NOTES ON GOMPF'S INFINITE ORDER CORKS

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Abstract. For any positive integer n we give a $\mathbb{Z}^n$-cork with a $\mathbb{Z}^n$-effective embedding in a 4-manifold being homeomorphic to $E(n)$. This means that a cork gives a subset $\mathbb{Z}^n$ in the differential structures on $E(n)$. Further, we describe handle decompositions of the twisted doubles (homotopy S^4) of Gompf's infinite order corks and show that they are Gluck twists and log transforms of S^4.

1. Introduction

1.1. **Twist and cork.** Let X be a smooth manifold and Y a codimension 0 submanifold with a smooth embedding $i : Y \hookrightarrow X$. Removing Y from X and regluing by a self-diffeomorphism $f : \partial Y \to \partial Y$, we obtain a new smooth manifold and denote the manifold by $X(i, Y, f)$ or simply $X(Y, f)$. We call the map f a *gluing map* or a *twist map.* This operation is called a *twist* and denote it by (Y, f). If the gluing map f extends to Y as a diffeomorphism, then we call (Y, f) a *trivial twist.*

Let Y be a contractible 4-manifold. We call a nontrivial twist (Y, f) a *cork.* Then the gluing map f is called a *cork map.* Corks play significant roles in studying exotic 4-manifolds. 'Exotic' means that the manifolds are homeomorphic but non-diffeomorphic each other. In fact, the following fact is well-known.

Fact 1.1 ([7], [2])**.** *Let X, X' be two closed simply connected exotic 4-manifolds. Then there exists a cork (C, τ) such that $X' = X(C, \tau)$ and $\tau^2 = e$.*

1.2. **Gompf's infinite order corks.** Gompf in [4] gave an infinite exotic family using infinite order corks as below. Let X be a certain 4-manifold (including a square zero torus with two vanishing cycles).

Fact 1.2 ([4])**.** *Suppose that K_n is the $2n$-twist knot. Then there exists an infinite order cork (C, f) satisfying $X_{K_n} = X(C, f^n)$.*

X_K is the (Fintushel-Stern) knot-surgery of X by K. The *cork order* of a cork is defined to be the minimal positive number whose power of the twist map is a trivial twist. Higher order corks are known to exist by [10] and [1].

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Definition 1.3 (G-effective embedding (defined in [1]))**.** *Let $\mathcal{C}$ be a 4-manifold with boundary. Let G be a group acting on $\partial\mathcal{C}$ effectively. If there exists an embedding i of $\mathcal{C}$ into a 4-manifold X such that $X(i,\mathcal{C},g)$ is not diffeomorphic to $X(i,\mathcal{C},g')$ for any distinct $g,g'\in G$, then we call the embedding i a G-effective.*

1.3. **Galaxy.** Here we give terminologies related to cork and cork twist. These terminologies make it easier to understand our results. Let X be a smooth 4-manifold. We call the set of exotic structures on X *the galaxy* of X and denote it by

$$\mathrm{gal}(X).$$

Our interest is to understand some kind of structures on the set $\mathrm{gal}(X)$. Any cork twist can be regarded as some relationship among subsets of $\mathrm{gal}(X)$.

Let C be a contractible 4-manifold. Let G be a nontrivial subgroup in $\mathrm{Diff}(\partial C)$. Let (C,G) be a G-cork and $C\hookrightarrow X$ a G-effective embedding. Then the collection $S=\{X(C,g)|g\in G\}$ is a subset of $\mathrm{gal}(X)$ with one to one correspondence $g\mapsto X(C,g)$. We call such a subset S a *(G-)constellation* and the embedding $G\overset{\cong}{\to} S\subset \mathrm{gal}(X)$ a *constellation embedding*.

Fact 1.1 means that a pair of every two points in $\mathrm{gal}(X)$ is a $\mathbb{Z}_2$-constellation. If $G\hookrightarrow \mathrm{gal}(X)$ is a constellation with respect to a G-cork (C,G), then any subgroup $e\neq H<G$ gives an H-constellation $H\hookrightarrow \mathrm{gal}(X)$ with respect to an H-cork (C,H). We call this constellation *a subconstellation*. The main theorem in [11] says that any infinite family in $\mathrm{gal}(X)$ is not always a constellation.

1.4. **Results.** In this subsecion, we digest the results (in Section 1.5 and 1.6) obtained in this article.

The first result (Theorem 1) gives a construction of $\mathbb{Z}^n$-cork. Furthermore, we show that this cork gives a $\mathbb{Z}^n$-constellation in $\mathrm{gal}(E(n))$ by knot-surgeries on a single fibered knot. This construction is due to an n-fold boundary sum of Gompf's C.

The twisted double $\mathbb{S}_{r,s,m,k}$ of Gompf's infinite order corks (C,f) is given by Gompf in [4]. We investigate the diffeomorphism type of $\mathbb{S}_{r,s,m,k}$. We prove $\mathbb{S}_{r,s,m,k}\cong\mathbb{S}_{r,s,m+2,k}$. We prove $\mathbb{S}_{r,s,0,k}$ is diffeomorphic to the standard S^4 and $\mathbb{S}_{r,s,1,k}$ is the Gluck twists of S^4 along a 2-knot in S^4. These are the second result (Theorem 2). The third result (Theorem 3) is a log transform construction of $\mathbb{S}_{r,s,m,k}$. As a result, $\mathbb{S}_{r,s,m,k}$ is a $(1/s)$-log transform (or $(-1/r)$-log transform) along a torus in S^4. As a result, $\mathbb{S}_{r,s,m,k}$ is a homotopy S^4 having three kinds of constructions: a cork twist, a Gluck twist and a log transform.

1.5. $\mathbb{Z}^n$**-corks.** In [4] Gompf defined infinite order corks (C,f) and asked in [4] whether there exists a $\mathbb{Z}^2$-cork by taking the full T^2 action of his corks. In [5] he partially gave a negative answer for this question. We construct a $\mathbb{Z}^n$-cork below, but it is not an answer of this question.

Theorem 1. *For any natural number n there exists a $\mathbb{Z}^n$-cork C_n. Furthermore, there exist a 4-manifold X_n (homeomorphic to $E(n)$) and $\mathbb{Z}^n$-effective embedding $C_n \hookrightarrow X_n$. This $\mathbb{Z}^n$-effective embedding gives a $\mathbb{Z}^n$-constellation $\mathbb{Z}^n \hookrightarrow \mathrm{gal}(E(n))$.*

Here C_n is the boundary sum of n copies of $C(1,1;-1)$ as defined in [4]. This construction is due to performing cork twisting at distinct two clasps as mentioned by Gompf in [4]. Here we give the following interesting questions:

Question 1.4. *Is there a 4-manifold X such that $\mathrm{gal}(X)$ includes a $\mathbb{Z}^n$-constellation for every natural number n?*

Question 1.5. *Does there exist a 4-manifold X with a G-constellation in $\mathrm{gal}(X)$ for an infinite non-abelian group?*

Related topics to this question will be written in a sequel.

1.6. **The twisted double of Gompf's (C, f).** Let C denote $C(r, s; m)$. The double $D(C) = C \cup_{\mathrm{id}} (-C)$ is the boundary of $C \times I = B$. Since, C is a Mazur type (consisting of one 1-handle and one 2-handle) contractible 4-manifold, B is diffeomorphic to the 5-ball. Because, the attaching sphere of the 5-dimensional 2-handle is S^1 in 4-space, it depends only on the homotopy type of S^1. Thus, the two handles are a canceling pair. In particular, the double $D(C)$ is diffeomorphic to S^4.

The twisted double $C \cup_{f^k} (-C) = S^4(C, f^k) =: \mathbb{S}_{r,s,m,k}$ are all homotopy 4-spheres. As proven above, since the untwisted double $\mathbb{S}_{r,s,m,0}$ is S^4. The problem that we concern is the diffeomorphism type of a general $\mathbb{S}_{r,s,m,k}$.

Conjecture 1.6 ([4])**.** *Let r, s, m be any integers with $r, s > 0 > m$. Let k be a nonzero integer. Then $\mathbb{S}_{r,s,m,k}$ is standard S^4.*

We will prove the following theorem on $\mathbb{S}_{r,s,m,k}$ in Section 3.

Theorem 2. *For any integer m, $\mathbb{S}_{r,s,m,k} \cong \mathbb{S}_{r,s,m+2,k}$ holds. If $m = 0$, then $\mathbb{S}_{r,s,0,k}$ is diffeomorphic to S^4. If $m = 1$, then $\mathbb{S}_{r,s,1,k}$ is the Gluck twist along a 2-knot $K_{r,s,k}$ in S^4*

We give another aspect of $\mathbb{S}_{r,s,m,k}$.

Theorem 3. *$\mathbb{S}_{r,s,m,k}$ is a $(1/s)$-log transform of S^4 along an embedded torus.*

Note that exchanging the roles of r and s, we also know that $\mathbb{S}_{r,s,m,k}$ is $(-1/r)$-log transform of S^4. From the proof of Theorem 3 we get a handle decomposition of $\mathbb{S}_{r,s,m,k}$.

Proposition 1.7. *$\mathbb{S}_{r,s,1,k}$ admits a handle decomposition with one 0-handle, two 1-handles, four 2-handles, two 3-handles, and one 4-handle.*

If the embedding of the torus in S^4 extends to a fishtail neighborhood, then the log transform does not change the diffeomorphism type as discussed in [5]. Other similar situations appear in [12] and [13]. In [13], it is shown that the knot-surgery of S^4 along a torus is trivial by using a fishtail neighborhood embedding in S^4. It, however, seems difficult to find a certain fishtail neighborhood in $\mathbb{S}_{r,s,m,k}$. Distinguishing the differential structures of $\mathbb{S}_{r,s,m,k}$ might be a hard work.

$\mathbb{S}_{r,s,m,k}$ is also considered as a 4-manifold obtained by log transforms along two tori in S^4 as written in Section 3.1. The tori have a symmetry, then we have $\mathbb{S}_{r,s,m,k} \cong \mathbb{S}_{-s,-r,m,k}$ due to the handle diagram in FIGURE 10.

Furthermore, the 0-log transform and $(0,0)$-log transform of S^4 with respect to the torus and the tori are homotopy $S^3 \times S^1 \# S^2 \times S^2$ and $\#^2 S^3 \times S^1 \#^2 S^2 \times S^2$ respectively. By the similar argument to the proof of Theorem 2, if m is even, then $\mathbb{S}_{\infty,s,m,k} \cong S^3 \times S^1 \# S^2 \times S^2$ and $\mathbb{S}_{\infty,\infty,m,k} \cong \#^2 S^3 \times S^1 \#^2 S^2 \times S^2$ hold. In the case where m is odd, $\mathbb{S}_{\infty,s,m,k}$ and $\mathbb{S}_{\infty,\infty,m,k}$ are the Gluck twists along the standard 4-manifolds.

Question 1.8. *Are $S^4[0]$ and $S^4[0,0]$ diffeomorphic to the standard $(S^3 \times S^1)\#(S^2 \times S^2)$ or $\#^2(S^3 \times S^1)\#^2(S^2 \times S^2)$ respectively?*

A similar construction is a *Scharlemann manifold* in [8], which is a surgery of $\Sigma \times S^1$ for a rational homology sphere Σ. The surgeries are done along normally generating loops of $\pi_1(\Sigma)$ in $\Sigma \times S^1$. In the case where Σ is a Dehn surgery of a knot and the loop is the meridian of the knot, the manifold is equivalent to a knot-surgery of the double of the fishtail neighborhood. For example see [13]. The general Scharlemann manifold gives a homotopy

$$\#(S^3 \times S^1)\#^l(S^2 \times S^2)\#^m(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}).$$

The case where $\Sigma = \Sigma(2,3,5) = S^3_{-1}(\text{left handed } 3_1)$ and the loop is the meridian of the trefoil is the original one in [8]. The author in [13] proved that some Scharlemann manifolds are standard. Here we give another question.

Question 1.9. *Are $S^4[0]$ and $S^4[0,0]$ diffeomorphic to a Scharlemann manifold and a connected-sum of those manifolds respectively?*

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2. GOMPF'S INFINITE ORDER CORK C.

2.1. **Knot-surgery.** Before drawing Gompf's infinite order corks, we review the definition of Fintushel-Stern knot-surgery. Let K be a knot in S^3. Let X be a 4-manifold with a square zero embedded torus T. Then the operation

$$X_K = [X - \nu(T)] \cup [(S^3 - \nu(K)) \times S^1]$$

is called a *(Fintushel-Stern) knot-surgery* along K. The gluing map is defined in [3]. The notation $\nu(\cdot)$ stands for a tubular neighborhood of a submanifold. Then the Seiberg-Witten invariant formula of knot-surgery is the following:

$$SW_{X_K} = SW_X \cdot \Delta_K(t),$$

where $\Delta_K(t)$ is the Alexander polynomial of K.

We remark Sunukjian's work in [9]. The paper says that if a 4-manifold X has a non-trivial Seiberg-Witten invariant, then two knot-surgeries X_{K_1} and X_{K_2} along knots K_1 and K_2 with $\Delta_{K_1} \neq \Delta_{K_2}$ cannot be diffeomorphic. In short, Alexander polynomial distinguishes smooth structures of knot surgeries.

2.2. **A handle diagram of C.** In this section we describe the handle diagram of Gompf's infinite order cork C. We call a handle diagram a *diagram* simply. In [6] the diagram is described by the different method. The manifold $C = C(r, s; m)$ can be built by attaching a single 2-handle to $I \times P$, where P is the complement of (r, s)-double twist knot $\kappa(r, -s)$ as in Figure 1 in [4].

Lemma 2.1. *The diagram of C is* Figure *1 and the cork map f is* Figure *2.*

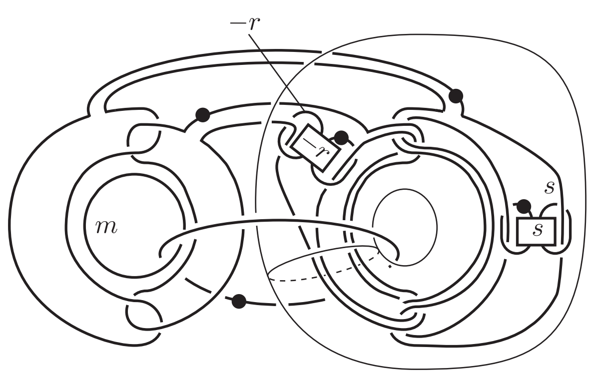

Figure 1. The handle decomposition of $C(r, s; m)$.

The boundary of Gompf's contractible 4-manifold C has a torus decomposition along an incompressible torus. The incompressible torus can be realized as a torus indicated in Figure 1. Note that in the diagram in Figure 1 the torus apparently cannot be realized as an embedded torus, because the torus meets the dotted 1-handles 4 times. However, by inserting 2 pairs of canceling 2/3-handles, and putting the torus over the 2-handle, we can avoid the intersections. As a result we can find our required embedded torus.

Proof. Let Σ be a punctured torus. $\Sigma \times S^1$ is diffeomorphic to the 0-surgered solid torus along the Bing double as in the left of Figure 3.

Thus by the fundamental method of handle calculus, the picture of the cylinder $I \times \Sigma \times S^1$ is the right diagram in Figure 3. Attaching a $-r$-framed

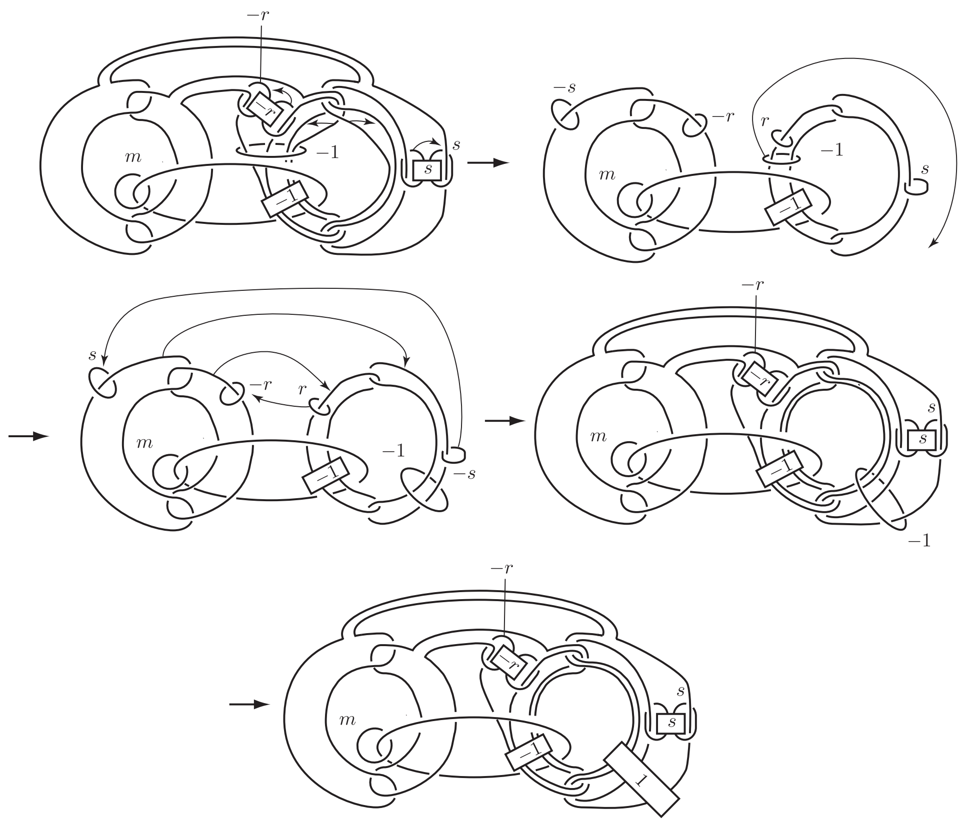

FIGURE 2. The diffeomorphism f on $\partial C(r, s; m)$.

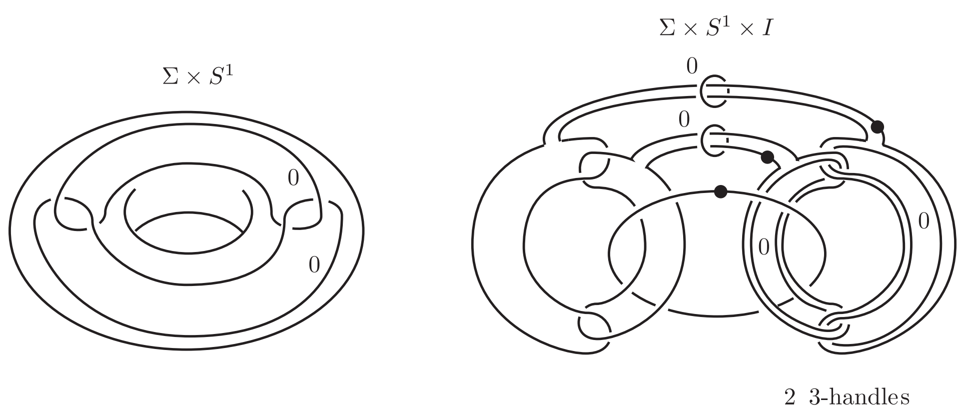

FIGURE 3. $\Sigma \times S^1$ and $\Sigma \times S^1 \times I$.

2-handle (an s-framed 2-handle on another side respectively) and removing the union of the core and the attaching sphere cross interval I, we obtain the other handle attachment and removal in FIGURE 1. The two 0-framed

2-handles in the right of FIGURE 3 are canceled out with the 3-handles when removing the cores. The similar situation appears in [12].

Hence, the handle diagram of C is as in the picture in FIGURE 1. Reducing the diagram, we get a ribbon 1-handle and m-framed 2-handle along the meridian of the ribbon knot. □

The map f is defined to be the right handed Dehn twist cross the identity on

$$(I \times \partial\Sigma) \times S^1.$$

This cork C produces a knot-surgery X_K for a 4-manifold X for a square zero torus T. The torus satisfies the following condition.

We assume $r = s = -m = 1$. Let V be the neighborhood of the Kodaira's singular fibration of type III, which V is also used in [14] in the same situation. Suppose that V is embedded in X. T is embedded as a general fiber of $V \subset X$. Then there exists an embedding $C \subset V \subset X$ such that

$$X_K = X(C, f^k),$$

where $K = \kappa(k, -1)$ is the $2k$-twist knot.

If C does not satisfy $r = s = -m = 1$, we can construct the knot surgery by the twist knot on an embedded torus under certain condition as mentioned in [4].

Due to [4], for any integer k the k-fold composition f^k cannot extend to the inside C as any diffeomorphism.

We describe in FIGURE 4 the local deformation of $X(C, f^2)$ according to the diffeomorphism f in FIGURE 2.

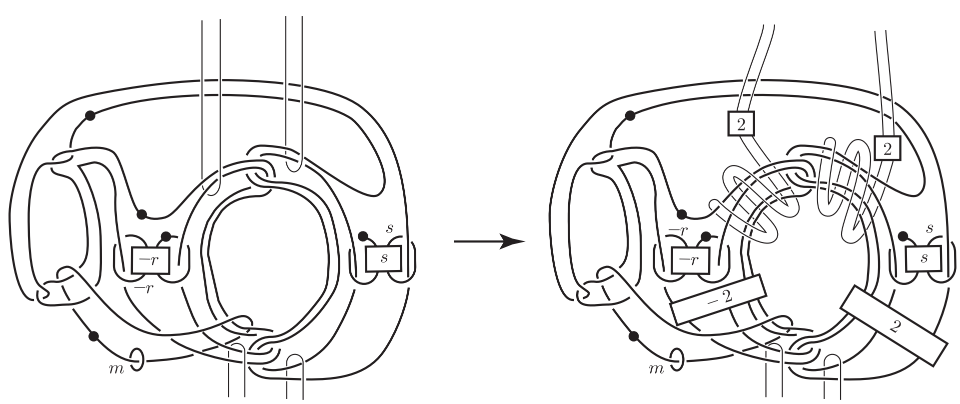

FIGURE 4. The cut and paste of $C(r, s; m)$ by f^2.

2.3. **2-bridge knots $K_{m,n}$.** In the next section we prove Theorem 1. First we prepare a 2-bridge knot $K_{m,n}$ as in FIGURE 5 for integers m, n. The knot

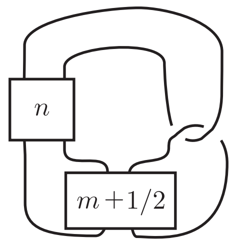

FIGURE 5. $K_{m,n}$.

$K_{m,n}$ is classified as follows:

$$K_{m,n}=\begin{cases}T_{2,2m-1} & n=0\\ \text{unknot} & (m,n)=(-1,-1)\\ T_{2,-3} & (m,n)=(0,1)\\ \text{non-torus 2-bridge knot} & \text{otherwise.}\end{cases} \tag{1}$$

The following equality holds

$$K_{-1,n}\approx K_{0,n+1}.$$

Here $T_{p,q}$ is the right handed $(|p|,|q|)$-torus knot if $pq>0$ and is the left handed $(|p|,|q|)$-torus knot if $pq<0$.

Let $\Delta_{m,n}$ denote the Alexander polynomial $\Delta_{K_{m,n}}$. The Alexander polynomial is computed as follows: If $m\geq 1$, then

$$\Delta_{m,n}(t)\doteq n(t^{m+1}+t^{-m-1})-3n(t^m+t^{-m})+(4n+1)\sum_{i=-m+1}^{m-1}(-1)^{i-m+1}t^i$$

and if $m=0$, then

$$\Delta_{0,n}(t)=n(t+t^{-1})-(2n-1).$$

This formula can be easily proven by using the skein relation of the Alexander polynomial.

We prove the following lemma:

Lemma 2.2. *If $m\geq 1$, then the polynomials $\Delta_{m,n}(t)$ are distinct each other in $\mathbb{Z}[t,t^{-1}]/\pm t^{\pm 1}$. In particular, for $(m,n)\neq(m',n')$ with $m,m'\geq 1$, $K_{m,n}$ and $K_{m',n'}$ are non-isotopic as unoriented knots.*

Proof. Suppose that $\Delta_{m,n}=\Delta_{m',n'}$. If $n\neq 0$ and $n'\neq 0$ hold, then, comparing the coefficients of top degree terms, we have $n=n'$ and $m=m'$. If either n or n' is 0 and the other is not 0, then the two polynomials do not agree. Suppose that $n'\neq 0$ and $n=0$ hold. Then we have $\Delta_{m,n}(t)=\Delta_{T_{2,2m-1}}(t)$. Comparing the top degrees of $\Delta_{m,n}(t)$ and $\Delta_{m',n'}(t)$, we have $n'=\pm 1$. However, comparing the values of the second top degree of $\Delta_{m',n'}(t)$ we must have $n'=\pm 3$. This is contradiction. If $n=n'=0$, then, comparing the top degrees of the two polynomials, we have $m=m'$. $\square$

2.4. **Proof of Theorem 1.** We remark that the natural number k in this proof corresponds to n in the statement.

Let $K(n_1,\cdots,n_k)$ be $K_{1,n_1}\#\cdots\#K_{k,n_k}$. Hence $K(0,\cdots,0)=\#_{i=1}^k T_{2,2i-1}$ holds. We embed k copies of Σ in the exterior E of $K(0,\cdots,0)$ disjointly. In the case of 2 copies see FIGURE 6. Thus, disjoint k copies of $I\times\Sigma\times S^1$

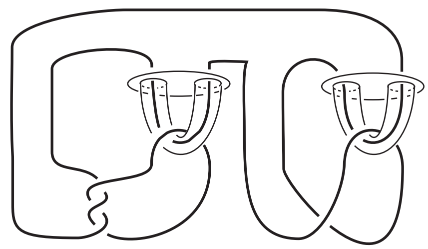

FIGURE 6. A disjoint embedding of two copies of Σ in E_2.

in $E\times S^1$ are also embedded. By attaching $2k$ (-1-framed) 2-handles on the meridians of $E\times S^1$ and k (-1-framed) 2-handles on the S^1-direction, we can embed k copies of $C=C(1,1;-1)$ in $X_k:=E(k)_{\#_{i=1}^k T_{2,2i-1}}$, because it has $12k$ vanishing cycles.

We take each point in the complements of the incompressible tori in $\partial C\cup\partial C$ and connect the two components by a 1-handle attached on the neighborhoods of the two points in X_k, which the 1-handle is disjoint from k C's. Embedding such $k-1$ 1-handles, we construct $\natural^k C\hookrightarrow X_k$.

Let f_i be a diffeomorphism on $\partial(\natural^k C)$ which acts as Gompf's f on the i-th component of $\#^k\partial C$ and acts as the identity on the other component of $\#^k\partial C$. Since those points are taken in the complement of the incompressible tori, the two maps f_i and f_j are commutative. The twist $(\natural^k C, f_1^{n_1}\cdots f_k^{n_k})$ of X_k produces $E(k)_{K(n_1,\cdots,n_k)}$.

Then, the computation of the Seiberg-Witten invariants is as follows:

$$SW_{E(k)_{K(n_1,\cdots,n_k)}}=SW_{E(k)}\prod_{i=1}^k\Delta_{i,n_i}$$

$$SW_{X_k}=SW_{E(k)}\prod_{i=1}^k\Delta_{i,0}.$$

Comparing the degrees of the two results, the two Seiberg-Witten invariants do not agree, unless $n_i=0$ for all i. Since X_k and $E(k)_{K(n_1,\cdots,n_k)}$ are exotic when $(n_1,\cdots,n_k)\neq(0,\cdots,0)$. $(\natural^k C, f_1^{n_1}\cdots f_k^{n_k})$ gives an exotic $E(k)$. Thus, $(\natural^k C, f_1^{n_1}\cdots f_k^{n_k})$ is a cork. This means that $(\natural^k C,\{f_1^{n_1}\cdots f_k^{n_k}|n_j\in\mathbb{Z}\})$ is a $\mathbb{Z}^k$-cork.

To prove that this embedding is a $\mathbb{Z}^k$-effective embedding, we have only to show that if $\prod_{i=1}^k\Delta_{i,n_i}(t)=\prod_{i=1}^k\Delta_{i,n_i'}(t)$, then $(n_1,\cdots,n_k)=(n_1',\cdots,n_k')$.

Claim 2.3. *Let k, p be integers with $k > 0$ and $p \geq 0$ and $n_i, n'_i (i = 1, \cdots, k)$ integers. If we have*

$$\prod_{i=1}^{k-p} \Delta_{p+i, n_{p+i}} = \prod_{i=1}^{k-p} \Delta_{p+i, n'_{p+i}}, \tag{2}$$

then we have $n_{p+1} = n'_{p+1}$.

If $\Delta_{p,q}$ were irreducible, then this claim would be easy. However, since some 2-bridge knots are ribbon, such Alexander polynomials are not always irreducible.

Proof. By the induction of the number k in (2) we prove this claim. Let σ_i and σ'_i be the i-th elementary symmetric polynomials in $n_{p+1}, \cdots, n_k$ and $n'_{p+1}, \cdots, n'_k$ respectively. For example, we see

$$\sigma_i = \sum_{\{\ell_1, \cdots, \ell_i\} \subset \{p+1, \cdots, k\}, \#\{\ell_1, \cdots, \ell_i\} = i} n_{\ell_1} \cdots n_{\ell_i}.$$

Let d be the degree of (2). Comparing the degree d of (2), we obtain

$$\sigma_{k-p} = \sigma'_{k-p}.$$

Let j_0 be an integer with $1 \leq j_0 \leq p+1$. Comparing the coefficients of the degree $d - 2j$ of (2) with $0 \leq j \leq j_0$ we have $\sigma_{k-p-j_0} = \sigma'_{k-p-j_0}$.

Further, the coefficients with degree $d - 2p - 3$ of the left hand side of (2) is

$$\begin{aligned} & S + (-3n_{p+1}) \prod_{j=p+2}^{k-p} n_j + \sum_{j=p+2}^{k-p} (-4n_j - 1) \prod_{\substack{\ell=p+1 \\ \ell \neq j}}^{k} n_\ell \\ = \quad & S_0 - \sum_{j=p+2}^{k} \prod_{\substack{\ell=p+1 \\ \ell \neq j}}^{k} n_\ell = S_0 - \sigma_{k-p-1} + n_{p+2} \cdots n_k, \end{aligned}$$

where S, S_0 are polynomials generated by $\sigma_{k-p}, \sigma_{k-p-1} \cdots, \sigma_{k-p-j}$.

Thus $n_{p+2} \cdots n_k = n'_{p+2} \cdots n'_k$ holds. By using $\sigma_{k-p} = \sigma'_{k-p}$, we have $n_{p+1} = n'_{p+1}$. □

We go back to the proof of Theorem 1. Suppose that $\prod_{i=1}^k \Delta_{i,n_i} = \prod_{i=1}^k \Delta_{i,n'_i}$. Then, by using Claim 2.3 in the case of $p = 0$ we have $n_1 = n'_1$. By dividing Δ_{1,n_1} from the both side of this equality, we have $\prod_{i=2}^k \Delta_{i,n_i} = \prod_{i=2}^k \Delta_{i,n'_i}$. Iterating this process by using Claim 2.3, we have $n_2 = n'_2, \cdots, n_k = n'_k$ holds.

Thus $\prod_{i=1}^k \Delta_{i,n_i} = \prod_{i=1}^k \Delta_{i,n'_i}$ implies $(n_1, \cdots, n_k) = (n'_1, \cdots, n'_k)$. Therefore we show that this embedding

$$\natural^k C := C_k \hookrightarrow X_k = E(k)_{\#_{i=1}^k T_{2,2i-1}}$$

is $\mathbb{Z}^k$-effective. □

This proof focuses on the case of $C = C(1,1;-1)$. However, Gompf's method in Remarks (a) in [4] would change our examples to $C = C(r,s;m)$ with $r, s > 0 > m$.

3. The twisted double $\mathbb{S}_{r,s,m,k}$ of C.

3.1. **The diagrams of twisted doubles.** Let $\mathbb{S}_{r,s,m,k}$ denote the homotopy S^4 defined in Section 1.6. We prove the following proposition first of all.

Proposition 3.1. *In the case of $k = 1$, the diagram of $\mathbb{S}_{r,s,m,1}$ is* Figure *9. The handle diagram of $\mathbb{S}_{r,s,m,k}$ is* Figure *10.*

Proof. The move of the meridians of 2-handles by f is described in Figure 7. Hence by f, the meridians of four 2-handles in Figure 1 are

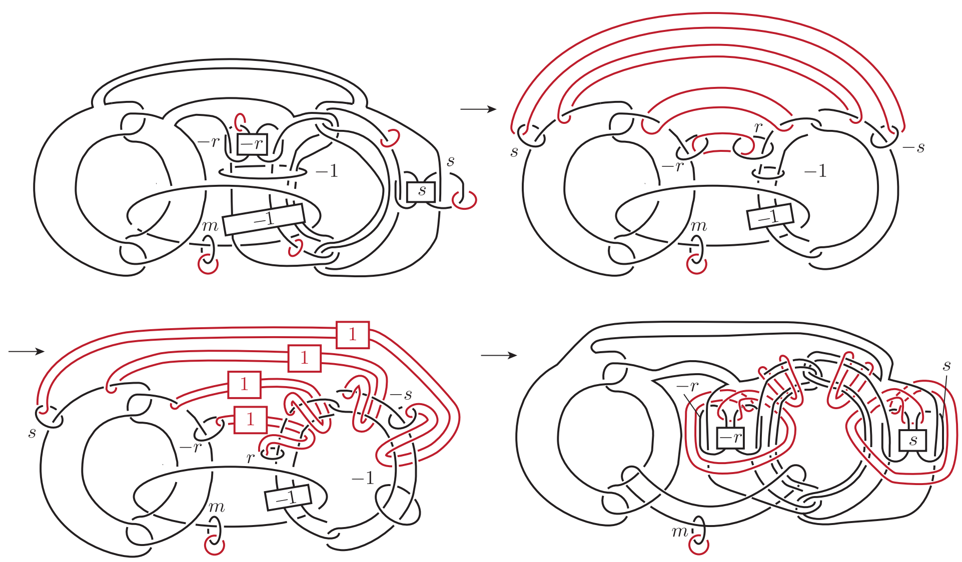

Figure 7. The move of meridians of 2-handles by the f.

moved to the link α, β, γ and δ in Figure 9.

Hence, this diagram in Figure 9 is the $k = 1$ case. In the general k case, the images of the meridians of the 2-handles by f^k are α, β, γ and δ in Figure 10. Thus the first diagram in Figure 10 describes $\mathbb{S}_{r,s,m,k}$. The β, γ and the meridian of the m-framed knot can be canceled with two 3-handles, because the calculation in Figure 8 proves that the three components are the unlink. After the canceling, β, γ and the meridian of the m-framed 2-handle are the 0-framed unlink on the handle decomposition of the 2-handlebody. Thus we cancel those together with three 3-handles. Then we obtain the second picture as in Figure 10.

□

Proof of Proposition 1.7. The union of 2-handles α, δ, two 3-handles and a 4-handle is diffeomorphic to C. Hence $\mathbb{S}_{r,s,m,k} = C \cup_{f^k} (-C)$ admits two 1-handles and four 2-handles. □

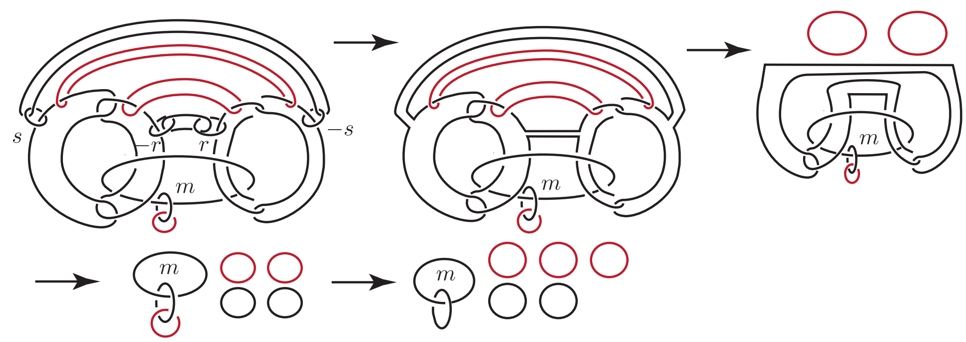

FIGURE 8. This calculation proves that the red three components is the unlink.

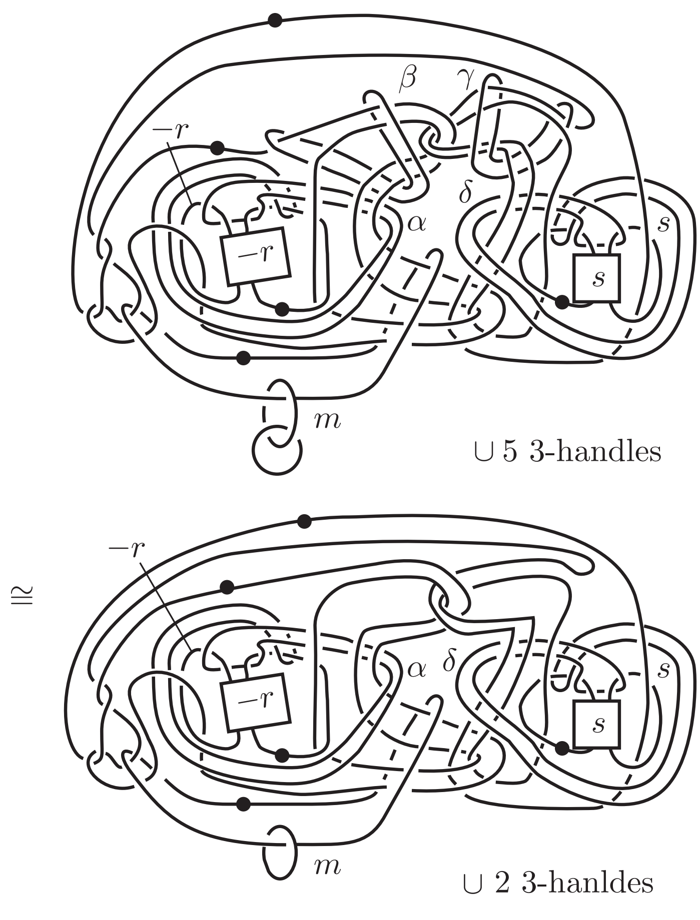

FIGURE 9. Twisted double $\mathbb{S}_{r,s,m,1}$.

In particular, in FIGURE 10, the four 2-handles are α, δ, s-framed 2-handle and $(-r)$-framed 2-handle.

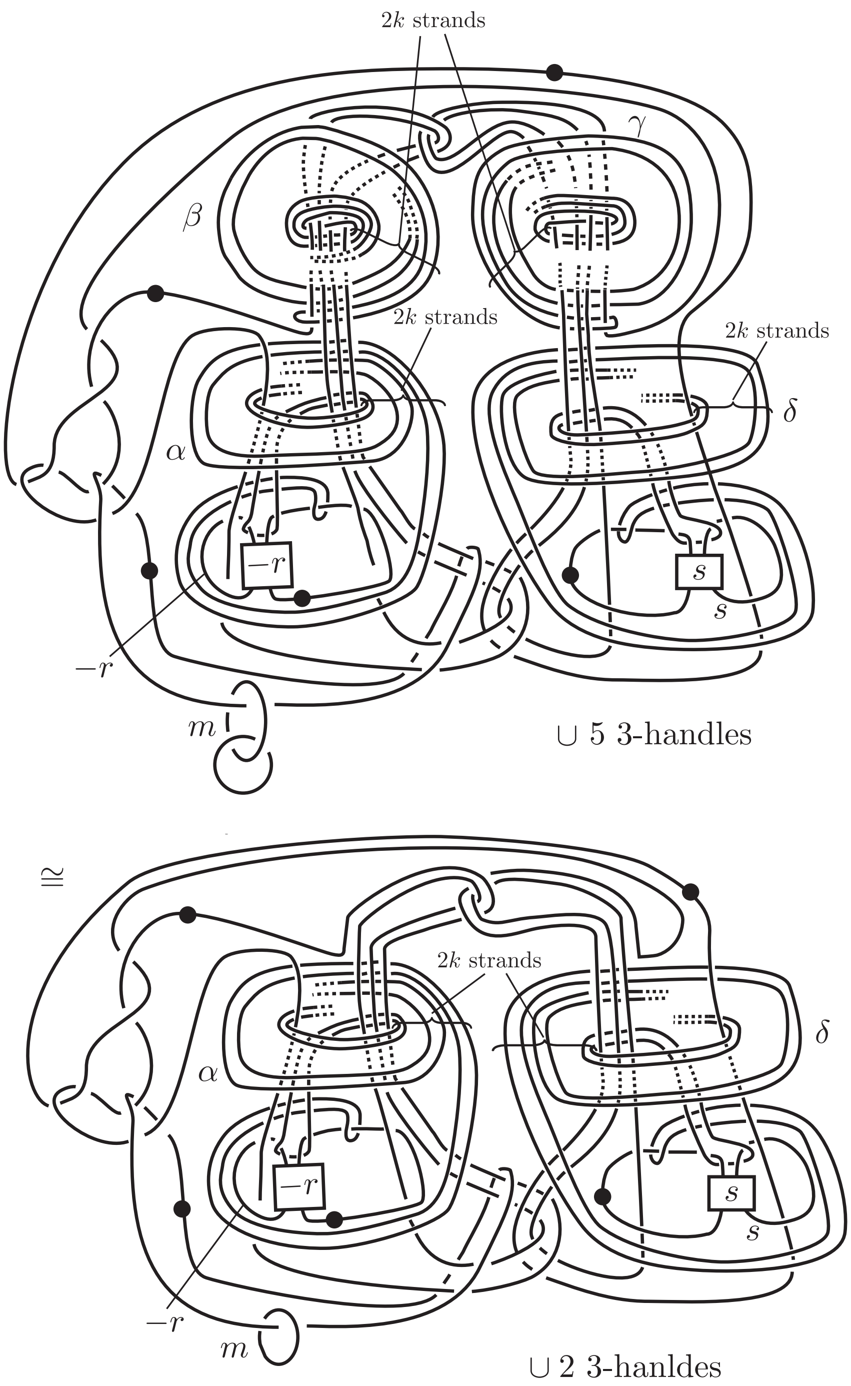

FIGURE 10. Twisted double $\mathbb{S}_{r,s,m,k}$ and a diagram after canceling.

If one proves that $\mathbb{S}_{r,s,m,k}$ is standard, the first diagram in FIGURE 10 would be useful as an auxiliary information of the handle decomposition.

Let X be a 4-manifold. We consider an embedded $S^2 \hookrightarrow X$ with trivial normal bundle. The homotopy types of self-diffeomorphisms on $S^2 \times S^1$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let τ be a self-diffeomorphism over $S^2 \times S^1$ with non-trivial homotopy class. The surgery $X \rightsquigarrow X(S^2 \times D^2, \tau)$ is called a *Gluck twist*.

3.2. **Proofs of Theorem 2.** First we prove that $\mathbb{S}_{r,s,0,k}$ is diffeomorphic to S^4. The deformation of the pictures in FIGURE 17 presents the $\mathbb{S}_{r,s,0,k} \cong S^4$.

In the diagram of $\mathbb{S}_{r,s,0,k}$ the union of the (lowest) 0-framed 2-handle and a 4-handle consists of $S^2 \times D^2$. By removing the $S^2 \times D^2$ and attaching an m-framed 2-handle and a 4-handle, we obtain the surgery $S^4 = \mathbb{S}_{r,s,0,k} \rightsquigarrow \mathbb{S}_{r,s,0,k}(S^2 \times D^2, \tau^m) = \mathbb{S}_{r,s,m,k}$. Thus $\mathbb{S}_{r,s,m,k}$ is the Gluck surgery of S^4. Hence, the diffeomorphism $\mathbb{S}_{r,s,m,k} \cong \mathbb{S}_{r,s,m-2,k}$ holds. □
The last diffeomorphism can be also verified by the calculus in FIGURE 11.

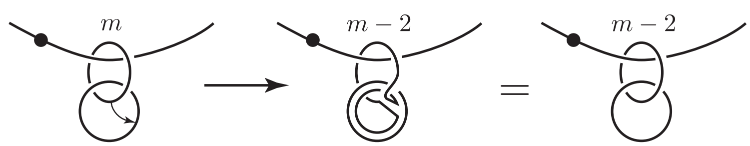

FIGURE 11. The diffeomorphism $\mathbb{S}_{r,s,m,k} \cong \mathbb{S}_{r,s,m-2,k}$.

3.3. **The log transform along several tori.** Here we define the log transform along n tori with the square zero. Let $e_i : T_i^2 \hookrightarrow X$ be disjoint embedded tori each other that the squares are all zero for $i = 1, \cdots n$. Let c_i be a curve presenting a primitive element in $H_1(T_i^2)$. Let p_i, q_i be several pairs of coprime integers. Let $\tilde{e}_i$ be an embedding of the tubular neighborhood of T^2 with respect to e_i. Suppose that a gluing diffeomorphism $g_{c_i,p_i,q_i} : T^2 \times \partial D^2 \to T_i^2 \times \partial D^2$ a diffeomorphism satisfying

$$\partial D^2 \mapsto p_i \cdot \partial D^2 + q_i \cdot c_i.$$

In fact, the image of ∂D^2 by the gluing map is the attaching sphere of the unique 2-handle in $T^2 \times D^2$ and the remaining handles of $T^2 \times D^2$ are two 3-handles and one 4-handle. The information of the image of ∂D^2 determines a diffeomorphism type. Hence, the diffeomorphism type of the log transform along the tori depends only on e_i, c_i, p_i and q_i.

We denote the log transform by

$$X[(e_i), (c_i), (p_i/q_i)].$$

and call (p_i/q_i)*-log transform along* (e_i) *with direction* (c_i). When embeddings (e_i) and curves (c_i) are clear in the context, we omit these items.

3.4. **Proof of Theorem 3.** First, we find $T^2 \times D^2$ in the diagrams in FIGURE 9 ($\mathbb{S}_{r,s,m,1}$), in general, FIGURE 10 ($\mathbb{S}_{r,s,m,k}$). Removing handles, we can show that the submanifold as in FIGURE 12 is embedded in $\mathbb{S}_{r,s,m,k}$. It contains two disjoint T^2s in the last picture.

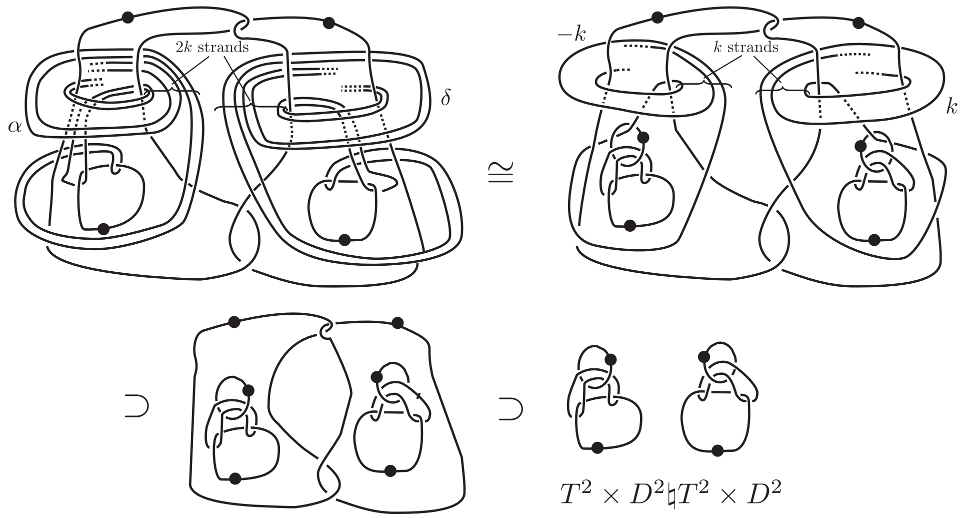

FIGURE 12. Disjoint embedded two T^2's in S^4.

One time (1/1)-log transform with direction ℓ corresponds to the change of the diagram which is given in FIGURE 13. In general, the $(1/s)$-log transform is the s-times iteration of this process. Hence, $\mathbb{S}_{r,s,m,k}$ is the $(1/s)$-log trans-

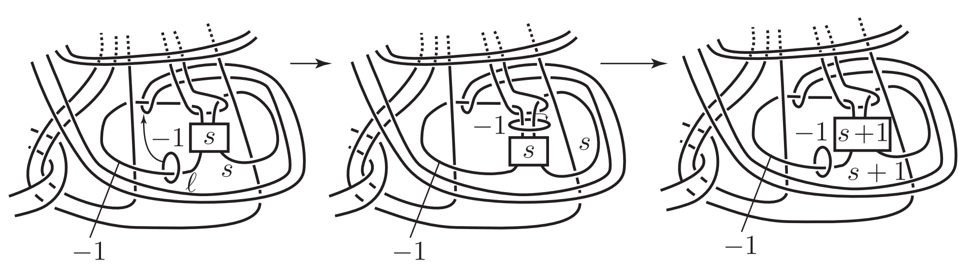

FIGURE 13. The (1/1)-log transform with direction ℓ.

form of $\mathbb{S}_{r,0,m,k}$. Since the knot $\kappa(r,0)$ is isotopic to the unknot, $C(r,0;m)$ is the standard 4-ball. Thus $\mathbb{S}_{r,0,m,k}$ is diffeomorphic to S^4.

Therefore, $\mathbb{S}_{r,s,m,k}$ is a $(1/s)$-log transform along a torus embedded in S^4. By exchanging the roles of r and s, we can show that $\mathbb{S}_{r,s,m,k}$ is a $(-1/r)$-log transform along another torus in S^4. □

Theorem 3 says that the twisted double $\mathbb{S}_{r,s,m,k}$ is obtained by two log transforms along two embedded disjoint tori in S^4 as in FIGURE 12. Namely, we have

$$\mathbb{S}_{r,s,m,k} = S^4[(e_{m,k,1}, e_{m,k,2}), (c_1, c_2), (-1/r, 1/s)] = S^4[(-1/r, 1/s)].$$

The two torus embeddings $e_{m,k,i} : T_i^2 \hookrightarrow S^4$ $(i = 1, 2)$ are embedded in such a way that each torus is embedded in each component $T^2 \times D^2$ in $\natural^2 T^2 \times D^2$. The $T^2 \times D^2 \natural T^2 \times D^2$ exterior in S^4 is described in FIGURE 14.

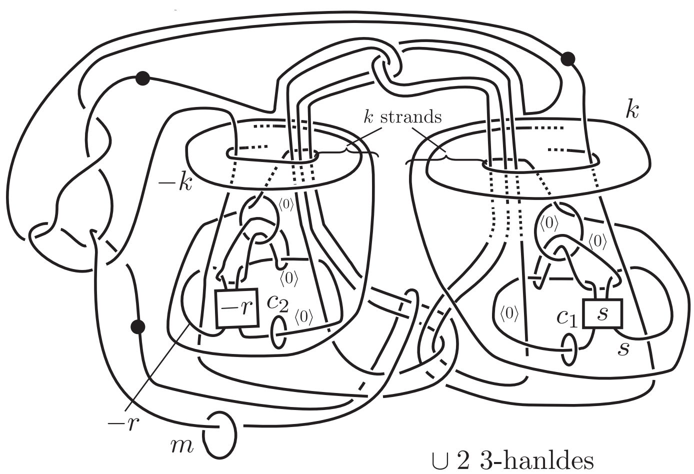

FIGURE 14. The $\natural^2 T^2 \times D^2$ exterior in S^4 and curves c_1, c_2 on the boundary.

3.5. **A remark for the curves** c_1 **and** c_2**.** As mentioned in Section 1.6, if either of curves c_1 or c_2 in the boundary has an embedded slice disk in the exterior with framing -1. However, it is difficult to find such a disk in terms of the following observation.

We consider a cobordism $\mathcal{C}$ from $\#^2 T^2 \times S^1$ to $\#^3 S^2 \times S^1$ by removing the three 3-handles and one 4-handle from the exterior which is described in FIGURE 14. We take an annulus in $\mathcal{C}$ beginning from either of c_1 or c_2. Suppose that the annulus has no critical points in $\mathcal{C}$, i.e., the annulus is the trace by the gradient flow of the Morse function for the handle decomposition. Let $\tilde{c}$ be the obtained knot in $\#^3 S^2 \times S^1$. Turning the union of three 3-handles and a 4-handle by the upside down calculus, we obtain a knot description in FIGURE 15. Clearly, this knot $\tilde{c}$ has no -1-framed disk in $\natural^3 D^3 \times S^1$. Because if there is such a disk, then by attaching a 2-handle on $\tilde{c}$ with 0-framing, we must find a (-1)-sphere in the attached manifold whose intersection form is $\langle 0 \rangle$. This is a contradiction.

This means that the easy way cannot be found a -1-framed embedded disk in the exterior for $e_{m,k,i}$.

3.6. **The cases of** r **or** $s = \infty$**.** The cases where either r or s is ∞ can be regarded as 0-log transforms along the tori from the equality $\mathbb{S}_{r,s,m,k} = S^4[-1/r, 1/s]$. Namely, $\mathbb{S}_{\infty,s,m,k} = S^4[0, 1/s]$ and $\mathbb{S}_{\infty,\infty,m,k} = S^4[0, 0]$.

$\mathbb{S}_{\infty,\infty,m,k}$ is obtained by exchanging two dots and two 0's in the sub-handle for $\natural^2 T^2 \times D^2$. The diagram of $\mathbb{S}_{\infty,\infty,m,k}$ is described in FIGURE 16.

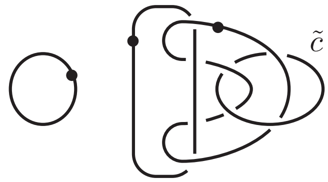

FIGURE 15. The isotopy class of $c_1 \subset \partial(\natural^3 D^3 \times S^1)$.

By computing the fundamental groups and homology groups, the manifolds $\mathbb{S}_{\infty,0,m,k}$ and $\mathbb{S}_{\infty,\infty,m,k}$ are homotopic to

$$(S^3 \times S^1)\#(S^2 \times S^2) \text{ and } \#^2(S^3 \times S^1)\#^2(S^2 \times S^2)$$

respectively. In the case of $m \equiv 0 \bmod 2$, by using the same move as FIGURE 17, we have $\mathbb{S}_{\infty,s,m,k} = (S^3 \times S^1)\#(S^2 \times S^2)$ and $\mathbb{S}_{\infty,\infty,m,k} = \#^2(S^3 \times S^1)\#^2(S^2 \times S^2)$. Thus essentially, we have to investigate the manifold $\mathbb{S}_{\infty,s,1,k}$ and $\mathbb{S}_{\infty,\infty,1,k}$. These are the Gluck twists of standard $(S^3 \times S^1)\#(S^2 \times S^2)$ or $\#^2(S^3 \times S^1)\#^2(S^2 \times S^2)$.

In [13] some homotopy $(S^3 \times S^1)\#(S^2 \times S^2)$'s are constructed according to Scharlemann's method. Some of those are diffeomorphic to the standard manifold. What one knows the relationship between these manifolds would be an interesting problem.

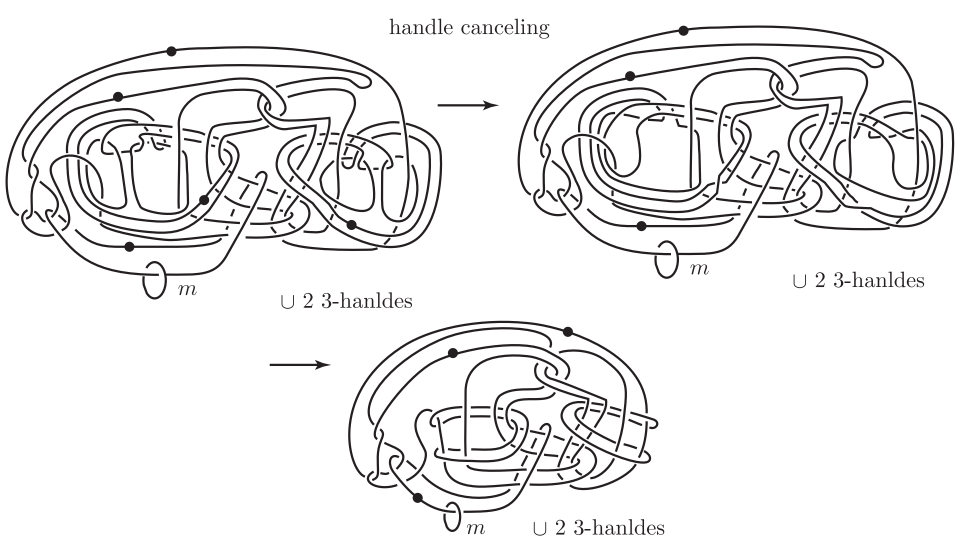

FIGURE 16. The handle diagram of $S^4[0,0]$.

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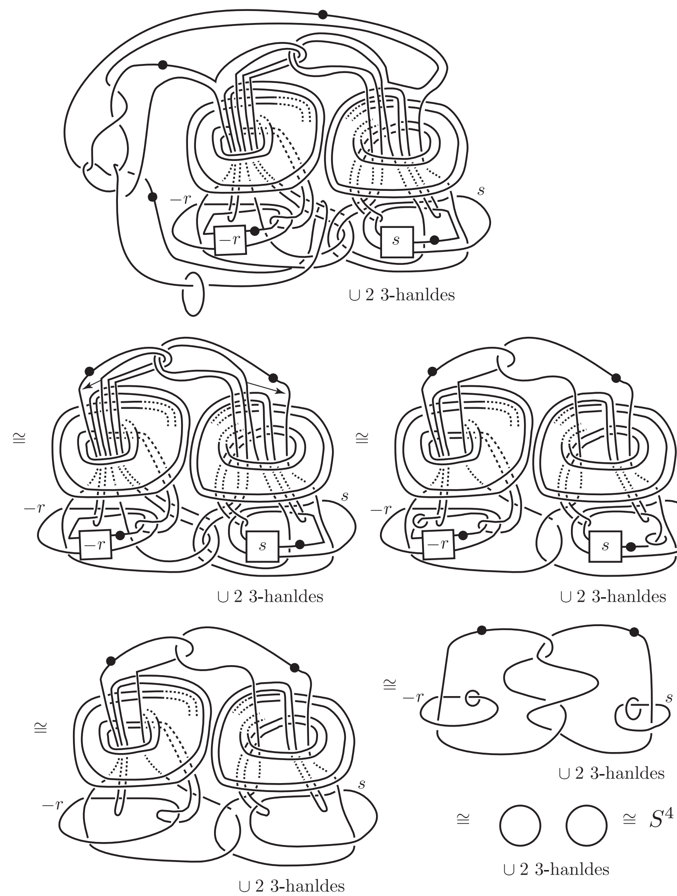

FIGURE 17. $\mathbb{S}_{r,s,0,k}$ is the standard S^4.

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