q}^{\delta,2}(t)+2ig+tg_{p,q}
\end{aligned}$$

$$\begin{aligned}
&\Upsilon_{K_{p,q}}(t)=-2\min_{iq-\delta<m\leq(i+1)q}\Phi(t,m)-tg_{K_{p,q}}\\
=\ &\max\left\{-2\min_{\nu=0,1\cdots,2g}\Phi_K(s,\nu)-tg_{K_{p,q}}-2\mu_i^1(t),\right.\\
&\left.-2\min_{\nu=1,2,\cdots,2g-1}\Phi_K(s,\nu)-tg_{K_{p,q}}-2\mu_i^2(t)\right\}\\
=\ &\max\left\{\max_{\nu=0,1\cdots,2g}\widetilde{\Upsilon}_K(s,\nu)+sg-t(pg+g_{p,q})+\Upsilon_{p,q}^{\delta,1}(t)+2ig+tg_{p,q},\right.\\
&\left.\min_{\nu=1,2,\cdots,2g-1}\widetilde{\Upsilon}_K(s,\nu)+sg-t(pg+g_{p,q})+\Upsilon_{p,q}^{\delta,2}(t)+2ig+tg_{p,q}\right\}\\
=\ &\max\left\{\Upsilon_K(s)+\Upsilon_{p,q}^{\delta,1}(t),\Upsilon_K^{tr}(s)+\Upsilon_{p,q}^{\delta,2}(t)\right\}
\end{aligned}$$

We suppose $2-\mu_k<s<2$. Then we have

$$\min_{iq-\delta<m\leq(i+1)q}\Phi(t,m)=\min_{iq<m\leq(i+1)q}\Phi^i(t,m).$$
$$=\min\left\{\min_{i=1,2,\cdots 2g}\left(\min_{iq+(\nu-1)p<m\leq iq+(\nu-1)p+\delta}\Phi^i(t,m)\right),\right.$$
$$\left.\min_{\nu=1,2,\cdots,2g-1}\left(\min_{iq+(\nu-1)p+\delta<m\leq iq+\nu p}\Phi^i(t,m)\right)\right\},$$

$$\begin{aligned}&\min_{\nu=1,2,\cdots,2g}\left(\min_{iq+(\nu-1)p<m\leq iq+(\nu-1)p+\delta}\Phi^i(t,m)\right)\\ =\ &\min_{\nu=1,2,\cdots,2g}\left(\Phi_K(s,\nu)+\mu_i^3(t)\right)=\min_{\nu=0,1,\cdots,2g}\Phi_K(s,\nu)+\mu_i^3(t),\end{aligned}$$

and

$$\begin{aligned}&\min_{i=1,2,\cdots 2g-1}\left(\min_{iq+(\nu-1)p+\delta<m\leq iq+\nu p}\Phi^i(t,m)\right)\\ =\ &\min_{\nu=1,2,\cdots,2g-1}\left(\Phi_K(s,\nu)+\mu_i^4(t)\right)=\min_{\nu=1,2,\cdots,2g-1}\Phi_K(s,\nu)+\mu_i^4(t).\end{aligned}$$

Here $\mu_i^3(t)$ and $\mu_i^4(t)$ are the minimal values of $\Phi^i(t,m)$ over $iq-p<m\leq iq-p+\delta$ and $iq-p+\delta<m\leq iq$ respectively.

Here we compute $-2\mu_i^3(t)$ and $-2\mu_i^4(t)$.

$$\begin{aligned}-2\mu_i^3(t)&=-2\min_{iq-p<m\leq iq-p+\delta}\Phi^i(t,m)=-2\min_{iq-p<m\leq iq-p+\delta}\Phi_{p,q}(t,m)+2ig\\ &=\Upsilon_{p,q}^{\delta,3}(t)+2ig+tg_{p,q}\end{aligned}$$

$$\begin{aligned}-2\mu_i^4(t)&=-2\min_{iq-p<m\leq iq}\Phi^i(t,m)=-2\min_{iq-p+\delta<m\leq iq}\Phi_{p,q}(t,m)+2ig\\ &=\Upsilon_{p,q}^{\delta,4}(t)+2ig+tg_{p,q}\end{aligned}$$

Applying these terms to $\Upsilon_{K_{p,q}}(t)$, we get the following:

$$\begin{aligned}\Upsilon_{K_{p,q}}(t)&=\max\left\{\max_{\nu=0,1,\cdots,2g}\widetilde{\Upsilon}_K(s,\nu)+sg-t(pg+g_{p,q})-2\mu_i^3(t)\right.\\ &\qquad\left.\max_{\nu=1,2,\cdots,2g-1}\widetilde{\Upsilon}_K(s,\nu)+sg-t(pg+g_{p,q})-2\mu_i^4(t)\right\}\\ &=\max\left\{\Upsilon_K(s)+sg-tpg+\Upsilon_{p,q}^{\delta,3}(t)+2ig\right.\\ &\qquad\left.\Upsilon_K^{tr}(s)+sg-tpg+\Upsilon_{p,q}^{\delta,4}(t)+2ig\right\}\\ &=\max\left\{\Upsilon_K(s)+\Upsilon_{p,q}^{\delta,3}(t),\Upsilon_K^{tr}(s)+\Upsilon_{p,q}^{\delta,4}(t)\right\}.\end{aligned}$$

By using Lemma 14, we get the required formulas. □

Here we prove a corollary stated in Section 1.

Proof of Corollary 6. If $0<s<\mu_K$ or $2-\mu_K<s<2$ hold, then for all $\nu \in \{1, 2, \cdots, 2g-1\}$ $\Phi_K(s,\nu) > 0 = \Phi_K(s,0) = \Phi(K,2)$ holds. Thus $\Upsilon^{tr}_K(s) < \Upsilon_K(s)$ holds. Therefore, we have

$$\begin{aligned}\Upsilon_{K_{p,q}}(t) &\leq \max\left\{\Upsilon_K(s) + \Upsilon^{\delta,1}_{p,q}(t), \Upsilon_K(s) + \Upsilon^{\delta,2}_{p,q}(t)\right\} \\ &= \Upsilon_K(s) + \max\left\{\Upsilon^{\delta,1}_{p,q}(t), \Upsilon^{\delta,2}_{p,q}(t)\right\} = \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t)\end{aligned}$$

Since for any $1 < \nu < 2g-1$, the inequality $\Phi^i(t,m) < \Phi^i(t, m+\nu p)$ for $iq - p < m \leq iq - \delta$ holds. This means that $\Upsilon_{K_{p,q}}(t) \leq \Upsilon_K(s) + \Upsilon_{T_{p,q}}(t)$. □

6. The example $(T_{3,7})_{3,35}$.

6.1. **Computation of $\Upsilon_{(T_{3,7})_{3,35}}$.** We come back to the example $K_{3,35}$ observed in the previous section again, where $K = T_{3,7}$. We apply the cabling formula in Theorem 5 to this example. The genera are computed as $g = 6$ and $g_{3,35} = 34$. Then $S_K = \{0, 3, 6, 7, 9, 10\} \cup \mathbb{Z}_{n \geq 12}$ holds. When $\nu = 0, 1, 2, \cdots, 12$, the sequence $\varphi_K(\nu)$ is as follows:

$$\varphi_K(\nu) : 0, 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 6, 6.$$

Hence, we have $\mu_K = 2/3$ and $\delta = 2$.

First, we consider $2/3 < t < 4/3$, namely this corresponds to the case of $i = 1$ in Theorem 5. Furthermore we assume $2 + s = 3t$ and $0 < s < 2/3$. This means $2/3 < t < 8/9$. Then we have

$$\widetilde{\Upsilon}_{T_{3,35}}(t,m) = -2\varphi_{T_{3,35}}(m) - (34-m)t.$$

Since $\varphi_{T_{3,35}}(34) = \varphi_{T_{3,35}}(35) = 12$, we have

$$\Upsilon^{\delta,1}_{3,35}(t) = \max_{33<m\leq 35} \widetilde{\Upsilon}_{T_{3,35}}(t,m) = \max\{-24, -24+t\} = -24 + t.$$

Since $\varphi_{T_{3,35}}(33) = 11$, we have

$$\Upsilon^{\delta,2}_{3,35}(t) = \max_{32<m\leq 33} \widetilde{\Upsilon}_{T_{3,35}}(t,m) = -22 - t.$$

If $0 < s < 2/3$, then we have

$$\begin{aligned}\Upsilon^{tr}_K(s) &= \max_{\nu\in\{1,\cdots,11\}} \{-2\varphi_K(\nu) - (6-\nu)s\} \\ &= \max_{\nu\in\{3,6,9\}} \{-2\varphi_K(\nu) - (6-\nu)s\} \qquad (13) \\ &= -2\varphi_K(3) - (6-3)s = -2 - 3s\end{aligned}$$

and while we have $\Upsilon_K(s) = -6s$.

Here we explain the second equality (13). We consider several candidates of functions which give the maximal in $\{-2\varphi_K(\nu) - (6-\nu)s | \nu = 1, 2, \cdots 11\}$. During the set of $N_i := \{s \in \{0, 1, \cdots, 11\} | \varphi_K(s) = i\}$ for $i \in \mathbb{N}$ the maximal function $-2\varphi_K(\nu) - (6-\nu)s$ is the one of the maximal ν in N_i. This coincides with $S_K \cap [1, 11] = \{3, 6, 7, 9, 10\}$.

Suppose that $\varphi_K(\nu-1) < \varphi_K(\nu)$. Then since $-2\varphi_K(\nu-1) - (g-\nu+1)s > -2\varphi_K(\nu) - (g-\nu)s$ for any $0 < s < 2$. The function for such $\nu \in S_K$ is not a candidate of the maximal function. Thus we have only to consider $\{3,6,7,9,10\} - \{4,7,8,10,11\} = \{3,6,9\}$.

As a result, we have

$$\Upsilon^{\delta,1}_{3,35}(t) + \Upsilon_K(s) = -12 - 17t$$

and

$$\Upsilon^{\delta,2}_{3,35}(t) + \Upsilon^{tr}_K(s) = -18 - 10t.$$

Hence, when $2/3 < t < 8/9$, the $\Upsilon_{K_{p,q}}(t)$ is the following:

$$\begin{aligned}\Upsilon_{K_{3,35}}(t) &= \max\{-12-17t, -18-10t\} \\ &= \begin{cases} -12-17t & 2/3 < t < 6/7 \\ -18-10t & 6/7 \leq t < 8/9. \end{cases}\end{aligned}$$

Secondly, in $4/9 < t < 2/3$, applying (5) in Theorem 5, we compute $\Upsilon_{K_{3,35}}(t)$ as follows:

$$\begin{aligned}\Upsilon_{K_{3,35}}(t) &= \max\{-35t + (-12+18t), -34t + (-8+9t)\} \\ &= \max\{-12-17t, -8-25t\} \\ &= \begin{cases} -8-25t & 4/9 < t < 1/2 \\ -12-17t & 1/2 \leq t < 2/3. \end{cases}\end{aligned}$$

7. Toward a further cabling formula

Let K be an L-space knot. When integers p, q satisfy $q < (2g(K)-1)p$, the cable knot $K_{p,q}$ is not an L-space knot. In this case, to compute the $\Upsilon_{K_{p,q}}$, we require the different formula. For example, consider the family $\Upsilon_{K_{2,q}}$ for $K = T_{2,3}$ and $q \in 2\mathbb{Z}+1$. Then the paper can give the following equalities

$$\Upsilon_{K_{2,2n+1}}(t) = \Upsilon_K(2t) + \Upsilon_{T_{2,2n+1}}(t) \quad (n > 1).$$

Furthermore, since we have $\Delta_{K_{2,3}}(t) = \Delta_{T_{3,4}}(t)$, we obtain

$$\Upsilon_{K_{2,3}}(t) = \Upsilon_{T_{3,4}}(t).$$

Furthermore, we obtain $\Upsilon_{K_{2,1}}(t)$ as the graph in Figure 8. This is due to Hedden's formula in [6]. This graph coincides with

$$\Upsilon_{K_{2,1}}(t) = \Upsilon_{T_{3,4}}(t) - \Upsilon_K(t),$$

because $K_{2,1}$ is ν^+-equivalent to $T_{3,4}\#(-K)$. These equalities can be gener-

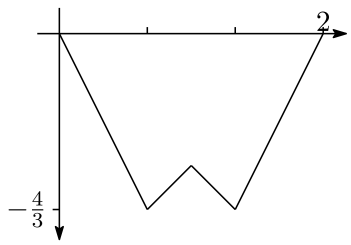

Figure 8. $\Upsilon_{(T_{2,3})_{2,1}}$

alized in other cases of cable knots of torus knots. For example, for $K = T_{2,5}$ and $g = 2$ we have

$$\Upsilon_{K_{2,2n+5}}(t) = \Upsilon_K(2t) + \Upsilon_{T_{2,2n+5}}(t) \quad (n > 1)$$

However, $\Upsilon_{K_{2,7}}(t)$ does not equal to the Υ-invariant of either of any torus knot or L-space cable knot of torus knot but $K_{2,7}$.

Here we raise the following question.

Question 15. *Let K be an L-space knot. Suppose that the integers q, Q satisfy $q < (2g(K) - 1)p < Q$. Does there exist the method to compute the $\Upsilon_{K_{p,q}}(t)$ by using $\Upsilon_{K_{p,Q}}(t)$ and so on?*

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